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Observability of Boolean control networks

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Abstract We show some new results on the observability of Boolean control networks (BCNs). First, to study the observability, we combine two BCNs with the same transition matrix into a new BCN. Then, we propose the concept of a reachable set that results in a given set of initial states, and we derive four additional necessary and sufficient conditions for the observability of BCNs. In addition, we present an algorithm and construct an observability graph to determine the observability of BCNs. Finally, we illustrate the obtained results using three numerical examples.

Keywords Boolean control network, observability, semi-tensor product

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1 Introduction

Boolean networks (BNs) are important models that are widely used in biological systems. They were first proposed by Kauffman [1] in 1969. In BNs, each gene is modeled as a binary device [1] or a node of the network with two possible levels of operation (i.e., ON: 1 or OFF: 0) at each discrete time point. The Boolean variables of nodes at time $t + 1$ are determined by the logical relationship with others at time t in the same network. Hence, BNs are autonomous systems, that is, if the initial Boolean states of all genes are assigned, the corresponding dynamical behavior is uniquely determined. Moreover, to manipulate BNs, Boolean control networks are extended models, which adds (Boolean) control inputs in the network.

Over the past few decades, Cheng et al. [2] proposed a new matrix product called the semi-tensor product (STP) of matrices, which has been successfully and widely employed to express and analyze BNs and BCNs. By utilizing the STP, the logical dynamics of BNs or BCNs can be converted into a standard discrete-time system with an algebraic form. In many recent literatures, the algebraic form of BCNs or BNs has been widely used to study different interesting issues, such as controllability [3–6], realization [7], observability [8], optimal control [9, 10], decomposition [11, 12], stabilization [13–15], synchronization [16, 17], and function perturbations [18]. The robust control invariance of BCNs was studied in [19], and the periodic trajectories and fault-detection analysis of BCNs were investigated in [20, 21]. The systematic analysis of switched Boolean networks (SBNs), probabilistic (control) Boolean networks (PBNs or PBCNs), and the tracking control of BCNs have been also considered in many studies [15, 22–

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25]. Moreover, many problems have been investigated in other fields using the STP, such as Morgen's problem [26] and evolutionary networked games [27, 28]. Recently, Lu et al. [29] reported a survey on the STP method along with its applications in logical networks and other finite-valued systems.

The observability of BNs (or BCNs) has become an interesting problem in recent years. The observability determines whether the initial state can be identified from outputs. To date, there are four main kinds of results pertaining to the observability of BCNs [8, 30–32]. These four kinds of definitions are essentially the same idea, and utilize control inputs to determine the observability of BCNs from output measurements under corresponding definitions. In [33], Zhang et al. studied the observability of BCNs under these definitions by utilizing finite automata. Moreover, Cheng et al. reported on the observability of BCNs and analyzed these four kinds of definitions, and this further resulted in a standard definition in [34]. They claimed that for any two distinct states, an input sequence can be injected, and would lead to different output trajectories; then, the BCN is observable [33]. Unfortunately, the computational complexity in [33] is a severe problem. Therefore, Cheng et al. proposed a simple alternative approach for the standard observability definition of BCNs. Motivated by Zhang and Cheng' studies, we propose some new results pertaining to the problem of the observability of BCNs.

In this study, we focus on the observability of BCNs considering the standard definition, and we obtained some new results. The main contributions of this paper can be summarized as follows.

(1) To investigate the observability, we utilize the STP of matrices to combine two BCNs with the same transition matrix into a new BCN. This leads to a concise algebraic form.

(2) We propose four necessary and sufficient conditions (Theorems 1, 3–5) for the observability of BCNs using simple approaches.

(3) Based on the knowledge of graph theory, we also propose an effective algorithm to determine the observability of BCNs.

The paper is organized as follows. In Section 2, we introduce some preliminaries on the STP and the algebraic form of BCNs. In Section 3, we investigate the observability of BCNs and present the main results. Moreover, we show the effectiveness of the obtained results by presenting two examples. In Section 5, we conclude the paper.

2 Preliminaries

2.1 Semi-tensor product (STP) of matrices

Notation. First, we introduce some preliminaries on the STP of matrices.

- $\mathbb{N}$: set of natural integers.
- $[1, n]$: set of integers $\{1, 2, \ldots, n\}$.
- $\mathrm{Col}_i(A)$ ($\mathrm{Row}_i(A)$): the ith column (row) of matrix A.
- δ_n^i: the ith column of identity matrix I_n.
- Δ_n: set of columns of identity matrix I_n.
- $\mathscr{D} := \{1, 0\}$. Identify each element in $\mathscr{D}$ with a vector as $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$; then $\mathscr{D} \sim \Delta_2$.
- A_{ij}: entry at the ith row and jth column of matrix A.
- Matrix $A_{m\times n}$ is called a logical (Boolean) matrix if its columns $\mathrm{Col}_i(A) \in \Delta_m$ ($A_{ij} \in \mathscr{D}$).
- $\mathcal{L}_{m\times n}$ ($\mathscr{B}_{m\times n}$): set of all $m \times n$ logical (Boolean) matrices.
- $\delta_m[i_1, i_2, \ldots, i_n]$: a matrix $[\delta_m^{i_1}, \delta_m^{i_2}, \ldots, \delta_m^{i_n}] \in \mathcal{L}_{m\times n}$.
- $x +_{\mathscr{B}} y := x \vee y$ and $x \times_{\mathscr{B}} y := x \wedge y$, $x, y \in \mathscr{D}$.
- $A +_{\mathscr{B}} B := (A_{ij} \vee B_{ij})$, $A, B \in \mathscr{B}_{m\times n}$. In particular, if $A, B \in \mathscr{B}_{n\times n}$, then $A \times_{\mathscr{B}} B := (\sum_{k=1}^{n}(A_{ik} \vee B_{kj}))$ and $A^{(k)} := \underbrace{A \times_{\mathscr{B}} \cdots \times_{\mathscr{B}} A}_{k}$.
- Matrix $A > 0$: $A_{ij} > 0$.
- $\mathrm{InCol}(A) := \{\delta_n^i | \mathrm{Col}_i(A) > 0\}$, where A is an $n \times m$ matrix.
- $|C|$: the cardinality of a set C.
- $\mathbf{0}_{m\times n}$: an $m \times n$ zero matrix.

