SCIENCE CHINA
Information Sciences

• HIGHLIGHT •

July 2018, Vol. 61 078201:1–078201:3
https://doi.org/10.1007/s11432-017-9304-7

# Hilger-type impulsive differential inequality and its application to impulsive synchronization of delayed complex networks on time scales

Zhenkun HUANG[1*], Jinde CAO[2] & Youssef N. RAFFOUL[3]

[1]*School of Science, Jimei University, Xiamen* 361021*, China;*
[2]*School of Mathematics, Southeast University, Nanjing* 210096*, China;*
[3]*Department of Mathematics, University of Dayton, Dayton, OH* 45469-2316*, USA*

Received 14 July 2017/Revised 17 October 2017/Accepted 30 November 2017/Published online 19 April 2018

**Citation** Huang Z K, Cao J D, Raffoul Y N. Hilger-type impulsive differential inequality and its application to impulsive synchronization of delayed complex networks on time scales. Sci China Inf Sci, 2018, 61(7): 078201, https://doi.org/10.1007/s11432-017-9304-7

The synchronization and synchronous control of complex networks [1] have rich dynamics and are lately receiving much needed attention [2–6]. Owing to the major role that impulsive control plays [6] in investigating synchronization problems and impulsive synchronization in dynamical networks, researchers have devoted more time researching them [2–6]. However, most of the existing literature concerning impulsive effects are confined to continuous or discrete time domains [2,6]. In this article, we establish Hilger-type impulsive differential inequality [7] which is a useful technical tool in investigating complex dynamic systems under impulsive disturbances. Meanwhile, we obtain scale-type synchronization criteria for complex networks with multiple delays. Our results not only improve existing results but also offer a new approach to understand the similarities and differences between synchronization of continuous-time and discrete-time cases, under impulsive effects.

*Preliminaries.* We recall some definitions and preliminary results [8] first. Let $\mathbb{T}$ be a nonempty closed subset of $\mathbb{R}$. The forward and backward jump operators $\sigma$, $\rho$: $\mathbb{T}\to\mathbb{T}$ and the graininess $\mu$: $\mathbb{T}\to\mathbb{R}^+$ are defined, respectively, by $\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}$, $\rho(t)=\sup\{s\in\mathbb{T}: s<t\}$ and $\mu(t)=\sigma(t)-t$. A point $t\in\mathbb{T}$ is called left-dense if $t>\inf\mathbb{T}$ and $\rho(t)=t$, left-scattered if $\rho(t)<t$, right-dense if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, and right-scattered if $\sigma(t)>t$. If $\mathbb{T}$ has a left-scattered maximum $m$, $\mathbb{T}^k=\mathbb{T}-\{m\}$, otherwise $\mathbb{T}^k=\mathbb{T}$. The notation $[a,b]_{\mathbb{T}}$ means $[a,b]_{\mathbb{T}}=[a,b]\bigcap\mathbb{T}$, where $b\leqslant+\infty$. $f:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. Let $C_{\rm rd}(\Omega,\mathbb{R})$ be the set of all rd-continuous functions on $\Omega\subseteq\mathbb{T}$, $C_{\rm rd}:=C_{\rm rd}(\mathbb{T},\mathbb{R})$ and $C_{\rm rd}^{\tau}:=C_{\rm rd}([-\tau,0]_{\mathbb{T}},\mathbb{R})$. The (delta) derivative of $f:\mathbb{T}\to\mathbb{R}$ is defined by

$$f^{\triangle}(t)=\frac{f(\sigma(t))-f(t)}{\sigma(t)-t},$$

if $f$ is continuous at $t$ and $t$ is right-scattered. If $t$ is not right-scattered, the derivative is defined by

$$f^{\triangle}(t)=\lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}=\lim_{s\to t}\frac{f(t)-f(s)}{t-s},$$

provided this limit exists. A function $p:\mathbb{T}\to\mathbb{R}$ is said to be regressive if and only if $1+\mu(t)p(t)\neq0$ for all $t\in\mathbb{T}^k$. Let $\mathcal{R}$ be the set of all rd-continuous and regressive functions on $\mathbb{T}$ and $\mathcal{R}^+=\{p\in\mathcal{R}: 1+\mu(t)p(t)>0 \text{ for all } t\in\mathbb{T}\}$. Denote $\mathscr{M}$ as the set

* Corresponding author (email: hzk974226@jmu.edu.cn)

 

of matrices with appropriate dimension and $\mathscr{M}_E$ as the set of matrices with equal sum of each row. $I_n$ is the identity matrix of order $n$.

