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Cooperative output regulation for linear uncertain MIMO multi-agent systems by output feedback

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Abstract In this paper, we study the cooperative output regulation problem for a class of general linear uncertain multi-input multi-output (MIMO) multi-agent systems subject to a well-defined vector relative degree. By proposing some suitable internal model, the problem is first converted into the auxiliary cooperative stabilization problem of the augmented system in the so-called strict feedback normal form. This auxiliary problem is then solved by some developed robust control techniques, such as multiple high-gain feedback and mixed distributed observer, leading to an effective distributed output feedback regulator synthesis for the original cooperative output regulation problem.

Keywords cooperative control, output regulation, MIMO, output feedback, vector relative degrees

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1 Introduction

In the past decade, control of multi-agent systems has drawn a broad interest owning to its wild applications, such as consensus of rigid spacecrafts [1], formation of Euler-Lagrange systems [2], assignment of dynamic traffics [3], schedule of multiple unmanned aerial vehicles [4], and synchronization of nonholonomic vehicles [5]. One of the significant topics is the so-called cooperative output regulation problem [6]. This problem aims to deal with the tracking of a class of reference trajectories and the rejection of a class of unmeasured disturbances for multi-agent systems, simultaneously. Because of the network constraint, it differs from the traditional output regulation problem [7], in which the tracking reference signal is usually unmeasurable making the tracking error infeasible for the feedback. Thus, in contrast to the traditional centralized control scheme, the control for multi-agent systems should be distributed without making use of the tracking error signal. The early attempt for this problem is based on the feedforward control method by employing several classes of distributed observers for the reference signal that is viewed as the leader system in a multi-agent system (see [8–10], to name just a few). This method, however, can only handle the deterministic multi-agent systems without uncertain parameter and unmeasured external disturbance.

As is known to all, it is quite necessary to consider robustness against plant uncertainties in systems and control. Actually, the output regulation for the uncertain plant can be achieved by the well-known internal model principle [7]. However, due to the infeasibility of the tracking error for the feedback, the construction of the internal model for the multi-agent system should also be of the distributed version.

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Some efforts have been made on the so-called distributed p-copy internal model (see [6, 11, 12], to name just a few), with which the cooperative output regulation problem can be converted into the eigenvalue placement problem. The benefit for this type of internal model lies in that we need only to work on the nominal model of the uncertain plant. As a result, several standard linear control methods, such as LQR-based eigenvalue placement [13], Lueberger observer [14], imperfect actuator [15], can be employed, thus leading to a structurally stable solution in the sense that the obtained p-copy internal model regulator is able to tolerate sufficiently small uncertainty that is the small perturbations to nominal agent dynamics.

Later, several efforts have been made on linear multi-agent systems with the uncertainty ranging over an arbitrarily large set which is usually assumed to be compact. Obviously, the p-copy internal model design that works only on the nominal agent dynamics is no longer applicable to such arbitrarily large uncertainty leading to the failure of the aforementioned linear control methods. The canonical internal model design has been shown to be a useful tool for this type of uncertainty [16, Chapter 7]. In particular, one has to work on the uncertain plant directly and adopt the robust control techniques by taking into account the internal structure of the control plant. For example, Ref. [17] adopts the standard high-gain observer and studies an output feedback design for the leaderless network with an artificial leader system. Ref. [18] combines a distributed observer for the leader system and the standard high-gain observer for the followers so as to deal with the general switching network. Ref. [19] presents an output feedback normal form, based on which, an output feedback controller can be synthesized by a recursive process. Ref. [20] explores a mixed distributed observer that is coupled by a distributed Lueberger observer and a modified high-gain observer, and establishes an output feedback design based on the so-called strict-feedback normal form. Notably, all the above mentioned results work only on the single-input single-output (SISO) agent dynamics with a well-defined relative degree. Regarding the general and significant multi-input multi-output (MIMO) plants, the study will be more complicated because of their more complex internal structure. To the best of our knowledge, the only recent attempt has been presented in [21] by developing their output feedback normal form subject to a well-defined strict relative degree.

Motivated by this situation, in this paper, we aim to extend the study of the cooperative output regulation problem to a class of general linear uncertain MIMO multi-agent systems subject to a well-defined vector relative degree. Notice that the vector relative degree condition is somewhat milder than the strict relative degree condition in the sense that it allows each output channel to have different relative degree [22]. We first propose some suitable internal model and convert the problem into the auxiliary cooperative stabilization problem of the augmented system in the strict feedback normal form. Then we solve the auxiliary problem by developing some novel robust control techniques, such as multiple high-gain feedback and mixed distributed observer, and obtain an effective distributed output feedback regulator synthesis for the original cooperative output regulation problem.

The rest of this paper is organized as follows. Section 2 introduces the problem formulation, as well as the problem convention. Section 3 presents our main result, and Section 4 gives an illustrative example. Finally, Section 5 concludes the paper.

2 Problem statement and preliminaries

2.1 Problem statement

Let us consider the following linear uncertain MIMO multi-agent systems:

$$\begin{aligned} \dot{x}_i &= A_i(w)x_i + B_i(w)u_i + E_i(w)v, \\ y_i &= C_i(w)x_i, \quad i = 1, \ldots, N, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^m$ are the state, input, and output of the i-th subsystem, respectively, and $w \in \mathbb{R}^{n_w}$ represents the parametric uncertainty. $v \in \mathbb{R}^q$ represents both the reference and disturbance signals, and is assumed to be generated by the linear autonomous system

$$\dot{v} = Sv,\ y_0 = F(w)v, \tag{2}$$

where $y_0 \in \mathbb{R}^m$ is the output of system (2). Then, for $i = 1, \ldots, N$, the tracking error for the i-th subsystem is defined as $e_i = y_i - y_0$. Here, without loss of generality, we assume that $S \in \mathbb{R}^{q\times q}$ has no eigenvalues with negative real parts, and all the matrices $A_i(w) \in \mathbb{R}^{n_i\times n_i}$, $B_i(w) \in \mathbb{R}^{n_i\times m}$, $E_i(w) \in \mathbb{R}^{n_i\times q}$, $C_i(w) \in \mathbb{R}^{m\times n_i}$, $F(w) \in \mathbb{R}^{m\times q}$ are continuous in w.

System (2) and N subsystems of (1) together are viewed as the leader and followers in a multi-agent system, respectively. To characterize the communication constraint among them, we can define a digraph $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$, where $\bar{\mathcal{V}} = \{0, 1, \ldots, N\}$ with the node 0 associated with the leader and the other N nodes associated with the N followers, respectively, and, for $j = 0, 1, \ldots, N$, $i = 1, \ldots, N$, $i \neq j$, $(j, i) \in \bar{\mathcal{E}}$, if and only if the control u_i can make use of the information from the j-th subsystem for the feedback control. Let $\bar{\mathcal{A}} = [a_{ij}]_{i,j=0}^N$ be any weighted adjacency matrix of $\bar{\mathcal{G}}$, i.e., for $i, j = 0, 1, \ldots, N$, $a_{ii} = 0$, $a_{ij} > 0$ if $(j, i) \in \bar{\mathcal{E}}$ and $a_{ij} = 0$ otherwise, and $a_{ij} = a_{ji}$ if (j, i) is a bidirected edge of $\bar{\mathcal{E}}$.

As in [23], we define the virtual tracking error as $e_{vi} \triangleq \sum_{j=0}^N a_{ij}(y_i - y_j)$, $i = 1, \ldots, N$. We aim to find a distributed output feedback controller of the following form:

$$u_i = g_{1i}(\varrho_i), \quad \dot{\varrho}_i = g_{2i}(\varrho_i, \varrho_{vi}, e_{vi}), \quad i = 1, \ldots, N, \tag{3}$$

where g_{1i} and g_{2i} to be determined are linear functions of their arguments vanishing at the origin, $\varrho_i \in \mathbb{R}^{n_{\varrho_i}}$ with n_{ϱ_i} to be specified later, and $\varrho_{vi} \triangleq \sum_{j=1}^N a_{ij}(\hat{C}_i\varrho_i - \hat{C}_j\varrho_j) + a_{i0}\hat{C}_i\varrho_i$ with $\hat{C}_i \in \mathbb{R}^{m\times n_{\varrho_i}}$. Then the mathematical formulation of the cooperative output regulation problem for system (1) can be detailed as follows.