- $\mathbf{1}_k = [\underbrace{1,1,\ldots,1}_{k}]^{\mathrm{T}}$.

Definition 1 ([2]). The STP of two matrices $A \in M_{m\times n}$ and $B \in M_{p\times q}$ is defined as

$$A \ltimes B = (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p}),$$

where $\alpha = \mathrm{lcm}(n,p)$ is the least common multiple of n and p, and $\otimes$ is the tensor (or Kronecker) product.

Lemma 1 ([2]). Any logical function $f(x_1,\ldots,x_n)$ with logical arguments $x_1,\ldots,x_n \in \Delta_2$ can be expressed in a multi-linear form as

$$f(x_1,\ldots,x_n) = M_f \ltimes x_1 \ltimes x_2 \ltimes \cdots \ltimes x_n,$$

where $M_f \in \mathcal{L}_{2\times 2^n}$ is the structure matrix of f.

2.2 Algebraic representation of Boolean control networks

A BCN with an output can be described as follows:

$$\begin{cases} x_1(t+1) = f_1(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t)), \\ \quad\vdots \\ x_n(t+1) = f_n(x_1(t),\ldots,x_n(t),u_1(t),\ldots,u_m(t)), \\ y_1(t) = h_1(x_1(t),\ldots,x_n(t)), \\ \quad\vdots \\ y_p(t) = h_p(x_1(t),\ldots,x_n(t)), \end{cases} \tag{1}$$

where $x_i \in \Delta_2$, $i \in [1,n]$, $u_l \in \Delta_2$, $l \in [1,m]$, $y_j \in \Delta_2$, $j \in [1,q]$, $t = 0,1,2,\ldots$ are the states, inputs (or controls), outputs and discrete time, respectively, $f_i : \Delta_2^n \to \Delta_2$, $i \in [1,n]$, and $h_j : \Delta_2^p \to \Delta_2$, $j \in [1,p]$ are logical functions.

Using Lemma 1, we describe a BN in algebraic form as follows:

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ y(t) = Hx(t), \end{cases} \tag{2}$$

where $x(t) = \ltimes_{i=1}^n x_i(t) \in \Delta_{2^n}$, $u(t) = \ltimes_{i=1}^m u_i(t) \in \Delta_{2^m}$, $y(t) = \ltimes_{i=1}^p y_i(t) \in \Delta_{2^p}$, $L \in \mathcal{L}_{2^n\times 2^{n+m}}$ and $H \in \mathcal{L}_{2^p\times 2^n}$. For additional details, please refer to [2].

3 Main results

3.1 Problem formulation

In this subsection, we investigate the observability for BCNs. Before giving the definition of observability, we denote the control sequence by

$$\pi^k = \{u(0),u(1),\ldots,u(k)\}, \quad k \geqslant 0.$$

In addition, $\prod^k$ denotes the set of all control sequences $\{u(0),u(1),\ldots,u(k)\}$, $k \geqslant 0$. For convenience, let $x(t;x_0,\pi^{t-1})$ and $y(t;x_0,\pi^{t-1})$ be the state and the output of system (1) at time t with initial state $x_0 \in \Delta_{2^n}$ under the control sequence $\pi^{t-1} \in \prod^{t-1}$. The following definition of observability is proposed in [32].

Definition 2 ([32]). Consider BCN (1). We call (x,x') the initial distinguishable state pair if $x \neq x'$ and $Hx \neq Hx'$. BCN (1) is observable if for $x \neq x'$ and $Hx = Hx'$, $x,x' \in \Delta_{2^n}$, there exists an integer $t > 0$, such that

$$y(t;x,\pi^{t-1}) \neq y(t;x',\pi^{t-1}).$$

Let $x'(t+1) = Lu(t)x'(t)$ be the same system as system (1), and the initial state of the new system may be different from the one in system (1). To facilitate the analysis, let $z(t) = x(t) \ltimes x'(t)$, which is obviously a one-one mapping from $\Delta_{2^n} \times \Delta_{2^n}$ to $\Delta_{2^{2n}}$. Therefore, using STP, we have

$$\begin{aligned} z(t+1) &= x(t+1) \ltimes x'(t+1) \\ &= Lu(t)x(t)Lu(t)x'(t) \\ &= L(I_{2^m 2^n} \otimes L)u(t)x(t)u(t)x'(t) \\ &= L(I_{2^m 2^n} \otimes L)u(t)W_{[2^m,2^n]}u(t)x(t)x'(t) \\ &= L(I_{2^m 2^n} \otimes L)(I_{2^m} \otimes W_{[2^m,2^n]})\Phi_m u(t)z(t) \\ &:= Eu(t)z(t), \end{aligned} \tag{3}$$

where $E = L(I_{2^m 2^n} \otimes L)(I_{2^m} \otimes W_{[2^m,2^n]})\Phi_m \in \mathcal{L}_{2^{2n} \times 2^{2n+m}}$.

System (3) is an algebraic expression for a constructed BCN, which combines two BCNs (1) with the same transition matrix. Denote by $C_0 = \{\delta_{2^{2n}}^{(i-1)2^n+i} | i \in [1, 2^n]\}$ the forbidden states, and let $C = \Delta_{2^{2n}} \backslash C_0$.