*System description and Hilger-type impusive differential inequality.* Consider the impulsive complex networks on time scales

$$\begin{cases} x^{\Delta}(t) = (I_N \otimes A)x(t) + \mathcal{F}(x(t), x(t-\tau(t))) \\ \qquad\qquad +c(B\otimes\Gamma)\displaystyle\int_{t-\delta(t)}^{t} H(t-s)x(s)\Delta s, \\ \Delta x(t_k) = (I_N \otimes C_k)(\eta^k \otimes I_n)x(t_k), \end{cases} \quad (1)$$

where $t \in \mathbb{T}$, the initial condition is given by $x(s) = (x_1(s), x_2(s), \ldots, x_n(s))^{\mathrm{T}} = \psi(s)$ $(s \in [-\tau, 0]_{\mathbb{T}})$, and $\mathcal{F}(x(t), x(t-\tau(t))) := (F(x_1(t), x_1(t-\tau(t))), \ldots, F(x_N(t), x_N(t-\tau(t))))^{\mathrm{T}}$. $A$ is a constant $n\times n$ matrix, $c>0$ is a coupling strength, $\Gamma$ is an inner coupling matrix, and $B = (b_{ij})_{N\times N} \in \mathscr{M}_E$ is a coupling matrix. $0 \leqslant \tau(t) \leqslant \tau_M$, $0 \leqslant \delta(t) \leqslant \delta_M$. $H : [0, \delta_M]_{\mathbb{T}} \to \mathbb{R}^+$ is a nonnegative function and $\int_0^{\delta_M} H(w)\Delta w < +\infty$. The map $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies $\|F(v, v_1) - F(\hat{v}, \hat{v}_1)\| \leqslant K\|v-\hat{v}\| + K_\tau\|v_1 - \hat{v}_1\|$. Let $\eta^k := (\eta^k_{ij})_{N\times N}$ be the impulsive instant sequence $\{t_k\}_{k=1}^{\infty} \subset [t_0, +\infty)_{\mathbb{T}}$ that satisfies $0 = t_0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $C_k$ be a $n \times n$ constant matrix. We assume all impulsive points are right-dense and $\tau := \max\{\tau_M, \delta_M\}$. Below, we state the technical lemma for Hilger-type impulsive differential inequality.

**Lemma 1.** Let $\upsilon(t)$ satisfy

$$\begin{cases} \upsilon^{\Delta}(t) = (a + b\mu(t))\upsilon(t), \\ \upsilon(t_k^+) = a_k \upsilon(t_k), \;\; k = 1, 2, \ldots, \end{cases} \quad (2)$$

where $a, b, a_k \in \mathbb{R}$. For any $t \in [t_{k-1}, t_k]_{\mathbb{T}}$, if there exists a constant $\alpha \in \mathbb{R}$ such that

$$(H_1) : \begin{cases} a + b\mu(t) \in \mathcal{R}^+, \;\; \sup\limits_{k\in\mathbb{N}} \left\{ \dfrac{\log a_k}{t_k - t_{k-1}} \right\} < \alpha, \\ \dfrac{\log a_k}{t_k - t_{k-1}} \mu(t) \leqslant \log(1 + \alpha\mu(t)) \end{cases}$$

holds, $\upsilon(t) \leqslant a^\natural \upsilon(t_0) e_{\alpha\oplus(a+b\mu)}(t, t_0)$ for all $t \in [t_0, +\infty)_{\mathbb{T}}$, where $a^\natural := \max_{k\in\mathbb{N}}\{1/a_k, 1\}$.

**Discussion 1.**

- When $\mathbb{T} = \mathbb{R}$, $a + b\mu(t) \in \mathcal{R}^+$ and $\alpha > -\frac{1}{\mu(t)}$ can be removed. Then $(H_1)$ is reduced to

$$(\widehat{H_1}) : \sup_{k\in\mathbb{N}} \left\{ \frac{\log a_k}{t_k - t_{k-1}} \right\} \leqslant \alpha,$$

and hence $\upsilon(t) \leqslant a^\natural \upsilon(t_0) e^{(\alpha+a)(t-t_0)}$ for all $t \in [t_0, +\infty)$.

- If all points in $\mathbb{T}$ are right-scattered, because $\alpha\mu(t) > \log(1 + \alpha\mu(t))$, $(H_1)$ is reduced to

$$(\widetilde{H_1}) : \begin{cases} a + b\mu(t) \in \mathcal{R}, \;\; t \in [t_0, +\infty)_{\mathbb{T}}, \\ \dfrac{\log a_k}{t_k - t_{k-1}} \mu(t) \leqslant \log(1 + \alpha\mu(t)), \end{cases}$$

where $t \in [t_{k-1}, t_k]_{\mathbb{T}}$. As $\mathbb{T} = \mathbb{Z}$, $(\widetilde{H_1})$ implies that $1 + a + b \neq 0$ and $a_k \leqslant (1+\alpha)e^{t_k - t_{k-1}}$. Hence one gets $\upsilon(t) \leqslant a^\natural \upsilon(t_0) \exp \int_{t_0}^{t} \log(1 + a + b + \alpha + \alpha(a + b))\Delta s$.