Problem 1. Given systems (1) and (2), as well as the associated digraph $\bar{\mathcal{G}}$, for any given arbitrarily large compact subset $\mathbb{W}$ of $\mathbb{R}^{n_w}$, find a distributed controller of the form (3), such that, for any $w \in \mathbb{W}$,

(1) The origin of the closed-loop system is exponentially stable when v is set to zero;

(2) For any initial states $x_i(0)$, $\varrho_i(0)$, and $v(0)$, the tracking error of each subsystem satisfies $\lim_{t\to\infty} e_i(t) = 0$, $i = 1, \ldots, N$.

2.2 Preliminaries

In this subsection, for making our paper self-contained, we repeat briefly the internal model framework for the output regulation problem, and convert our problem into a trackable cooperative stabilization problem.

As shown in [23, Section 3] or [16, Section 7.5], the i-th subsystem of system (1) admits a well-defined internal model provided that for any $w \in \mathbb{W}$ and any $E_i(w)$, $F(w)$, the so-called regulator equations,

$$\begin{aligned} X_i(w)S &= A_i(w)X_i(w) + B_i(w)U_i(w) + E_i(w), \\ 0 &= C_i(w)X_i(w) + F(w), \end{aligned} \tag{4}$$

have a unique solution $(X_i(w), U_i(w))$. Here $X_i(w)v$ and $U_i(w)v$ are to characterize the steady-state of x_i and u_i, respectively, at which the tracking error $e_i \equiv 0$. Assume that the minimum polynomial of S is $\lambda^l + \alpha_l\lambda^{l-1} + \cdots + \alpha_2\lambda^1 + \alpha_1$. Let

$$\Phi = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -\alpha_1 & -\alpha_2 & \cdots & -\alpha_l \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^{\mathrm{T}}, \tag{5}$$

and (M, Q) be any controllable pair with M being Hurwitz. It is shown in [24] that the Sylvester equation $T\Phi - MT = Q\Gamma$ has the unique nonsingular solution matrix T. Furthermore, let $\Theta_i(w) =$

$\mathrm{col}(\Theta_i^{[1]}(w),\ldots,\Theta_i^{[m]}(w))$, where $\Theta_i^{[s]}(w) \triangleq T^{-1}\mathrm{col}(U_i^{[s]}(w),U_i^{[s]}(w)S,\ldots,U_i^{[s]}(w)S^{l-1})$, $s = 1,\ldots,m$, with $U_i^{[s]}(w)$ being the s-th row of the matrix $U_i(w)$. This fact results in the following so-called steady-state generator

$$\begin{aligned}\Theta_i(w)S &= (I_m\otimes M)\Theta_i(w) + (I_m\otimes Q)U_i(w),\\ U_i(w) &= (I_m\otimes\Psi)\Theta_i(w),\end{aligned}$$

where $\Psi = \Gamma T^{-1}$. As a consequence, for $i = 1,\ldots,N$, we can define

$$\dot{\eta}_i = (I_m\otimes M)\eta_i + (I_m\otimes Q)u_i, \tag{6}$$

with output $(I_m\otimes\Psi)\eta_i$ to estimate the signal $U_i(w)v$ that is not allowed in the feedback control due to the existence of w. The system

$$\begin{aligned}\dot{x}_i &= A_i(w)x_i + B_i(w)u_i + E_i(w)v,\\ \dot{\eta}_i &= (I_m\otimes M)\eta_i + (I_m\otimes Q)u_i,\\ e_i &= C_i(w)x_i + F(w)v,\end{aligned} \tag{7}$$

composed of systems (1) and (6), is called the augmented system. System (7) has an output zeroing subspace $\{(x_i,\eta_i,u_i) : x_i = X_i(w)v,\ \eta_i = \Theta_i(w)v,\ u_i = U_i(w)v\}$ that is invariant under the control input $U_i(w)v$. For this reason, we can perform on system (7) the following coordinate and input transformation:

$$\bar{x}_i = x_i - X_i(w)v,\quad \bar{\eta}_i = \eta_i - \Theta_i(w)v,\quad \bar{u}_i = u_i - (I_m\otimes\Psi)\eta_i,$$

resulting in the equivalent augmented system

$$\begin{aligned}\dot{\bar{x}}_i &= A_i(w)\bar{x}_i + B_i(w)\bar{u}_i + B_i(w)(I_m\otimes\Psi)\bar{\eta}_i,\\ \dot{\bar{\eta}}_i &= (I_m\otimes(M+Q\Psi))\bar{\eta}_i + (I_m\otimes Q)\bar{u}_i,\\ e_i &= C_i(w)\bar{x}_i,\end{aligned} \tag{8}$$

which has the output zeroing subspace exactly at the origin. According to the internal model principle [16, Secition 7.4], the solvability of the stabilization problem of system (8) implies the solvability of the cooperative output regulation problem of system (1). More specifically, if we can find a distributed controller

$$\bar{u}_i = -G_{1i}\zeta_i,\quad \dot{\zeta}_i = G_{2i}\zeta_i + G_{3i}\zeta_{vi} + G_{4i}e_{vi},\quad i = 1,\ldots,N, \tag{9}$$

where $\zeta_i \in \mathbb{R}^{n_{\zeta_i}}$ to be defined later and $\zeta_{vi} = \sum_{j=1}^N a_{ij}(\bar{C}_i\zeta_i - \bar{C}_j\zeta_j) + a_{i0}\bar{C}_i\zeta_i$ with $\bar{C}_i \in \mathbb{R}^{m\times n_{\zeta_i}}$, that stabilize the augmented system (8) for all $w \in \mathbb{W}$, then the controller

$$\begin{aligned}u_i &= -G_{1i}\zeta_i + (I_m\otimes\Psi)\eta_i,\\ \dot{\eta}_i &= (I_m\otimes M)\eta_i + (I_m\otimes Q)u_i,\\ \dot{\zeta}_i &= G_{2i}\zeta_i + G_{3i}\zeta_{vi} + G_{4i}e_{vi},\quad i = 1,\ldots,N,\end{aligned} \tag{10}$$

solves the cooperative output regulation problem of system (1) as described in Problem 1. In particular, the controller (10) is of the distributed form (3) with $\varrho_i = \mathrm{col}(\eta_i,\zeta_i)$. In this sense, we say Problem 1 can be converted into the stabilization problem of system (8). This stabilization problem is still cooperative as the stabilizer (9) is also required to be distributed.

Normally, by (4), for the given w, it is shown that the augmented system (8) is stabilizable and detectable provided that the original system (1) is stabilizable and detectable. Nevertheless, as shown in [16, Section 7.5], since the parametric uncertainty w is assumed to range over the arbitrarily large compact set $\mathbb{W}$, the stabilization problem of system (8) is in general untractable. Specifically, by employing the standing minimum phase and relative degree assumptions in the robust control field, most of early efforts have been made for SISO control plants (see [19, 20] to name just a few). Regarding MIMO control plants, the study will be more complicated because of the various descriptions of relative degrees. One recent attempt in [21] has considered the MIMO plants with strict relative degree by developing their output feedback normal form. In what follows, we further study the MIMO plants with vector relative degree. Alternatively, we adopt from [22] the strict-feedback normal form and develop a class of mixed distributed observer for estimating those infeasible states.

3 Main results

In this section, we first adopt the strict-feedback normal form for system (1) and obtain a stabilization problem of the equivalent augmented system that is shown to be stabilizable by multiple high gain feedback. Then we develop a class of distributed high-gain observer inspired by the SISO case studied in [20]. With the aid of this novel observer, we can establish a distributed output feedback stabilizer for the augmented system (8), thus, leading to the solvability of the cooperative output regulation problem for system (1).

3.1 Strict-feedback normal form

Let us give some standing assumptions.

Assumption 1. The digraph $\bar{\mathcal{G}}$ contains a directed spanning tree with the node 0 as the root.

Assumption 2. For $i = 1, \ldots, N$, each subsystem of system (1) satisfies the following properties.

(1) It is minimum phase.