Lemma 2. Given a BCN (1), denote by B_0 the set of the initial distinguishable state pairs. If $(\delta_{2^n}^i, \delta_{2^n}^j) \in B_0$, then $(\delta_{2^n}^j, \delta_{2^n}^i) \in B_0$ and

$$(H^{\mathrm{T}}H)_{ij} = 0 = (H^{\mathrm{T}}H)_{ji}.$$

Proof. If $(\delta_{2^n}^i, \delta_{2^n}^j)$ is an initial distinguishable pair, that is, the corresponding outputs $H\delta_{2^n}^i$ and $H\delta_{2^n}^j$ are different, it is implied that $(\delta_{2^n}^j, \delta_{2^n}^i)$ is also an initial distinguishable pair, and $(H\delta_{2^n}^i)^{\mathrm{T}}(H\delta_{2^n}^j) = 0 = (H\delta_{2^n}^j)^{\mathrm{T}}(H\delta_{2^n}^i)$, i.e., $(H^{\mathrm{T}}H)_{ij} = 0 = (H^{\mathrm{T}}H)_{ji}$. The lemma is proved.

By multiplying two elements in each pair in B_0, we can simply express the set B_0 with a new set defined as $B = \{\delta_{2^n}^{i_1}\delta_{2^n}^{j_1}, \delta_{2^n}^{i_2}\delta_{2^n}^{j_2}, \ldots, \delta_{2^n}^{i_b}\delta_{2^n}^{j_b}\} = \{\delta_{2^{2n}}^{c_1}, \delta_{2^{2n}}^{c_2}, \ldots, \delta_{2^{2n}}^{c_b}\}$. Let $b = |B_0| = |B|$ and $\Theta = \{c_1, c_2, \ldots, c_b\}$. Hence, $C' = C \backslash B = \{\delta_{2^{2n}}^{e_1}, \delta_{2^{2n}}^{e_2}, \ldots, \delta_{2^{2n}}^{e_d}\}$ is the set of indistinguishable states at the initial time. In other words, for all $z_0 = \delta_{2^n}^i \delta_{2^n}^j \in C'$, $H\delta_{2^n}^j = H\delta_{2^n}^i$ holds. Hereafter, let $d = |C'|$ and $\Xi = \{i | i \in [1, 2^{2n}],\ \delta_{2n}^i \in C'\}$. Denote by $R_k(z_0)$ the set of states reaching the terminal state $z_0 \in B$ within k steps. Then, the set ending up with the initial distinguishable state set B within k steps can be defined as

$$R_k(B) = R_k(\delta_{2^{2n}}^{c_1}) \cup R_k(\delta_{2^{2n}}^{c_2}) \cup \cdots \cup R_k(\delta_{2^{2n}}^{c_b}).$$

Denote by $S_k(B) = R_k(B) \backslash R_{k-1}(B)$ the set of states that reach B in exactly k steps. The next lemma is obvious, and hence the proof is omitted.

Lemma 3. (1) Denote $R_0(B) = B$, then $B = R_0(B) \subseteq R_1(B) \subseteq \cdots \subseteq R_k(B) \subseteq \cdots \subseteq C$.

(2) If for some $k \in \mathbb{N}$, $R_k(B) = R_{k+1}(B)$, then $R_t(B) = R_k(B)$ or $S_{k+1}(B) \subseteq R_k(B)$ holds for all $t \geqslant k$.

Using Lemma 3, we obtain the following proposition.

Proposition 1. If $k^* \in \mathbb{N}$ is the minimum integer, such that $R_{k+1}(B) = R_k(B) \subseteq C$ or $S_{k+1}(B) \subseteq R_k(B)$, then $0 \leqslant k^* \leqslant \frac{d}{2}$.

Proof. First, k is obviously a nonnegative integer. It holds that $|C'| = d$ and the fact that $\delta_{2^n}^i \delta_{2^n}^j$ is equivalent to $\delta_{2^n}^j \delta_{2^n}^i$ from the perspective of observability. Hence, C' has $\frac{d}{2}$ equivalent states. According to the pigeonhole principle and Lemma 3, there exists a minimum integer $k^* \leqslant \frac{d}{2}$, such that $R_{k+1}(B) = R_k(B)$. Hence, $R_{k^*+1}(B) = R_{k^*}(B)$, which implies $S_{k^*+1}(B) \subseteq R_{k^*}(B)$. The proof is completed.

3.2 Some necessary and sufficient conditions

The following result is straightforward based on Lemma 3 and Proposition 1.

Theorem 1. BCN (1) is observable if and only if there exists an integer k, $0 \leqslant k \leqslant \frac{d}{2}$, such that

$$R_k(B) = C.$$

As is already known, the state vector $\delta_{2^{2n}}^{c_i} \in B$ $(i \in [1, b])$ is a column of $I_{2^{2n}}$, and by summing up these b state vectors, the state set B can be simply expressed with a new vector defined as

$$\tilde{B} = (0, \dots 0, \overset{(c_1)}{1}, 0, \dots, 0, \overset{(c_b)}{1}, 0, \dots, 0)^{\mathrm{T}},$$

where the $c_i (i \in [1, b])$ element of $\tilde{B}$ is 1 and others are 0. Similarly, we can do the same operations on sets C and C' that result in $\tilde{C} \sim C$ and $\tilde{C}' \sim C'$. Let $\tilde{R}_k(\cdot)$ be the vector form of $R_k(\cdot)$. Then, we have

$$\tilde{R}_k(B) \sim R_k(B) = \bigcup_{i=1}^{b} R_k(\delta_{2^{2n}}^{c_i}) \sim \sum_{i=1}^{b}{}_{\mathscr{B}} \tilde{R}_k(\delta_{2^{2n}}^{c_i}),$$

which implies that

$$\tilde{R}_k(B) = \sum_{i=1}^{b}{}_{\mathscr{B}} \tilde{R}_k(\delta_{2^{2n}}^{c_i}). \tag{4}$$

Let $\tilde{S}_k(B) = \tilde{R}_k(B) \backslash \tilde{R}_{k-1}(B)$.