**Proposition 1.** For any arbitrary $\epsilon > 0$, let $\upsilon(t)$ be the unique solution of

$$\begin{cases} \upsilon^{\Delta}(t) = (a + b\mu(t))\upsilon(t) + c\upsilon(t - \tau(t)) \\ \qquad\qquad +d\displaystyle\int_{t-\delta(t)}^{t} H(t-s)\upsilon(s)\Delta s + \epsilon, \\ \upsilon(t_k^+) = a_k \upsilon(t_k), \end{cases} \quad (3)$$

with $\upsilon(s) = \psi(s) \in C_{\mathrm{rd}}([t_0 - \tau, t_0]_{\mathbb{T}}, \mathbb{R})$. Assume there exist constants $\alpha$ and $\lambda < 0$ such that $(H_1)$ and

$$(H_2) : \begin{cases} 1 + \lambda\mu(s) > 0, \\ 1 \geqslant a^\natural \sup\limits_{s\in[t_0,+\infty)_{\mathbb{T}}} \left\{ \dfrac{c e_{\ominus\lambda}(s, s-\tau(s))}{\lambda - p(s)} \right. \\ \qquad \left. + \dfrac{d e_{\ominus\lambda}(s, s-\delta(s))}{\lambda - p(s)} \displaystyle\int_0^{\delta(s)} H(w)\Delta w \right\} \\ \quad > 0, \\ \Xi := \inf\limits_{s\in[t_0,+\infty)_{\mathbb{T}}} \{-p(s)\} \\ \quad > a^\natural \left( c + d\displaystyle\int_0^{\delta_M} H(w)\Delta w \right) \end{cases}$$

hold, where $p(s) := \alpha \oplus (a + b\mu(s))$ and $a^\natural := \max_{k\in\mathbb{N}}\{1/a_k, 1\}$. Then we have (i) $\upsilon(t) = \mathcal{G}(t, t_0)\psi(t_0) + \int_{t_0}^{t} \mathcal{G}(t, \sigma(s))[c\upsilon(s - \tau(s)) + d\int_{s-\delta(s)}^{s} H(s-w)\upsilon(w)\Delta w + \epsilon]\Delta s$, where $\mathcal{G}(t, t_0) = \prod_{t_0<t_k<t} a_k e_{a+b\mu}(t, t_0)$; (ii) $\upsilon(t) < \Upsilon e_\lambda(t, t_0) + \frac{a^\natural \epsilon}{\Xi - a^\natural(c + d\int_0^{\delta_M} H(w)\Delta w)}$ for all $t \in [t_0, +\infty)_{\mathbb{T}}$, where $\Upsilon := a^\natural \sup_{s\in[t_0-\tau, t_0]_{\mathbb{T}}} |\psi(s)|$.

**Discussion 2.** Let $\mathbb{T} = \mathbb{R}$. One can get $p(s) \equiv \alpha + a$. As a consequence, $(H_2)$ can be reduced to

$$(\widehat{H_2}) : \begin{cases} \dfrac{1}{a^\natural} \geqslant \sup\limits_{s\in[t_0,+\infty)} \left\{ \dfrac{c e^{-\lambda\tau(s)}}{\lambda - (\alpha + a)}, \right. \\ \qquad \left. + \dfrac{d e^{-\lambda\delta(s)}}{\lambda - (\alpha + a)} \displaystyle\int_0^{\delta(s)} H(w)\mathrm{d}w \right\}, \\ \Xi := -(\alpha + a) \\ \quad > a^\natural \left( c + d\displaystyle\int_0^{\delta_M} H(w)\mathrm{d}w \right). \end{cases}$$

Based on the above discussion, we arrive at Corollary 1.

**Corollary 1.** For any arbitrary $\epsilon > 0$, let $\upsilon(t)$ be the unique solution of

$$\begin{cases} \dfrac{\mathrm{d}\upsilon(t)}{\mathrm{d}t} = a\upsilon(t) + c\upsilon(t-\tau(t)) \\ \qquad +d\displaystyle\int_{t-\delta(t)}^{t} H(t-s)\upsilon(s)\mathrm{d}s + \epsilon, \\ \upsilon(t_k^+) = a_k\upsilon(t_k), \end{cases} \quad (4)$$

with initial condition $\upsilon(s) = \psi(s) \in C([t_0-\tau, t_0], \mathbb{R})$. If there exist constants $\alpha$ and $\lambda < 0$ such that $(\widehat{H_1})$ and $(\widehat{H_2})$ hold, $\upsilon(t) < \Upsilon e^{\lambda(t-t_0)} - \frac{a^\natural\epsilon}{\alpha+a+a^\natural(c+d\int_0^{\delta_M}H(w)\mathrm{d}w)}$ holds for all $t \in [t_0, +\infty)$, where $\Upsilon$, $a^\natural$ are as those defined in Proposition 1.