(2) It has vector relative degree $[r_{1i}, \ldots, r_{mi}]$, in the sense that, for all $w \in \mathbb{W}$,

(a) For all $s = 1, \ldots, m$ and $k = 1, \ldots, r_{si} - 2$ (if $r_{si} \geqslant 2$), $C_i^{[s]}(w)[A_i(w)]^k B_i(w) = 0$, where $C_i^{[s]}(w)$ denotes the s-th row of $C_i(w)$;

(b) The high-frequency gain matrix $\Gamma_i(w)$ defined by

$$\Gamma_i(w) = \begin{bmatrix} C_i^{[1]}(w)[A_i(w)]^{r_{1i}-1}B_i(w) \\ C_i^{[2]}(w)[A_i(w)]^{r_{2i}-1}B_i(w) \\ \vdots \\ C_i^{[m]}(w)[A_i(w)]^{r_{mi}-1}B_i(w) \end{bmatrix},$$

satisfies $\mathrm{rank}(\Gamma_i(w)) = m$.

(3) There exists a known matrix $L_i \in \mathbb{R}^{m\times m}$ such that the matrix $\Gamma_i(w)L_i = [\Gamma_i(w)L_i]^{\mathrm{T}}$ is positive definite.

Remark 1. Assumptions 1 and 2 are borrowed from [21] except for replacing the strict relative degree condition with the vector relative degree condition. Their rationality has been well discussed in [21]. In particular, property (2) of Assumption 2 can guarantee the stabilizability and detectability of system (1), thus of the augmented system (8). It can also guarantee the solvability of the regulator equations (4). Here the vector relative degree condition is somewhat milder than the strict relative degree condition in the sense that it is allowed that r_{si}, $s = 1, \ldots, m$, can be different.

Remark 2. Define a matrix $H = [h_{ij}]_{i,j=1}^N$ with $h_{ii} = \sum_{j=0}^N a_{ij}$ and $h_{ij} = -a_{ij}$, for any $i \neq j$, $i, j = 1, \ldots, N$, where the elements a_{ij} are obtained from be any weighted adjacency matrix $\bar{\mathcal{A}} = [a_{ij}]_{i,j=0}^N$. Under Assumption 1, recalling from [25, Theorem 2.3], there exists a diagonal matrix $O = \mathrm{diag}(o_1, \ldots, o_N)$ with $o_i > 0$, $i = 1, \ldots, N$, such that $I_N - OH - H^{\mathrm{T}}O$ is negative definite.

With the aid of property (2) of Assumption 2, recalling the strict-feedback normal form as given by [22, Theorem 2.4], we have that, for each subsystem of (8), there exists an invertible matrix $T_i(w) \in \mathbb{R}^{n_i\times n_i}$, such that under the linear transformation $T_i(w)\bar{x}_i \triangleq \mathrm{col}(\bar{z}_i, \bar{\xi}_i, \hat{\xi}_i)$, where $\bar{z}_i \in \mathbb{R}^{n_i - \sum_{s=1}^m r_{si}}$, $\bar{\xi}_i \triangleq \mathrm{col}(\bar{\xi}_i^{[1]}, \bar{\xi}_i^{[2]}, \ldots, \bar{\xi}_i^{[m]}) \in \mathbb{R}^{\sum_{s=1}^m r_{si} - m}$ with $\bar{\xi}_i^{[s]} \triangleq \mathrm{col}(\bar{\xi}_{1i}^{[s]}, \ldots, \bar{\xi}_{(r_{si}-1)i}^{[s]}) \in \mathbb{R}^{r_{si}-1}$ for $s = 1, \ldots, m$, and

$\hat{\xi}_i = \mathrm{col}(\bar{\xi}^{[1]}_{r_{1i}i}, \bar{\xi}^{[2]}_{r_{2i}i}, \ldots, \bar{\xi}^{[m]}_{r_{mi}i}) \in \mathbb{R}^m$, system (8) is equivalent to the following form:

$$\begin{aligned}
\dot{\bar{z}}_i &= R_{zi}(w)\bar{z}_i + Y_{zi}(w)D_{zi}\bar{\xi}_i,\\
\dot{\bar{\xi}}^{[1]}_{ki} &= \bar{\xi}^{[1]}_{(k+1)i}, \quad k = 1, \ldots, r_{1i} - 1,\\
\dot{\bar{\xi}}^{[2]}_{ki} &= \bar{\xi}^{[2]}_{(k+1)i}, \quad k = 1, \ldots, r_{2i} - 1,\\
&\vdots\\
\dot{\bar{\xi}}^{[m]}_{ki} &= \bar{\xi}^{[m]}_{(k+1)i}, \quad k = 1, \ldots, r_{mi} - 1,\\
\dot{\hat{\xi}}_i &= R_{\xi i}(w)\bar{\xi}_i + Y_{\xi i}(w)\hat{\xi}_i + Z_{\xi i}(w)\bar{z}_i + \Gamma_i(w)\bar{u}_i + \Gamma_i(w)(I_m \otimes \Psi)\bar{\eta}_i,\\
\dot{\bar{\eta}}_i &= (I_m \otimes (M + Q\Psi))\bar{\eta}_i + (I_m \otimes Q)\bar{u}_i,\\
e_i &= \mathrm{col}\left(\bar{\xi}^{[1]}_{1i}, \ldots, \bar{\xi}^{[m]}_{1i}\right), \quad i = 1, \ldots, N,
\end{aligned} \tag{11}$$

where $R_{\xi i}(w)$ and $Y_{\xi i}(w)$ are matrices of compatible dimension, and $D_{zi} = \mathrm{diag}(D^{[1]}_{zi}, \ldots, D^{[m]}_{zi})$ with $D^{[s]}_{zi} = [1, 0, \ldots, 0] \in \mathbb{R}^{1\times(r_{si}-1)}$. Notice that property (1) of Assumption 2 implies that $R_{zi}(w)$ is Hurwitz for all $w \in \mathbb{W}$.

Next, for making the stabilization problem trackable, we further perform another coordinate transformation

$$\begin{aligned}
\theta_i &= \begin{bmatrix}
\gamma^{[1]}_{1i}\bar{\xi}^{[1]}_{1i} + \gamma^{[1]}_{2i}\bar{\xi}^{[1]}_{2i} + \cdots + \gamma^{1}_{(r_{1i}-1)i}\bar{\xi}^{[1]}_{(r_{1i}-1)i} + \bar{\xi}^{[1]}_{r_{1i}i}\\
\gamma^{[2]}_{1i}\bar{\xi}^{[2]}_{1i} + \gamma^{[2]}_{2i}\bar{\xi}^{[2]}_{2i} + \cdots + \gamma^{[2]}_{(r_{2i}-1)i}\bar{\xi}^{[2]}_{(r_{2i}-1)i} + \bar{\xi}^{[2]}_{r_{2i}i}\\
\vdots\\
\gamma^{[m]}_{1i}\bar{\xi}^{[m]}_{1i} + \gamma^{[m]}_{2i}\bar{\xi}^{[m]}_{2i} + \cdots + \gamma^{[m]}_{(r_{mi}-1)i}\bar{\xi}^{[m]}_{(r_{mi}-1)i} + \bar{\xi}^{[m]}_{r_{mi}i}
\end{bmatrix},\\
\tilde{\eta}_i &= \bar{\eta}_i - (I_m \otimes Q)\Gamma_i(w)^{-1}\theta_i,
\end{aligned}$$

where, for $s = 1, \ldots, m,\ k = 1, 2, \ldots, r_{si} - 1$, the coefficients $\gamma^{[s]}_{ki}$ are chosen such that the polynomial $\lambda^{(r_{si}-1)} + \gamma^{[s]}_{(r_{si}-1)i}\lambda^{(r_{si}-2)} + \cdots + \gamma^{[s]}_{2i}\lambda + \gamma^{[s]}_{1i}$ is stable. Let $\chi_i = \mathrm{col}(\bar{\xi}_i, \bar{z}_i, \tilde{\eta}_i)$. Then the augmented system (11) can be further put into the form

$$\dot{\chi}_i = A_{\chi i}(w)\chi_i + B_{\chi i}(w)\theta_i, \tag{12a}$$

$$\dot{\theta}_i = A_{\theta i}(w)\chi_i + B_{\theta i}(w)\theta_i + \Gamma_i(w)\bar{u}_i, \quad i = 1, \ldots, N. \tag{12b}$$