The following result provides a simple algebraic expression for $\tilde{R}_k(B)$. Because matrix E in (3) is an $2^{2n} \times 2^m 2^{2n}$ logical matrix, we split it into 2^m square blocks as $E = [E_1, E_2, \dots, E_{2^m}]$, where $E_j \in \mathcal{L}_{2^{2n} \times 2^{2n}}$, $j \in [1, 2^m]$.

Theorem 2. Consider system (3). Then, we have

$$\tilde{R}_k(B) = \begin{cases} \tilde{B}, & k = 0, \\ \displaystyle\sum_{i=1}^{k}{}_{\mathscr{B}} \sum_{j=1}^{b}{}_{\mathscr{B}} \left(\mathrm{Row}_{c_j} \left(\Gamma^{(i)} \right) \right)^{\mathrm{T}} +_{\mathscr{B}} \tilde{B}, & k > 0, \end{cases} \tag{5}$$

where $\Gamma = \sum_{i=1}^{2^m}{}_{\mathscr{B}} E_i$.

Proof. We will prove this theorem by mathematical induction on the integer k. When $k = 1$ and given the terminal state $\delta_{2^{2n}}^{c_i}$, based on (3), we have

$$\begin{aligned} \tilde{S}_1(\delta_{2^{2n}}^{c_i}) =& \left(\mathrm{Row}_{c_i}(E \ltimes \delta_{2^m}^1)\right)^{\mathrm{T}} +_{\mathscr{B}} \cdots +_{\mathscr{B}} \left(\mathrm{Row}_{c_i} \left(E \ltimes \delta_{2^m}^{2^m} \right) \right)^{\mathrm{T}} \\ =& \sum_{j=1}^{2^m}{}_{\mathscr{B}} (\mathrm{Row}_{c_i}(E_j))^{\mathrm{T}} \\ =& (\mathrm{Row}_{c_i}(\Gamma))^{\mathrm{T}}. \end{aligned} \tag{6}$$

Eq. (6) together with (4) yields

$$\tilde{S}_1(B) = \sum_{j=1}^{b}{}_{\mathscr{B}} \tilde{S}_1(\delta_{2^{2n}}^{c_j}) = \sum_{j=1}^{b}{}_{\mathscr{B}} \left(\mathrm{Row}_{c_i}(\Gamma)\right)^{\mathrm{T}},$$

and $\tilde{R}_1(B) = \tilde{S}_1(B) +_{\mathscr{B}} \tilde{B}$, which means that Eq. (5) holds for the case of $k = 1$.

Now, suppose that Eq. (5) holds for $k = p - 1$. According to the induction, the case of $k = p$ is considered. Then, $\tilde{R}_p(B) = \tilde{R}_{p-1}(B) \cup \tilde{S}_p(B)$. Note that $\tilde{S}_p(B)$ refers to the set of states that arrive at the state $\delta_{2^{2n}}^{c_i}$ in exactly p steps. For the entry Γ_{ij}^p, if $\Gamma_{ij}^p = 1$, it signifies that there exists at least a control sequence driving the state $\delta_{2^{2n}}^{j}$ to the state $\delta_{2^{2n}}^{i}$ in exactly p steps, and then

$$\tilde{S}_p(B) = \sum_{j=1}^{b}{}_{\mathscr{B}} \tilde{S}_p(\delta_{2^{2n}}^{c_j}) = \sum_{j=1}^{b}{}_{\mathscr{B}} \left(\mathrm{Row}_{c_i} \left(\Gamma^{(p)} \right) \right)^{\mathrm{T}}.$$

Therefore,

$$\begin{aligned}\tilde{R}_p(B) =&\tilde{R}_{p-1}(B) \cup \tilde{S}_p(B)\\ =&\sum_{i=1}^{p-1}{}_{\mathscr{B}}\sum_{j=1}^{b}{}_{\mathscr{B}} \left(\mathrm{Row}_{c_j}\left(\Gamma^{(i)}\right)\right)^{\mathrm{T}} +_{\mathscr{B}} \tilde{B} +_{\mathscr{B}} \sum_{j=1}^{b}{}_{\mathscr{B}} \left(\mathrm{Row}_{c_j}\left(\Gamma^{(p)}\right)\right)^{\mathrm{T}}\\ =&\sum_{i=1}^{p}{}_{\mathscr{B}}\sum_{j=1}^{b}{}_{\mathscr{B}} \left(\mathrm{Row}_{c_j}\left(\Gamma^{(i)}\right)\right)^{\mathrm{T}} +_{\mathscr{B}} \tilde{B}.\end{aligned}$$

The theorem is proved.

With respect to Theorem 2, Theorem 1 can be restated in a different and intuitive form.

Theorem 3. BCN (1) is observable if and only if there exists an integer k with $0 \leqslant k \leqslant \frac{d}{2}$, such that

$$\sum_{i=1}^{k}{}_{\mathscr{B}}\sum_{j=1}^{b}{}_{\mathscr{B}} \left(\mathrm{Row}_{c_j}\left(\Gamma^{(i)}\right)\right)^{\mathrm{T}} +_{\mathscr{B}} \tilde{B} = \tilde{C}.$$

Laschov and Margaliot [35] defined a matrix $Q = E \ltimes \mathbf{1}_{2^m}$ (which is similar yet different from the matrix Γ defined in Theorem 2) to investigate the controllability of BNs. Based on matrix Q, we present the following result from [35] before establishing another necessary and sufficient condition for the observability of BCN (1).

