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Locality preserving projection on SPD matrix Lie group: algorithm and analysis

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Appendix A Proof of main theorem

Theorem: Covariance matrix used as the feature descriptor of images preserves the spatial geometrical structure of image pixels, that is in local neighborhoods there are two positive constants $c_1 < c_2$ satisfing the following inequality:

$$c_1 \cdot d\left(I_i, I_j\right) \leqslant D_{le}\left(S_i, S_j\right) \leqslant c_2 \cdot d\left(I_i, I_j\right)$$

Proof:
Let I be the original image pixel matrix, where $p_j = \left(p_j^x, p_j^y\right)$ is the location expressed w.r.t the origin of this patch and is the same for all patches. In this proof, we just consider the gray scale image pixels. In image recognition, one usually considers image pixel matrix as a high dimensional row vector where sample images are the corresponding points in the high dimensional vector space. According to the properties of a manifold, the distance between two adjacent image points can be approximately viewed as Euclidean distance. Suppose I_i, I_j are two adjacent images, the corresponding pixel dimension is $m \times m$. The Euclidean distance can be written as:

$$d\left(I_i, I_j\right) = \sum_{k=1}^{m\times m} \|I_i\left(p_k\right) - I_j\left(p_k\right)\|^2. \tag{A1}$$

Suppose the corresponding covariance matrix descriptors of I_i, I_j are S_i, S_j which lie on a SPD matrix Lie group. The geodesic distance $D_{le}\left(S_i, S_j\right)$ between S_i and S_j is shown as:

$$D_{le}\left(S_i, S_j\right) = \|log\left(S_i\right) - log\left(S_j\right)\|_F^2. \tag{A2}$$

According to the definition of logarithmic map on matrix Lie group, we can write the geodesic distance $D_{le}\left(S_i, S_j\right)$ as:

$$\begin{aligned} &\|\sum_{k=1}^{\infty} \frac{\left(I - S_j\right)^k}{k} - \sum_{k=1}^{\infty} \frac{\left(I - S_i\right)^k}{k}\| \\ &= \|\left(S_i - S_j\right) M\|_F^2 \leqslant \|S_i - S_j\|_F^2 \|M\|_F^2, \end{aligned} \tag{A3}$$

where we define:

$$M := \left(3I - \frac{3}{2}\left(S_i + S_j\right) + \frac{1}{3}\left(S_i^2 + S_iS_j + S_j^2\right) + \cdots\right)$$

As the definition of covariance matrix:

$$\begin{aligned} &S_i\left(f_k, f_l\right) - S_j\left(f_k, f_l\right) \\ &= \frac{1}{n}\sum_{h=1}^{n}\left(I_i\left(t_h + p_k\right) I_i\left(t_h + p_l\right) - I_j\left(t_h + p_k\right) I_j\left(t_h + p_l\right)\right), \end{aligned} \tag{A4}$$

where t_h is the first point location of the h^{th} patch, and f_k is the feature vector defined as in [1].
The Frobenius norm of covariance matrix is defined as follows:

$$\|S_i - S_j\|_F^2 = \sum_{k=1}^{d}\sum_{l=1}^{d} \|\left(S_i - S_j\right)(k, l)\|^2, \tag{A5}$$

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where d is the number of patches of each image.

$$\|\left(S_i - S_j\right)(k,l)\|^2 = \|S_i\left(f_k, f_l\right) - S_j\left(f_k, f_l\right)\|^2. \tag{A6}$$

From Eq.A5 and Eq.A6, we give the Frobenius norm of covariance matrix:

$$\begin{aligned} &\|S_i - S_j\|_F^2 \\ &= \frac{1}{n}\|f_i \cdot f_i^T - f_j \cdot f_j^T\|_F^2 \\ &\leqslant \frac{1}{n^2}\|\left(f_i - f_j\right) \cdot \left(f_i^T + f_j^T\right)\|_F^2 + \frac{1}{n^2}\|f_j f_i^T - f_i f_j^T\|_F^2 \\ &\leqslant \frac{1}{n^2}\|f_i - f_j\|_F^2 \cdot \left(\|f_i^T + f_j^T\|_F^2 + 2\|f_i^T\|_F^2\right), \end{aligned} \tag{A7}$$

where $f_i = (f_{i1}, f_{i2}, \cdots, f_{in})$, $f_j = (f_{j1}, f_{j2}, \cdots, f_{jn})$, and f_{ik} represent the k^{th} block subimage feature vector of the i^{th} image.
Then

$$\begin{aligned} &\|f_i - f_j\|^2 = \sum_{k=1}^{n}\|f_{ik} - f_{jk}\|^2 \\ &= \sum_{k=1}^{n}\sum_{l=1}^{d}\|I_i\left(p_{kl}\right) - I_j\left(p_{kl}\right)\|^2 \\ &\leqslant d \cdot \sum_{k=1}^{m\times m}\|I_i\left(p_k\right) - I_j\left(p_k\right)\|^2 = d \cdot d\left(I_i, I_j\right). \end{aligned} \tag{A8}$$

From Eq.A7 and Eq.A8, we obtain:

$$D_{le}\left(S_i, S_j\right) \leqslant c_2 \cdot d\left(I_i, I_j\right),$$

where c_2 is a positive constant. And simultaneously $D_{le}\left(S_i, S_j\right) \geqslant 0$, so there are two constants $0 \leqslant c_1 \leqslant c_2$ satisfied:

$$c_1 \cdot d\left(I_i, I_j\right) \leqslant D_{le}\left(S_i, S_j\right) \leqslant c_2 \cdot d\left(I_i, I_j\right).$$

In summary, we can prove that covariance descriptor preserves the geometrical structure of images.

References

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