Here, the matrices $A_{\chi i}(w)$, $B_{\chi i}(w)$, $A_{\theta i}(w)$, and $B_{\theta i}(w)$ in system (12) are described by

$$\begin{aligned}
A_{\chi i}(w) &= \begin{bmatrix}
\Lambda_i & 0 & 0\\
Y_{zi}(w)D_{zi} & R_{zi}(w) & 0\\
-(I_m \otimes Q)\Gamma_i(w)^{-1}(D_{\theta i}\Lambda_i + R_{\xi i}(w) - Y_{\xi i}(w)D_{\theta i}) & -(I_m \otimes Q)\Gamma_i(w)^{-1}Z_{\xi i}(w) & I_m \otimes M
\end{bmatrix},\\
B_{\chi i}(w) &= \begin{bmatrix}
D_{\bar{\xi} i}\\
0\\
(I_m \otimes (M + Q\Psi)Q)\Gamma_i(w)^{-1} - (I_m \otimes Q)\Gamma_i(w)^{-1}(D_{\theta i}D_{\bar{\xi} i} + Y_{\xi i}(w) + \Gamma_i(w)(I_m \otimes \Psi Q)\Gamma_i(w)^{-1})
\end{bmatrix},\\
A_{\theta i}(w) &= [D_{\theta i}\Lambda_i + R_{\xi i}(w) - Y_{\xi i}(w)D_{\theta i} \quad Z_{\xi i}(w) \quad \Gamma_i(w)(I_m \otimes \Psi)],\\
B_{\theta i}(w) &= D_{\theta i}D_{\bar{\xi} i} + Y_{\xi i}(w) + \Gamma_i(w)(I_m \otimes \Psi Q)\Gamma_i(w)^{-1},
\end{aligned}$$

where $\Lambda_i = \mathrm{diag}(\Lambda^{[1]}_i, \ldots, \Lambda^{[m]}_i)$, $D_{\bar{\xi} i} = \mathrm{diag}(D^{[1]}_{\bar{\xi} i}, \ldots, D^{[m]}_{\bar{\xi} i})$, $D_{\theta i} = \mathrm{diag}(D^{[1]}_{\theta i}, \ldots, D^{[m]}_{\theta i})$, and, for $s =$

$1, \ldots, m$,

$$\Lambda_i^{[s]} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -\gamma_{1i}^{[s]} & -\gamma_{2i}^{[s]} & \cdots & -\gamma_{(r_{si}-1)i}^{[s]} \end{bmatrix} \in \mathbb{R}^{(r_{si}-1)\times(r_{si}-1)},\ D_{\bar{\xi}i}^{[s]} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{(r_{si}-1)\times 1},\ D_{\theta i}^{[s]} = \begin{bmatrix} \gamma_{1i}^{[s]} \\ \gamma_{2i}^{[s]} \\ \vdots \\ \gamma_{(r_{si}-1)i}^{[s]} \end{bmatrix}^{\mathrm{T}}.$$

According to the choice of $\gamma_{ki}^{[s]}$, we know Λ_i is Hurwitz. Since M and $R_{zi}(w)$ are also Hurwitz, so is $A_{\chi i}(w)$ for all $w \in \mathbb{W}$.

3.2 Distributed state feedback synthesis

We first synthesize a distributed state feedback controller to robustly stabilize the system (12) via multiple high gain feedback technique.

Lemma 1. Under Assumptions 1 and 2, there exist sufficiently large positive real numbers K_i, $i = 1, \ldots, N$, such that the distributed state feedback controller

$$\bar{u}_i = -K_i L_i \theta_{vi},\ \theta_{vi} = \sum_{j=1}^{N} a_{ij}(\theta_i - \theta_j) + a_{i0}\theta_i \tag{13}$$

stabilizes system (8) for every $w \in \mathbb{W}$.

Proof. Firstly, consider the χ_i-subsystem, $i = 1, \ldots, N$. Since the matrix $A_{\chi i}(w)$ is Hurwitz, there exists a positive matrix $P_{1i}(w) \in \mathbb{R}^{(n_i+m(l-1))\times(n_i+m(l-1))}$, such that $P_{1i}(w)A_{\chi i}(w) + A_{\chi i}(w)^{\mathrm{T}}P_{1i}(w) \leqslant -I_{n_i+m(l-1)}$. Define the positive function $V_1(\chi) = \sum_{i=1}^{N} \chi_i^{\mathrm{T}} P_{1i}(w)\chi_i$ with $\chi = \mathrm{col}(\chi_1, \ldots, \chi_N)$. Then the derivative of $V_1(\cdot)$ along the subsystem (12a) is given by

$$\begin{aligned} \dot{V}_1(\chi) &= \sum_{i=1}^{N} 2\chi_i^{\mathrm{T}} P_{1i}(w)(A_{\chi i}(w)\chi_i + B_{\chi i}(w)\theta_i) \\ &\leqslant -||\chi||^2 + 2\sum_{i=1}^{N} \chi_i^{\mathrm{T}} P_{1i}(w) B_{\chi i}(w)\theta_i \\ &\leqslant -\frac{1}{2}||\chi||^2 + 2\rho_{\chi 1}||\theta||^2, \end{aligned} \tag{14}$$

where $\rho_{\chi 1} = \max_{i=1,\ldots,N} ||P_{1i}(w)B_{\chi i}(w)||^2$.

Next, consider the θ_i-subsystem, $i = 1, \ldots, N$. Let $\theta = \mathrm{col}(\theta_1, \ldots, \theta_N)$. According to the definition of θ_{vi} in (13) and the definition of H given in Remark 2, we have $\theta_v = (I_m \otimes H)\theta$ with $\theta_v = \mathrm{col}(\theta_{v1}, \ldots, \theta_{vN})$. Define the positive function $V_2(\theta) = \sum_{i=1}^{N} V_{2i}(\theta_i)$ with $V_{2i}(\theta_i) = o_i K_i \theta_{vi}^{\mathrm{T}}[\Gamma_i(w)L_i]\theta_{vi}$. Letting $\sigma_i = K_i[\Gamma_i(w)L_i]\theta_{vi}$, we have the derivative of $V_{2i}(\cdot)$ along the subsystem (12b), which satisfies

$$\dot{V}_{2i}(\theta_i) = 2o_i K_i \theta_{vi}^{\mathrm{T}}[\Gamma_i(w)L_i]\dot{\theta}_{vi} = 2o_i\sigma_i^{\mathrm{T}}(\Omega_i^{\theta} - \Omega_i) \leqslant \frac{1}{2}||\sigma_i||^2 + 2o_i^2||\Omega_i^{\theta}||^2 - 2o_i\sigma_i^{\mathrm{T}}\Omega_i, \tag{15}$$

where $\Omega_i^{\theta} = \sum_{j=0}^{N} a_{ij}(B_{\theta i}(w)\theta_i - B_{\theta j}(w)\theta_j + A_{\theta i}(w)\chi_i - A_{\theta j}(w)\chi_j)$, and $\Omega_i = \sum_{j=0}^{N} a_{ij}(\sigma_i - \sigma_j)$. Notice that

$$\sum_{i=1}^{N} ||\Omega_i^{\theta}||^2 \leqslant \rho_{\theta 2}||\theta||^2 + \rho_{\chi 2}||\chi||^2, \tag{16}$$

where $\rho_{\theta 2} = ||H||^2 \max_{i=1,\ldots,N} ||B_{\theta i}(w)||^2$, and $\rho_{\chi 2} = ||H||^2 \max_{i=1,\ldots,N} ||A_{\theta i}(w)||^2$. According to the property of the matrix O given in Remark 2, we have

$$2\sum_{i=1}^{N} o_i\sigma_i^{\mathrm{T}}\Omega_i = \sigma^{\mathrm{T}}[I_m \otimes (OH + H^{\mathrm{T}}O)]\sigma \geqslant ||\sigma||^2, \tag{17}$$

with $\sigma \triangleq \mathrm{col}(\sigma_1, \ldots, \sigma_N)$. By (15)–(17), we have

$$\dot{V}_2(\theta) \leqslant -\frac{1}{2}||\sigma||^2 + 2\bar{o}(\rho_{\theta 2}||\theta||^2 + \rho_{\chi 2}||\chi||^2), \tag{18}$$

with $\bar{o} \triangleq \max_{i=1,\ldots,N}\{o_i^2\}$.