Lemma 4 ([35])**.** Let $\delta_{2^{2n}}^i$, $\delta_{2^{2n}}^j \in \Delta_{2^{2n}}$, $l(k; i, j)$ denote the number of controls from $\delta_{2^{2n}}^i$ to $\delta_{2^{2n}}^j$ in exactly k steps, and set $Q = E \ltimes 1_{2^m}$. Then, we can conclude that

$$l(k; i, j) = \left(\delta_{2^{2n}}^j\right)^{\mathrm{T}} Q^k \left(\delta_{2^{2n}}^i\right) = \left(Q^k\right)_{ji},$$

and

$$\sum_{j=1}^{2^{2n}} \left(Q^k\right)_{ji} = 2^{km}, \quad \forall\, i \in [1, 2^{2n}].$$

Based on Lemma 4 and Proposition 1, the following result answers the observability problem of BCN (1).

Theorem 4. BCN (1) is observable if and only if there exists an integer k satisfying $0 \leqslant k \leqslant \frac{d}{2}$, such that,

$$\begin{cases} |B| = 2^{2n} - 2^n, \quad k = 0,\\ \displaystyle\sum_{j \notin \Theta} \bar{Q}_{ji}^k \neq \frac{2^{(k+1)m} - 2^m}{2^m - 1}, \quad \forall i \in \Xi, \quad k \neq 0, \end{cases} \tag{7}$$

where $\bar{Q}^k = \sum_{\alpha=1}^{k} Q^\alpha$.

Proof. (Sufficiency). Suppose that Eq. (7) holds. If $k = 0$ and $|B| = 2^{2n} - 2^n$, then it can be induced that $B = C$. In other words, all pairs $(\delta_{2^{2n}}^i, \delta_{2^{2n}}^j)$ with $i \neq j$ are initial distinguishable. Using Lemma 4, the sum of all elements in every column of Q^k is 2^{km}. Then, we can construct the matrix $\bar{Q}^k$ by adding all the Q^α, $\alpha \in [1, k]$. Further, we can conclude that the sum of all elements in every column of $\bar{Q}^k$ is $\frac{2^{(k+1)m} - 2^m}{2^m - 1}$. Therefore, from (7), we find that for all $i \in \Xi$, there exists at least one element in $\bar{Q}^k$ with row index $j \in \Theta$ being nonzero. In other words, for any $i \in \Xi$, there is a control sequence π^{t-1}, $t \in [1, k]$ converting $\delta_{2^{2n}}^i$ to some initial distinguishable state $\delta_{2^{2n}}^j \in B$. Assume that $\delta_{2^{2n}}^i = \delta_{2^n}^{i_1} \delta_{2^n}^{i_2}$. Then, $\delta_{2^n}^{i_1}$ and $\delta_{2^n}^{i_2}$ are observable. Hence, the BCN (1) is observable.

(Necessity). Suppose that BCN (1) is observable. If it is initial observable, then $k = 0$, $B = \Delta_{2^{2n}} \backslash C_0 = C$, namely, $|B| = 2^{2n} - 2^n$. From Definition 2, when $i \neq j$ and $H\delta_{2^n}^i = H\delta_{2^n}^j$, there exists a control sequence $\pi^{k_{ij}-1}$ such that $y(k_{ij}; \delta_{2^n}^i, \pi^{k_{ij}-1}) \neq y(k_{ij}; \delta_{2^n}^j, \pi^{k_{ij}-1})$. In other words, there exists a control sequence $\pi^{k_{ij}-1}$ that converts $\delta_{2^n}^i \delta_{2^n}^j$ to some initial distinguishable state $v \in \Theta$. Then, we can obtain the maximum value of $\{k_{ij} | i \neq j$ and $H\delta_{2^n}^i = H\delta_{2^n}^j\}$ as $k = \max\{k_{ij}\}$. Therefore, from Proposition 1, Eq. (7) holds.

Remark 1. In this paper, we investigate the observability of BCN (1) by multiplying x and x'. It leads a concise algebraic form to study the observability of BCNs. Moreover, we propose two necessary and sufficient conditions for the observability of BCNs in simple ways in Theorems 3 and 4. Specifically, Theorem 4 gives a numerical form for the observability of BCNs.

Theorems 3 and 4 have given necessary and sufficient conditions for the observability of BCN (1). Moreover, another necessary and sufficient condition is given to determine the observability of BCN (1) as follows.

Theorem 5. BCN (1) is observable if there does not exist a permutation matrix $P \in \mathcal{L}_{2^{2n}\times 2^{2n}}$, and an integer k with $k > 0$ such that

(a) $PQP^{\mathrm{T}} = \begin{pmatrix} \mathcal{F} & \mathbf{0}_{2k\times 2^n} & \mathcal{A}_2 \\ \mathcal{A}_1 & \mathcal{G} & \mathcal{A}_3 \\ \mathbf{0}_{(2^{2n}-2^n-2k)\times 2k} & \mathbf{0}_{(2^{2n}-2^n-2k)\times 2^n} & \mathcal{J} \end{pmatrix}$,