Finally, we definite the positive function $V(\chi,\theta) = \mu V_1(\chi) + V_2(\theta)$, with $\mu \geqslant 2 + 4\bar{o}\rho_{\chi 2}$, for the closed-loop system consisted of (12) and (13). Choose K_i in order, such that

$$K_i \geqslant \sqrt{\frac{2 + 4\bar{o}\rho_{\theta 2} + 4\mu\rho_{\chi 1}}{\lambda_m \delta_m}}, \tag{19}$$

where λ_m and δ_m denote the minimal eigenvalues of the matrices $H^{\mathrm{T}}H$ and $[\Gamma(w)L]^2$, respectively, with $\Gamma(w)L = \mathrm{diag}(\Gamma_1(w)L_1, \ldots, \Gamma_N(w)L_N)$. By (14) and (18), the derivative of V along the closed-loop system composed of (12) and (13) satisfies

$$\dot{V}(\chi,\theta) \leqslant -||\chi||^2 - ||\theta||^2.$$

Therefore, the origin of the closed-loop system is exponentially stable for any $w \in \mathbb{W}$. The proof is thus completed.

According to Lemma 1, for the special case that every subsystem of system (1) has the vector relative degree $[1,\ldots,1]$, system (8) can be stabilized, for every $w \in \mathbb{W}$, by the distributed controller $\bar{u}_i = -K_i L_i e_{vi}$. In this situation, it is not necessary to introduce the observer dynamics ζ_i, and with the argument in Subsection 2.2, the cooperative output regulation problem of system (1) as described in Problem 1 can be solved by a distributed controller

$$u_i = -K_i L_i e_{vi} + (I_m \otimes \Psi)\eta_i, \quad \dot{\eta}_i = (I_m \otimes M)\eta_i + (I_m \otimes Q)u_i, \quad i = 1,\ldots,N.$$

However, in general, θ_i and hence θ_{vi} rely on the high order derivatives of e_i. These variables are not permitted for the feedback. Thus, we have to develop some suitable observers for estimating them and utilize the observer states for the feedback instead. For this purpose, let us give Corollary 1.

Corollary 1. Under Assumption 2, there exist sufficiently large positive real numbers K_i, $i = 1,\ldots,N$, the centralized state feedback controller

$$\bar{u}_i = -K_i L_i \theta_i \tag{20}$$

stabilizes system (8) for every $w \in \mathbb{W}$.

Proof. Corollary 1 is a special case of Lemma 1 by viewing that $H = I_N$, and Assumption 1 is satisfied automatically. In particular, we can obtain a simplified Lyapunov function $V(\chi,\theta) = \mu V_1(\chi) + V_2(\theta)$ with $V_1(\chi)$ the same as that in Lemma 1 and $V_2(\theta) = \sum_{i=1}^N K_i \theta_i^{\mathrm{T}}[\Gamma_i(w)L_i]\theta_i$.

3.3 Distributed output feedback synthesis

Besides the stabilizability of the strict-feedback normal form (11), the other benefit is its detectability. In particular, the output components of (11) are completely decoupled. This fact enables us to construct a distributed observer with a similar way as the SISO case.

Let $\zeta_i = \mathrm{col}(\zeta_i^{[1]}, \zeta_i^{[2]}, \ldots, \zeta_i^{[m]})$ with $\zeta_i^{[s]} \triangleq \mathrm{col}(\zeta_{1i}^{[s]}, \ldots, \zeta_{r_{si}i}^{[s]})$. Inspired by [20], we give the following distributed observer with two distinct parts. For $s = 1,\ldots,m$, the first part of the observer is of the form

$$\dot{\zeta}_{1i}^{[s]} = \zeta_{2i}^{[s]} + h_i\beta_{(r_{si}-1)i}^{[s]}\zeta_{1i}^{[s]} + \tau_i\left(e_{vi}^{[s]} - \zeta_{vi}^{[s]}\right), \tag{21}$$

and the second part of the observer is of the form

$$\begin{aligned}
\dot{\zeta}_{2i}^{[s]} &= \zeta_{3i}^{[s]} + h_i^2\beta_{(r_{si}-2)i}^{[s]}\zeta_{1i}^{[s]} - h_i\beta_{(r_{si}-1)i}^{[s]}\left(\zeta_{2i}^{[s]} + h_i\beta_{(r_{si}-1)i}^{[s]}\zeta_{1i}^{[s]}\right),\\
\dot{\zeta}_{3i}^{[s]} &= \zeta_{4i}^{[s]} + h_i^3\beta_{(r_{si}-3)i}^{[s]}\zeta_{1i}^{[s]} - h_i^2\beta_{(r_{si}-2)i}^{[s]}\left(\zeta_{2i}^{[s]} + h_i\beta_{(r_{si}-1)i}^{[s]}\zeta_{1i}^{[s]}\right),
\end{aligned}$$

$$
\begin{aligned}
&\vdots\\
&\dot{\zeta}^{[s]}_{(r_{si}-1)i} = \zeta^{[s]}_{r_i i} + h_i^{r_{si}-1}\beta^{[s]}_{1i}\zeta^{[s]}_{1i} - h_i^{r_{si}-2}\beta^{[s]}_{2i}\left(\zeta^{[s]}_{2i} + h_i\beta^{[s]}_{(r_{si}-1)i}\zeta^{[s]}_{1i}\right),\\
&\dot{\zeta}^{[s]}_{r_{si}i} = -h_i^{r_{si}-1}\beta^{[s]}_{1i}\left(\zeta^{[s]}_{2i} + h_i\beta^{[s]}_{(r_{si}-1)i}\zeta^{[s]}_{1i}\right),
\end{aligned}
\tag{22}
$$

where the coefficients $\beta^{[s]}_{ji}$, $s = 1,\ldots,m$, $j = 1,\ldots,r_{si}-1$, $i = 1,\ldots,N$, are chosen so that the polynomials $\lambda^{r_{si}-1} + \beta^{[s]}_{(r_{si}-1)i}\lambda^{(r_{si}-2)} + \cdots + \beta^{[s]}_{2i}\lambda + \beta^{[s]}_{1i}$ are all stable. Here $\zeta^{[s]}_{1i}$ is to estimate $\bar{\xi}^{[s]}_{1i}$, and $\mathrm{col}(\bar{\zeta}^{[2]}_{2i}, \bar{\zeta}^{[s]}_{3i}, \ldots, \bar{\zeta}^{[s]}_{r_{si}i})$ is to estimate the state $\mathrm{col}(\bar{\xi}^{[s]}_{2i} - h_i\beta^{[s]}_{(r_{si}-1)i}\bar{\xi}^{[s]}_{1i}, \bar{\xi}^{[s]}_{3i} - h_i^2\beta^{[s]}_{(r_{si}-2)i}\bar{\xi}^{[s]}_{1i}, \ldots, \bar{\xi}^{[s]}_{r_{si}i} - h_i^{r_{si}-1}\beta^{[s]}_{1i}\bar{\xi}^{[s]}_{1i})$. Thus, inspired by the state feedback controller (20), we consider an output feedback control law of the form

$$
\bar{u}_i = -K_iL_i\varphi_i, \tag{23}
$$

where

$$
\varphi_i = \begin{pmatrix}
\gamma^{[1]}_{1i}\zeta^{[1]}_{1i} + \cdots + \gamma^{[1]}_{(r_{1i}-1)i}(h_i^{r_{1i}-2}\beta^{[1]}_{2i}\zeta^{[1]}_{1i} + \zeta^{[1]}_{(r_{1i}-1)i}) + (h_i^{r_{1i}-1}\beta^{[1]}_{1i}\zeta^{[1]}_{1i} + \zeta^{[1]}_{r_{1i}i})\\
\gamma^{[2]}_{1i}\zeta^{[2]}_{1i} + \cdots + \gamma^{[2]}_{(r_{2i}-1)i}(h_i^{r_{2i}-2}\beta^{[2]}_{2i}\zeta^{[2]}_{1i} + \zeta^{[2]}_{(r_{2i}-1)i}) + (h_i^{r_{2i}-1}\beta^{[2]}_{1i}\zeta^{[2]}_{1i} + \zeta^{[2]}_{r_{2i}i})\\
\vdots\\
\gamma^{[m]}_{1i}\zeta^{[m]}_{1i} + \cdots + \gamma^{[m]}_{(r_{mi}-1)i}(h_i^{r_{mi}-2}\beta^{[m]}_{2i}\zeta^{[m]}_{1i} + \zeta^{[m]}_{(r_{mi}-1)i}) + (h_i^{r_{mi}-1}\beta^{[m]}_{1i}\zeta^{[m]}_{1i} + \zeta^{[m]}_{r_{mi}i})
\end{pmatrix}. \tag{24}
$$

Now we will depict the closed-loop system under some proper coordination. Let $\nu_{1i} = \mathrm{col}(\nu^{[1]}_{1i}, \ldots, \nu^{[m]}_{1i})$ with

$$
\begin{aligned}
\nu^{[1]}_{1i} &= h_i^{r_{1i}-1}\left(\bar{\xi}^{[1]}_{1i} - \zeta^{[1]}_{1i}\right),\\
&\vdots\\
\nu^{[m]}_{1i} &= h_i^{r_{mi}-1}\left(\bar{\xi}^{[m]}_{1i} - \zeta^{[m]}_{1i}\right).
\end{aligned}
\tag{25}
$$

Let $\nu_{ai} = \mathrm{col}(\nu^{[1]}_{ai}, \nu^{[2]}_{ai}, \ldots, \nu^{[m]}_{ai})$ and $\nu^{[s]}_{ai} = \mathrm{col}(\nu^{[s]}_{2i}, \ldots, \nu^{[s]}_{r_{si}i})$, with

$$
\begin{aligned}
\nu^{[s]}_{2i} &= h_i^{r_{si}-2}\left(\bar{\xi}^{[s]}_{2i} - h_i\beta^{[s]}_{(r_{si}-1)i}\bar{\xi}^{[s]}_{1i} - \zeta^{[s]}_{2i}\right),\\
\nu^{[s]}_{3i} &= h_i^{r_{si}-3}\left(\bar{\xi}^{[s]}_{3i} - h_i^2\beta^{[s]}_{(r_{si}-2)i}\bar{\xi}^{[s]}_{1i} - \zeta^{[s]}_{3i}\right),\\
&\vdots\\
\nu^{[s]}_{r_{si}i} &= \bar{\xi}^{[s]}_{r_{si}i} - h_i^{r_{si}-1}\beta^{[s]}_{1i}\bar{\xi}^{[s]}_{1i} - \zeta^{[s]}_{r_{si}i}.
\end{aligned}
\tag{26}
$$

From (25) and (26), we have (21)–(23) are equivalent to

$$
\bar{u}_i = K_iL_i(-\theta_i + D_{2i}(h_i)\nu_{ai} + D_{3i}(h_i)\nu_{1i}), \tag{27a}
$$

$$
\dot{\nu}_{1i} = -\tau_i\left(\sum_{j=1}^{N} a_{ij}(\nu_{1i} - \nu_{1j}) + a_{i0}\nu_{1i}\right) + h_iD_{4i}\nu_{1i} + h_iD_{5i}\nu_{ai}, \tag{27b}
$$

$$
\dot{\nu}_{ai} = h_iD_{6i}\nu_{ai} + D_{7i}\dot{\theta}_i + h_iD_{8i}\nu_{1i}, \tag{27c}
$$

where $D_{2i}(h_i) = \mathrm{diag}[D^{[1]}_{2i}(h_i), \ldots, D^{[m]}_{2i}(h_i)]$, $D_{3i}(h_i) = \mathrm{diag}[D^{[1]}_{3i}(h_i), \ldots, D^{[m]}_{3i}(h_i)]$, $D_{4i} = \mathrm{diag}[\beta^{[1]}_{(r_{1i}-1)i}, \beta^{[2]}_{(r_{2i}-1)i}, \ldots, \beta^{[m]}_{(r_{mi}-1)i}]$, $D_{5i} = \mathrm{diag}[D^{[1]}_{5i}, \ldots, D^{[m]}_{5i}]$, $D_{6i} = \mathrm{diag}[D^{[1]}_{6i}, \ldots, D^{[m]}_{6i}]$, $D_{7i} = \mathrm{diag}[D^{[1]}_{7i}, \ldots, D^{[m]}_{7i}]$, and $D_{8i} = \mathrm{diag}[D^{[1]}_{8i}, \ldots, D^{[m]}_{8i}]$, with, for $s = 1,\ldots,m$,

$$
D^{[s]}_{2i}(h_i) = \left[\frac{\gamma^{[s]}_{2i}}{h_i^{r_{si}-2}}, \ldots, \frac{\gamma^{[s]}_{(r_{si}-1)i}}{h_i}, 1\right],\ D^{[s]}_{3i}(h_i) = \beta^{[s]}_{1i} + \frac{\gamma^{[s]}_{(r_{si}-1)i}\beta^{[s]}_{2i}}{h_i} + \cdots + \frac{\gamma^{[s]}_{1i}}{h_i^{r_{si}-1}},\ D^{[s]}_{5i} = [1, 0_{1\times(r_{si}-2)}],
$$

$$D_{6i}^{[s]} = \begin{bmatrix} -\beta_{(r_{si}-1)i}^{[s]} & 1 & 0 & \cdots & 0 \\ -\beta_{(r_{si}-2)i}^{[s]} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\beta_{2i}^{[s]} & 0 & 0 & \cdots & 1 \\ -\beta_{1i}^{[s]} & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad D_{7i}^{[s]} = \begin{bmatrix} 0_{(r_{si}-1)\times 1} \\ 1 \end{bmatrix}, \quad D_{8i}^{[s]} = \begin{bmatrix} \beta_{(r_{si}-2)i}^{[s]} - (\beta_{(r_{si}-1)i}^{[s]})^2 \\ \beta_{(r_{si}-3)i}^{[s]} - \beta_{(r_{si}-2)i}^{[s]}\beta_{(r_{si}-1)i}^{[s]} \\ \vdots \\ \beta_{1i}^{[s]} - \beta_{2i}^{[s]}\beta_{(r_{si}-1)i}^{[s]} \\ -\beta_{1i}^{[s]}\beta_{(r_{si}-1)i}^{[s]} \end{bmatrix}.$$

Then we have Lemma 2.

Lemma 2. Under Assumptions 1 and 2, there exist sufficiently large positive real numbers K_i, h_i, and τ_i, $i = 1, \ldots, N$, such that, the origin of the closed-loop system composed of (12) and (27) is exponentially stable for every $w \in \mathbb{W}$. That is to say, the distributed output feedback controller (23) with the observer consisted of (21) and (22) stabilizes system (8) for every $w \in \mathbb{W}$.

Proof. Firstly, we consider the $\mathrm{col}(\chi_i, \theta_i)$-subsystem, $i = 1, \ldots, N$. It is noted that for any $h_i \geqslant 1$, $||D_{2i}(h_i)|| \leqslant ||D_{2i}(1)|| \triangleq \prod_{s=1}^{m} \varepsilon_{1i}^{[s]} \triangleq \varepsilon_{1i}$, $||D_{3i}(h_i)|| \leqslant \prod_{s=1}^{m}(|\beta_{1i}^{[s]}| + |\gamma_{(r_{si}-1)i}^{[s]}\beta_{2i}^{[s]}| + \cdots + |\gamma_{2i}^{[s]}\beta_{(r_{si}-1)i}^{[s]}| + |\gamma_{1i}^{[s]}|) \triangleq \varepsilon_{2i}$. Let $W_1(\chi,\theta) = 4V(\chi,\theta)$ with $V(\chi,\theta)$ be the same as that in Corollary 1. Then the derivative of $W_1(\chi,\theta)$ along $\mathrm{col}(\chi_i,\theta_i)$-subsystem satisfies

$$\begin{aligned} \dot{W}_1(\chi,\theta) &\leqslant -4||\chi||^2 - 4||\theta||^2 + 8\sum_{i=1}^{N} K_i^2 \theta_i^{\mathrm{T}} [\Gamma_i(w)L_i]^2 (D_{2i}(h_i)\nu_{ai} + D_{3i}(h_i)\nu_{1i}) \\ &\leqslant -4||\chi||^2 - 2||\theta||^2 + 16\sum_{i=1}^{N} K_i^4 ||\Gamma_i(w)L_i||^4 (\varepsilon_{1i}^2||\nu_{ai}||^2 + \varepsilon_{2i}^2||\nu_{1i}||^2). \end{aligned} \tag{28}$$