(b) $\mathrm{InCol}(P^{\mathrm{T}}\bar{\mathcal{F}}P) \subset C\backslash B$,

(c) $\mathrm{InCol}(P^{\mathrm{T}}\bar{\mathcal{G}}P) = C_0$,

where $\mathcal{F} \in \mathbb{R}^{2k\times 2k}$, $\mathcal{G} \in \mathbb{R}^{2^n\times 2^n}$, $\mathcal{J} \in \mathbb{R}^{(2^{2n}-2^n-2k)\times(2^{2n}-2^n-2k)}$, $\mathcal{A}_1 \in \mathbb{R}^{2^n\times 2k}$, $\mathcal{A}_2 \in \mathbb{R}^{2k\times(2^{2n}-2^n-2k)}$, $\mathcal{A}_3 \in \mathbb{R}^{2^n\times(2^{2n}-2^n-2k)}$,

$$\bar{\mathcal{F}} = \begin{pmatrix} \mathcal{F} & \mathbf{0}_{2k\times(2^{2n}-2k)} \\ \mathbf{0}_{(2^{2n}-2k)\times 2k} & \mathbf{0}_{(2^{2n}-2k)\times(2^{2n}-2k)} \end{pmatrix},$$

$$\bar{\mathcal{G}} = \begin{pmatrix} \mathbf{0}_{2k\times 2k} & \mathbf{0}_{2k\times 2^n} & \mathbf{0}_{2k\times(2^{2n}-2^n-2k)} \\ \mathbf{0}_{2^n\times 2k} & \mathcal{G} & \mathbf{0}_{2^n\times(2^{2n}-2^n-2k)} \\ \mathbf{0}_{(2^{2n}-2^n-2k)\times 2k} & \mathbf{0}_{(2^{2n}-2^n-2k)\times 2^n} & \mathbf{0}_{(2^{2n}-2^n-2k)\times(2^{2n}-2^n-2k)} \end{pmatrix}.$$

Proof. (Sufficiency). Suppose that conditions (a)–(c) are satisfied. According to condition (a) and considering the new network of matrix PQP^{T}, it can be concluded that node $\delta_{2^{2n}}^i$, $i \in [1, 2k]$ cannot be driven to node $\delta_{2^{2n}}^j$, $j \in [2^n + 2k + 1, 2^{2n}]$ in the new network (owing to $(P^{\mathrm{T}}QP)_{ji} = 0$). Further, from condition (b), it means that using the permutation matrix P as an inverse transformation on the new network into the origin network (1), $P\delta_{2^{2n}}^i \in C\backslash B$, $i \in [1, 2k]$. Therefore, for any $\delta_{2^{2n}}^q \in \mathrm{InCol}(P^{\mathrm{T}}\bar{\mathcal{F}}P)$, it cannot be driven to any node of $\mathrm{InCol}(P^{\mathrm{T}}\bar{\mathcal{J}}P)$, or B ($B \subset \mathrm{InCol}(P^{\mathrm{T}}\bar{\mathcal{J}}P)$ obviously holds). Then, assuming for any node $\delta_{2^{2n}}^q = \delta_{2^n}^{q_1}\delta_{2^n}^{q_2} \in \mathrm{InCol}(P^{\mathrm{T}}\bar{\mathcal{F}}P)$, pair $(\delta_{2^n}^{q_1}, \delta_{2^n}^{q_2})$ is indistinguishable in BCN (1), which implies that BCN (1) is not observable.

(Necessity). Suppose that BCN (1) is not observable. Then, there exists $2k$, $k > 0$ indistinguishable pairs $(\delta_{2^{2n}}^i, \delta_{2^{2n}}^j)$ (also $(\delta_{2^{2n}}^i, \delta_{2^{2n}}^j)$, so are $2k$ pairs), which indicates that $\delta_{2^{2n}}^i\delta_{2^{2n}}^j$ cannot be driven to any node of B, but some node of C_0. Therefore, we can find a permutation matrix P such that conditions (a)–(c) are satisfied.

Example 1. The following BCN is considered with $L = \delta_4[1\ 3\ 2\ 1\ 1\ 3\ 2\ 3]$ and $H = \delta_4[1\ 2\ 2\ 2]$.

$$\begin{aligned} x(t+1) &= Lu(t)x(t), \\ y(t) &= Hx(t). \end{aligned}$$

Using the variable substitution $z(t) = x(t)x'(t)$, we can obtain the corresponding structural matrix E of (3):

$$E = \delta_{16}[1\ 3\ 2\ 1\ 9\ 11\ 10\ 9\ 5\ 7\ 6\ 5\ 1\ 3\ 2\ 1\ 1\ 3\ 2\ 3\ 9\ 11\ 10\ 11\ 5\ 7\ 6\ 7\ 9\ 11\ 10\ 11].$$

This indicates that $C_0 = \{\delta_{16}^1, \delta_{16}^6, \delta_{16}^{11}, \delta_{16}^{16}\}$. From Lemma 2, we obtained $B = \{\delta_{16}^2, \delta_{16}^3, \delta_{16}^4, \delta_{16}^5, \delta_{16}^9, \delta_{16}^{13}\}$.

And $C=\Delta_{16}\backslash B=\{\delta_{16}^{2},\delta_{16}^{3},\delta_{16}^{4},\delta_{16}^{5},\delta_{16}^{7},\delta_{16}^{8},\delta_{16}^{9},\delta_{16}^{10},\delta_{16}^{12},\delta_{16}^{13},\delta_{16}^{14},\delta_{16}^{15}\}$. Then, we have

$$Q=E\ltimes\mathbf{1}_{16}=\begin{pmatrix}
2&0&0&1&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&2&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&2&0&1&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&2&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&0&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&2&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&2&0&1&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

Moreover, we can find a permutation matrix $P=\delta_{16}[3\ 10\ 7\ 16\ 11\ 4\ 1\ 8\ 9\ 2\ 5\ 12\ 13\ 14\ 15\ 6]$ such that

$$PQP^{\mathrm{T}}=\left(\begin{array}{cc|cccc|cccccccccc}
0&2&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
2&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
\hline
0&0&2&0&0&1&0&0&0&0&0&0&1&0&0&1\\
0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2&0&1&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&2&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&2&0&1&0&0&0\\
0&0&0&0&0&0&2&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&2&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right),$$

where

$$\mathcal{F}=\begin{pmatrix}0&2\\2&0\end{pmatrix},\quad \mathcal{G}=\begin{pmatrix}2&0&0&1\\0&0&2&0\\0&2&0&1\\0&0&0&0\end{pmatrix}.$$

Then, we have $\mathrm{InCol}(P^{\mathrm{T}}\bar{\mathcal{F}}P)=\{\delta_{16}^{7},\delta_{16}^{10}\}\subset C\backslash B$ and $\mathrm{InCol}(P^{\mathrm{T}}\bar{\mathcal{G}}P)=C_0$. Therefore, the permutation matrix P satisfies the conditions in Theorem 5, and it implies that the considered BCN is not observable.

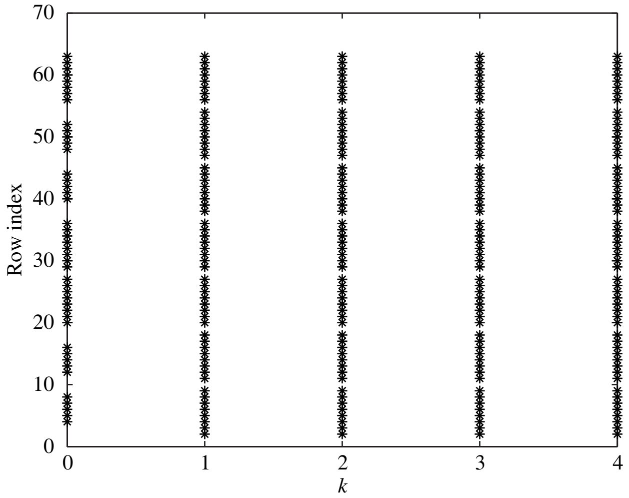

Figure 1 Row index of the column vector $\tilde{R}_k(B)$. Each point is the row index of $\tilde{R}_k(B)$. It holds that $\tilde{R}_0(B) \subset \tilde{R}_1(B) = \tilde{R}_2(B) = \tilde{C}$.

Example 2. Consider a BCN as follows:

$$\Sigma: \begin{cases} A(t+1) = B(t) \leftrightarrow C(t), \\ B(t+1) = C(t) \vee u_1(t)), \\ C(t+1) = A(t) \wedge u_2(t)), \end{cases} \quad \begin{cases} y_1(t) = A(t), \\ y_2(t) = B(t) \vee C(t). \end{cases}$$

Using the STP method, we can calculate the structure matrix L and output matrix H of Σ, and system Σ has the following algebraic form:

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ y(t) = Hx(t), \end{cases}$$

where

$$L = \delta_8[1\ 5\ 5\ 1\ 2\ 6\ 6\ 2\ 2\ 6\ 6\ 2\ 2\ 6\ 6\ 2\ 1\ 7\ 5\ 3\ 2\ 8\ 6\ 4\ 2\ 8\ 6\ 4\ 2\ 8\ 6\ 4],$$

and

$$H = \delta_4[1\ 1\ 1\ 2\ 3\ 3\ 3\ 4].$$

Next, by calculating matrix $H^{\mathrm{T}}H$, we can obtain the initial distinguishable state set B. Consequently, index set $\Theta = \{4\ 5\ 6\ 7\ 8\ 12\ 13\ 14\ 15\ 16\ 20\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 29\ 30\ 31\ 32\ 33\ 34\ 35\ 36\ 40\ 41\ 42\ 43\ 44\ 48\ 49\ 50\ 51\ 52\ 56\ 57\ 58\ 59\ 60\ 61\ 62\ 63\}$, $b = |\Theta| = 44$ and $B = \{\delta_{64}^i | i \in \Theta\}$.

Using the variable substitution $z(t) = x(t)x'(t)$, the corresponding structural matrix E of (3) can be obtained with $E = \delta_{64}[1\ 5\ 5\ 1 \cdots 26\ 32\ 30\ 28] \in \mathcal{L}_{64\times 256}$.

Our objective is to determine whether the BCN (1) is observable or not. From Theorem 2, we obtain matrix Γ, and Figure 1 shows the row index of the column vector $\tilde{R}_k(B)$ versus the time $t = k$. Then, it is easy to check that

$$\sum_{i=1}^{1}{}_{\mathscr{B}} \sum_{j=1}^{44}{}_{\mathscr{B}} \left(\mathrm{Row}_{c_j}(\Gamma^{(i)})\right)^{\mathrm{T}} +_{\mathscr{B}} \tilde{B} = \tilde{C}.$$

Hence, BCN Σ is observable.

3.3 Algorithm for the observability of BCNs

Our theoretical results give simple ways to determine the observability of BCNs. However, it is still affected by the computational complexity. Cheng et al. [34] reported a novel and effective method for the observability of BCNs. In this subsection, to study the observability of BCNs, we present a graphical algorithm whose computational complexity is in agreement with the method proposed by Cheng et al. [34].

Definition 3. We denote by $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$ a weighted directed graph, where $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{W}$ are the node set, edge set, and weight function, respectively. For system (3) (equivalently BCN (1)), $\mathcal{G}$ is a weighted directed graph while avoiding C_0, if $\mathcal{V} = C$, $\mathcal{E} = \{(z, z')|\exists u \in \Delta_{2^m}, Euz = z'\} \subset \mathcal{V} \times \mathcal{V}$ and $\mathcal{W} : \mathcal{E} \to 2^{\Delta_{2^m}}$, $(z, z') \mapsto \{u|\exists u \in \Delta_{2^m}, Euz = z'\}$.

Forbidden set C_0 is unrelated to the observability of BCN (1). Hence, the weighted directed graph $\mathcal{G}$ in Definition 3 does not consider forbidden states in C_0. Furthermore, we can see that the space complexity of graph $\mathcal{G}$ is at most $O(2^{2n}2^m)$ because in graph $\mathcal{G}$, the maximum number of nodes is $2^{2n} - 2^n$, and for every node, there are at most 2^m edges.