Secondly, we consider the ν_{ai}-subsystem, $i = 1, \ldots, N$. Let $\nu_a = \mathrm{col}(\nu_{a1}, \ldots, \nu_{aN})$. From the choice of $\beta_{ji}^{[s]}$, $s = 1, \ldots, m$, $j = 1, \ldots, r_{si} - 1$, we know $D_{6i}^{[s]}$ is Hurwitz. Therefore, there exists a positive matrix $P_{2i}^{[s]} \in \mathbb{R}^{(r_{si}-1)\times(r_{si}-1)}$, such that $[D_{6i}^{[s]}]^{\mathrm{T}} P_{2i}^{[s]} + P_{2i}^{[s]} D_{6i}^{[s]} \leqslant -I_{r_{si}-1}$. Let $W_2(\nu_a) = \sum_{i=1}^{N} \nu_{ai}^{\mathrm{T}} P_{2i} \nu_{ai}$ with $P_{2i} = \mathrm{diag}[P_{2i}^{[1]}, \ldots, P_{2i}^{[m]}]$. Then the derivative of $W_2(\cdot)$ along the ν_{ai}-subsystem satisfies

$$\begin{aligned} &\dot{W}_2(\nu_a) \\ &= \sum_{i=1}^{N} 2\nu_{ai}^{\mathrm{T}} P_{2i}(h_i D_{6i}\nu_{ai} + D_{7i}\dot{\theta}_i + h_i D_{8i}\nu_{1i}) \\ &\leqslant -\sum_{i=1}^{N} h_i||\nu_{ai}||^2 + 2\sum_{i=1}^{N} \nu_{ai}^{\mathrm{T}} P_{2i} D_{7i}(A_{\theta i}(w)\chi_i + B_{\theta i}(w)\theta_i + K_i\Gamma_i(w)L_i(-\theta_i + D_{2i}(h_i)\nu_{ai} + D_{3i}(h_i)\nu_{1i})) \\ &\quad + 2\sum_{i=1}^{N} h_i \nu_{ai}^{\mathrm{T}} P_{2i} D_{8i}\nu_{1i} \\ &= -\sum_{i=1}^{N}(h_i - 2 - 2K_i\varepsilon_{1i}||P_{2i}D_{7i}\Gamma_i(w)L_i|| - ||P_{2i}D_{7i}A_{\theta i}(w)||^2 - ||P_{2i}D_{7i}(B_{\theta i}(w) - K_i\Gamma_i(w)L_i)||^2)||\nu_{ai}||^2 \\ &\quad + ||\chi||^2 + ||\theta||^2 + \sum_{i=1}^{N}(K_i^2\varepsilon_{2i}^2||P_{2i}D_{7i}\Gamma_i(w)L_i||^2 + h_i^2||P_{2i}D_{8i}||^2)||\nu_{1i}||^2. \end{aligned} \tag{29}$$

Thirdly, we consider the ν_{1i}-subsystem, $i = 1, \ldots, N$. Let $\nu_1 = \mathrm{col}(\nu_1^{[1]}, \ldots, \nu_1^{[m]})$ with $\nu_1^{[s]} = \mathrm{col}(\nu_{11}^{[s]}, \ldots, \nu_{1N}^{[s]})$. Then the N ν_{1i}-subsystem can be put into the following compact form

$$\nu_1 = -(I_m \otimes \tau H)\nu_1 + D_9\nu_1 + D_{10}, \tag{30}$$

where $\tau = \mathrm{diag}[\tau_1 \ldots, \tau_N]$, $D_9 = \mathrm{diag}[D_9^{[1]}, \ldots, D_9^{[m]}]$, $D_{10} = \mathrm{col}(D_{10}^{[1]}, \ldots, D_{10}^{[m]})$, with, for $s = 1, \ldots, m$,

$D_9^{[s]} = \mathrm{diag}[h_1\beta_{(r_{s1}-1)1}^{[s]}, \ldots, h_N\beta_{(r_{sN}-1)N}^{[s]}]$, $D_{10}^{[s]} = \mathrm{col}(h_1 D_{51}^{[s]}\nu_{a1}^{[s]}, \ldots, h_N D_{5N}^{[s]}\nu_{aN}^{[s]})$. According to Remark 2, under Assumption 1, there exists a positive diagonal matrix $P_3 \in \mathbb{R}^{N\times N}$ such that $I_{mN} - I_m \otimes (H^{\mathrm{T}}P_3 + P_3 H)$ is negative definite. Let $W_3(\nu_1) = \nu_1^{\mathrm{T}}(I_m \otimes H)^{\mathrm{T}}(I_m \otimes \tau P_3)(I_m \otimes H)\nu_1$. Then the derivative of $W_3(\cdot)$ along the subsystem (30) satisfies

$$\begin{aligned}\dot{W}_3(\nu_1) =& 2\nu_1^{\mathrm{T}}(I_m \otimes H)^{\mathrm{T}}(I_m \otimes \tau P_3)(I_m \otimes H)[-(I_m \otimes \tau H)\nu_1 + D_9\nu_1 + D_{10}] \\ \leqslant & -\frac{\tau_m^2\delta_m}{2}||\nu_1||^2 + 4||(I_m \otimes P_3H)||^2||D_9||^2||\nu_1||^2 + 4||(I_m \otimes P_3H)||^2\sum_{i=1}^{N} h_i^2||D_{5i}||^2||\nu_{ai}||^2, \end{aligned} \quad (31)$$

where $\tau_m = \min_{i=1,\ldots,N}\{\tau_1, \ldots, \tau_N\}$ and δ_m can be found in (19).

Finally, we definite the positive function $W(\chi, \theta, \nu_a, \nu_1) = W_1(\chi, \theta) + W_2(\nu_a) + \frac{1}{h_m^2}W_3(\nu_1)$ with $h_m = \max_{i=1,\ldots,N}\{h_1, \ldots, h_N\}$ for the closed-loop system composed of (12) and (27). By (28), (29), and (31), choosing h_i such that

$$\begin{aligned} h_i \geqslant & \max_{w\in\mathbb{W}}\{2K_i\varepsilon_{1i}||P_{2i}D_{7i}\Gamma_i(w)L_i|| + ||P_{2i}D_{7i}A_{\theta i}(w)||^2 + ||P_{2i}D_{7i}(B_{\theta i}(w) - K_i\Gamma_i(w)L_i)||^2 \\ & + 16K_i^4\varepsilon_{1i}^2||\Gamma_i(w)L_i||^4\} + 4||(I_m \otimes P_3H)||^2||D_{5i}||^2 + 3, \end{aligned} \quad (32)$$

and τ_i such that

$$\tau_i \geqslant \tau_m \geqslant \sqrt{\frac{2h_m^2}{\kappa_m}\max_{i=1,\ldots,N}\max_{w\in\mathbb{W}} c_i(w)}, \quad (33)$$

with

$$c_i(w) = 16K_i^4\varepsilon_{2i}^2||\Gamma_i(w)L_i||^4 + K_i^2\varepsilon_{2i}||P_{2i}D_{7i}\Gamma_i(w)L_i||^2 + h_i^2||P_{2i}D_{8i}||^2 + 4||(I_m \otimes P_3H)||^2||D_9||^2 + 1,$$

gives that the derivative of $W(\cdot)$ along the closed-loop system composed of (12) and (27) satisfies

$$\dot{W}(\chi, \theta, \nu_a, \nu_1) \leqslant -(||\chi||^2 + ||\theta||^2 + ||\nu_1||^2 + ||\nu_a||^2).$$

Therefore, the origin of the closed-loop system is exponentially stable for every $w \in \mathbb{W}$. The proof is thus completed.