Let $D_i = \{z|\exists u, Euz = \delta_{2^{2n}}^i\}$ denote the set of states that can be transferred to $\delta_{2^{2n}}^i$ by control u at exactly one step for $\delta_{2^{2n}}^i \in C$. Intuitively, if no node can reach node $\delta_{2^{2n}}^i$ at exactly one step, then $D_i = \emptyset$. Moreover, for all $\delta_{2^{2n}}^i \in C_0$, we also set $D_i = \emptyset$.

Consider BCN (1), we design an effective algorithm to construct a graph, called the observability graph, to determine the observability. For the sake of simplicity, we define a virtual node "0", and $D_0 = B$. Set P is called a queue, if an element can be added in the back of P and can be removed from the front of P. In other words, a queue P is a first-in first-out (FIFO) data structure. Then, some notions are listed. P.empty() denotes a Boolean value (True or False), where true indicates that P is empty and vice versa. P.front() and P.pop() respectively represent an action for returning and deleting the first element of queue P, while P.push($\cdot$) signifies that an element is added at the end of queue P.

Based on Theorem 1, if there exists an integer k with $0 \leqslant k \leqslant \frac{d}{2}$ such that $R_k(B) = C$, then the BCN (1) is observable. Hence, our goal is to find such a k that satisfies $R_{k+1}(B) = R_k(B)$ or $S_{k+1}(B) \subseteq R_k(B)$, and to determine whether $R_k(B) = C$. We present Algorithm 1.

Algorithm 1 This algorithm determines the observability of BCNS (1). If the algorithm returns "Yes", then BCN (1) is observable; otherwise, it is unobservable.

vis[0] := True, step[0] $= -1$.
vis[i] := False, $\delta_{2^{2n}}^i \in C$.
$S := \emptyset$, $P := \{0\}$, where P is a queue.
while P.empty() = False **do**
  $i := P$.front().
  for $\delta_{2^{2n}}^j \in \mathcal{D}_i$ **do**
    if $\delta_{2^{2n}}^j \in C$ and vis[j] = False **then**
      vis[j] = True, step[j] := step[i] + 1.
      $S := S \cup \{\delta_{2^{2n}}^j\}$.
      P.push(j).
    end if
  end for
  P.pop().
end while
if $S = C$ **then**
  Return "Yes".
else
  Return "No".
end if

Remark 2. In the worst case, Algorithm 1 may traverse all nodes and edges of the weighted directed graph $\mathcal{G}$. According to our analysis for the weighted directed graph, $\mathcal{G}$ space is $O(2^{2n}2^m)$, so does the computational complexity of Algorithm 1, which is as effective as the method in [34].

Example 3. The following BCN is considered with $L = \delta_4[1\ 2\ 2\ 3\ 1\ 1\ 3\ 4]$ and $H = \delta_4[1\ 2\ 2\ 2]$.

$$x(t+1) = Lu(t)x(t),$$
$$y(t) = Hx(t).$$

Using the variable substitution $z(t) = x(t)x'(t)$, the corresponding structural matrix E of (3) can be obtained with

$$E = \delta_{16}[1\ 2\ 2\ 3\ 5\ 6\ 6\ 7\ 5\ 6\ 6\ 7\ 9\ 10\ 10\ 11\ 1\ 1\ 3\ 4\ 1\ 1\ 3\ 4\ 9\ 9\ 11\ 12\ 13\ 13\ 15\ 16].$$

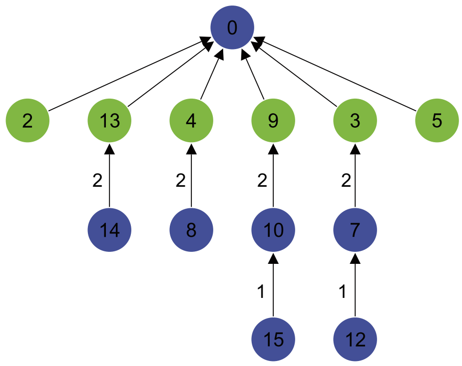

Figure 2 (Color online) The observability graph of the considered BCN. 0 is a virtual node, green circles denote the initial distinguishable states, and blue circles denote otherwise. For simplicity, number i in each circle denotes the node δ_{16}^{i}, and the weight j beside each edge denotes the weight δ_{2}^{j} of the edge.

According to Algorithm 1, we can construct the observability graph to determine whether the objective BCN is observable or not, and the observability graph is depicted in Figure 2.

From the constructed graph, we find obtained that $B = \{\delta_{16}^{2}, \delta_{16}^{3}, \delta_{16}^{4}, \delta_{16}^{5}, \delta_{16}^{9}, \delta_{16}^{13}\}$, $R_0(B) = B$, $R_1(B) = R_0(B) \cup \{\delta_{16}^{7}, \delta_{16}^{8}, \delta_{16}^{10}, \delta_{16}^{14}\}$ and $R_2(B) = R_1(B) \cup \{\delta_{16}^{12}, \delta_{16}^{15}\} = S = C$. In other words, for all $z = xx'$ and $Hx = Hx'$, there exists a control sequence π^{t-1} that transfers state z to some initial distinguishable state $z_0 = x_0x_0' \in B$ such that $Hx_0 \neq Hx_0'$. Therefore, the considered BCN is observable.

4 Conclusion

In this paper, we presented some new results to study the observability of BCNs. Based on the properties of the STP, we combined two BCNs with the same transition matrix into a new BCN. Then, we obtained a simple algebraic expression for the reachable set, resulting in a given set of initial states in no more than a given number of time steps. Based on this expression, we have derived some computable algebraic criteria for the observability of BCNs. The obtained results infer that to determine whether a BCN is observable, it requires only a finite number of time steps. Based on the knowledge of graph theory, we can construct the observability graph using Algorithm 1 to determine the observability of BCNs. By performing three numerical examples, we verify our theoretical results.

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Conflict of interest The authors declare that they have no conflict of interest.

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