Notice that the controller (23) with the observer consisted of (21) and (22) can be put into the form of (9) with the matrices G_{1i}, G_{2i}, G_{3i}, G_{4i} determined as

$$\begin{aligned} G_{1i} &= L_i\mathrm{diag}\left[G_{1i}^{[1]}, G_{1i}^{[2]}, \ldots, G_{1i}^{[m]}\right], \quad & G_{2i} = \mathrm{diag}\left[G_{2i}^{[1]}, G_{2i}^{[2]}, \ldots, G_{2i}^{[m]}\right], \\ G_{3i} &= \mathrm{diag}\left[G_{3i}^{[1]}, G_{3i}^{[2]}, \ldots, G_{3i}^{[m]}\right], \quad & G_{4i} = \mathrm{diag}\left[G_{4i}^{[1]}, G_{4i}^{[2]}, \ldots, G_{4i}^{[m]}\right], \end{aligned} \quad (34)$$

where, for $s = 1, \ldots, m$,

$$\begin{aligned} G_{1i}^{[s]} &= -K_i\left[h_i^{r_{si}-1}D_{3i}^{[s]}(h_i)\ \gamma_{2i}^{s}\ \cdots\ \gamma_{r_{si}-1}^{[s]}\ 1\right], \\ G_{2i}^{[s]} &= \begin{bmatrix} h_i\beta_{(r_{si}-1)i}^{[s]} & 1 & \cdots & 0 \\ h_i^2(\beta_{(r_{si}-1)i}^{[s]} - (\beta_{(r_{si}-1)i}^{[s]})^2) & -h_i\beta_{(r_{si}-1)i}^{[s]} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -h_i^{r_{si}-1}(\beta_{1i}^{[s]} - \beta_{2i}^{[s]}\beta_{(r_{si}-1)i}^{[s]}) & -h_i^{r_{si}-2}\beta_{2i}^{[s]} & \cdots & 1 \\ -h_i^{r_{si}}\beta_{1i}^{[s]}\beta_{(r_{si}-1)i}^{[s]} & -h_i^{r_{si}-1}\beta_{1i}^{[s]} & \cdots & 0 \end{bmatrix}, \\ G_{3i}^{[s]} &= \begin{bmatrix} -\tau_i \\ 0_{(r_{si}-1)\times 1} \end{bmatrix}, \quad G_{4i}^{[s]} = \begin{bmatrix} \tau_i \\ 0_{(r_{si}-1)\times 1} \end{bmatrix}. \end{aligned} \quad (35)$$

Therefore, with the argument in Subsection 2.2, we can immediately obtain Theorem 1.

Theorem 1. Under Assumptions 1 and 2, the cooperative output regulation problem of system (1) as described in Problem 1 can be solved by a distributed output feedback controller of the form (10) with the matrices G_{1i}, G_{2i}, G_{3i}, G_{4i} given by (34) and (35).

According to the above argument, we can obtain the design process of controller (10), which is summarized by Algorithm 1.

Algorithm 1 Synthesis for distributed output feedback controller (10)

Ensure: Assumptions 1 and 2 hold for systems (1) and (2).

1: Find the minimum polynomial of S and determine the matrix Φ, Γ; {by (5)}
2: Find any controllable pair M and Q; {by the Hurwitz matrix M}
3: Get T and $\Psi = \Gamma T^{-1}$; {by solving $T\Phi - MT = Q\Gamma$}
4: Define the internal model and form the augmented system; {by (6) and (7)}
5: Determine the distributed state feedback gain K_i; {by (19)}
6: Determine the observer gains h_i and τ_i in sequence; {by (32) and (33)}
7: Find matrices G_{1i}, G_{2i}, G_{3i}, G_{4i}. {by (34) and (35)}

4 A simulation example

In this section, to illustrate the result, we consider a group of linear heterogenous MIMO multi-agent systems of the form (1) with $m = 2$ and $N = 4$. The system matrices are given as, for $i = 1, 2$,

$$A_i(w) = \begin{bmatrix} 0 & 1 & 0 & 0 & \frac{1}{3} \\ c_{1i}(w) & c_{2i}(w) & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 & c_{3i}(w) \end{bmatrix},\ B_i(w) = \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix},\ C_i(w) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}},\ E_i(w) = \begin{bmatrix} c_{4i}(w) & 0 \\ 0 & c_{5i}(w) \\ c_{4i}(w) & 0 \\ 0 & 0 \\ 0 & c_{5i}(w) \end{bmatrix},$$

for $i = 3, 4$,

$$A_i(w) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ c_{1i}(w) & -1 & 1 & 0 & -2 & \frac{1}{3} \\ 1 & c_{2i}(w) & c_{3i}(w) & 2 & 0 & 0 \\ -1 & 2 & 0 & c_{1i}(w) & -1 & 3 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & c_{2i}(w) & c_{3i}(w) \end{bmatrix},\ B_i(w) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 3 \end{bmatrix},\ C_i(w) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

and

$$E_i(w) = \begin{bmatrix} c_{4i}(w) & 0 & 0 & c_{4i}(w) & 0 & 0 \\ 0 & 0 & c_{5i}(w) & 0 & 0 & c_{5i}(w) \end{bmatrix}^{\mathrm{T}}.$$

The leader is in the form of (2) with $S = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $F(w) = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. We assume that the network topology $\bar{\mathcal{G}}$ of the whole system is illustrated in Figure 1, which obviously satisfies Assumption 1. We assume that, for $i = 1, 2, 3, 4$, the parameters $c_{1i}(w) = -2 + w_{1i}^{\frac{3}{2}}$, $c_{ji}(w) = 2(1 - w_{ji}) + w_{ji}^{\frac{5}{2}}$, $j = 2, 3, 4, 5$. Determine the compact set $\mathbb{W} = \{w_{1i} \in [-3, 1], w_{3i} \in [1, 3], w_{ji} \in [-3, 3], j = 2, 4, 5\}$. It can be verified that agents 1 and 2 have vector relative degree $[2, 3]$, agents 3 and 4 have vector relative degree $[3, 2]$.

According to the Algorithm 1, we have $\Phi = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $\Gamma = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]^{\mathrm{T}}$. Here we define the controllable pair (M, Q) with $M = \left[\begin{smallmatrix} 0 & 1 \\ -2 & -2 \end{smallmatrix}\right]$, $Q = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and obtain $\Psi = [1\ 2]$. Define the gains as $K_1 = K_2 = K_3 = K_4 = 10$,

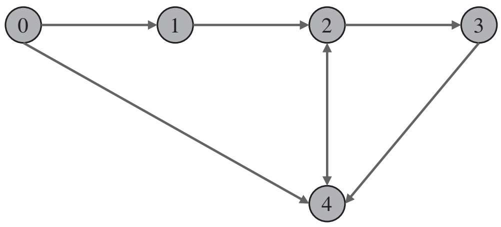

Figure 1 The digraph $\bar{\mathcal{G}}_i$.

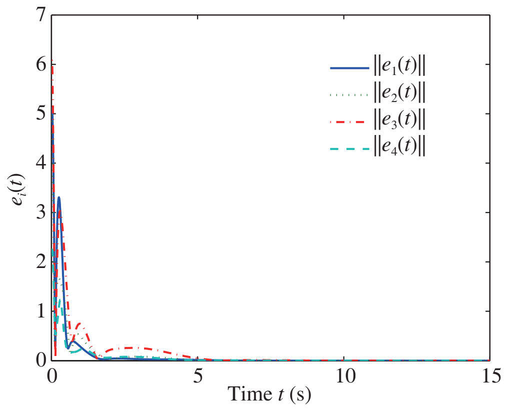

Figure 2 (Color online) Response of regulated errors.

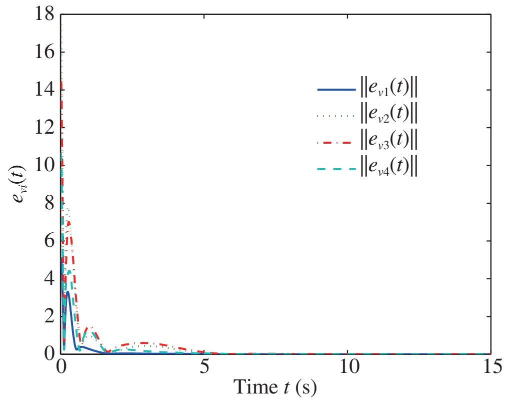

Figure 3 (Color online) Response of virtual regulated errors.

$h_1 = h_2 = h_3 = h_4 = 40$, $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 50$. The simulation results are shown in Figures 2 and 3, with the initial conditions of the closed-loop system are chosen randomly. Both figures verify that the regulated errors and the virtual regulated errors of all the followers converge to zero asymptotically. Thus our control law has successfully solved the cooperative robust regulation problem of linear MIMO multi-agent systems defined in Problem 1.

5 Conclusion

In this paper, we have considered the cooperative output regulation problem for a class of general linear uncertain MIMO systems with the vector relative degrees. An effective distributed output feedback regulator synthesis has been presented based on several robust control techniques, such as the internal model principle, the multiple high-gain feedback and the mixed distributed observer. The future work will focus on the same problem for nonlinear MIMO multi-agent systems. It is worth mentioning that we may adopt the adaptive dynamic high gain for handling the more general case that the set $\mathbb{W}$ is non-compact, which we leave as one of another future topics.

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