**Discussion 3.** Let $\mathbb{T} = \mathbb{Z}$. From Discussion 1, $(H_1)$ implies that $1+a+b \neq 0$ and $a_k \leqslant (1+\alpha)e^{t_k - t_{k-1}}$. Then one gets $p(s) \equiv \alpha+a+b+\alpha(a+b)$. Meanwhile, $(H_2)$ will be reduced to

$$(\widetilde{H_2}): \begin{cases} \dfrac{1}{a^\natural} + \lambda > 0, \\ 1 \geqslant \sup\limits_{s\in[t_0,+\infty)_{\mathbb{T}}} \left\{ \dfrac{c\exp\int_{s-\tau(s)}^{s}\log\frac{1}{1+\lambda}\Delta s}{\lambda - (\alpha+a+b+\alpha(a+b))} \right. \\ \quad + \dfrac{d\exp\int_{s-\delta(s)}^{s}\log\frac{1}{1+\lambda}\Delta s}{\lambda-(\alpha+a+b+\alpha(a+b))} \\ \quad \left. \cdot \displaystyle\int_0^{\delta(s)} H(w)\Delta w \right\}, \\ \Xi := -(\alpha+a+b+\alpha(a+b)) \\ \quad > a^\natural\left(c + d\displaystyle\sum_{i=0}^{\delta_M} H(i)\right), \end{cases}$$

and so we have Corollary 2.

**Corollary 2.** For any arbitrary $\epsilon > 0$, let $\upsilon(t)$ be the unique solution of

$$\begin{cases} \upsilon(t+1) = (a+b+1)\upsilon(t) + c\upsilon(t-\tau(t)) \\ \qquad +d\displaystyle\int_{t-\delta(t)}^{t} H(t-s)\upsilon(s)\Delta s + \epsilon, \\ \upsilon(t_k^+) = a_k\upsilon(t_k), \end{cases}$$

with initial condition $\upsilon(s) = \psi(s) \in C_{\mathrm{rd}}([t_0-\tau, t_0]_{\mathbb{Z}}, \mathbb{R})$. If there exist constants $\lambda < 0$ and $\alpha$ such that $1+a+b \neq 0$, $a_k \leqslant (1+\alpha)e^{t_k-t_{k-1}}$ and $(\widetilde{H_2})$ hold, $\upsilon(t) < \Upsilon(1+\lambda)^{t-t_0} - \frac{a^\natural\epsilon}{\alpha+a+b+\alpha(a+b)+a^\natural(c+d\sum_{i=0}^{\delta_M}H(i))}$ holds for all $t \in [t_0, +\infty)_{\mathbb{Z}}$, where $\Upsilon, a^\natural$ are defined as those in Proposition 1.

*Main results.* Let $\mathcal{A}_N := I_N \otimes A$, $\mathcal{A}_P := I_P \otimes A$, $\mathcal{C}_k^N := I_N \otimes C_k$, $\mathcal{D}_k := \eta^k \otimes I_n$ and $\mathcal{B}_\Gamma := c(B\otimes\Gamma)$. For $A \in \mathbb{R}^{n\times n}$, the matrix norm is defined as $\|A\| = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$, where $\lambda_{\max}(A)$ is the maximum eigenvalue of $A$. The main results are stated in Theorem 1.

**Theorem 1.** Suppose there exist constants $\alpha$ and $\lambda < 0$ such that $(H_1)$ and $(H_2)$ hold. Then system (1) is exponentially synchronized under impulsive disturbance, where $a := 2K + \hat{\delta} + \varrho + \lambda_{\max}(\mathcal{A}_P^{\mathrm{T}} + \mathcal{A}_P)$, $b := 2K^2(\frac{1}{\delta} + \hat{\varrho} + 1) + (\delta + \varrho + 1)\|\mathcal{A}_P\|^2$, $c := \frac{K_\tau^2}{\hat{\delta}} + 2K_\tau^2(\frac{1}{\delta} + \hat{\varrho} + \frac{1}{K_\tau})\mu_M$, $d := \frac{2\|\hat{\mathcal{B}}_\Gamma\|^2}{\varrho} + (\frac{1}{\hat{\varrho}}+1)\|\hat{\mathcal{B}}_\Gamma\|^2\mu_M$, $a_k := \lambda_{\max}\{(I_P\otimes I_n + (I_P\otimes\mathcal{C}_k)(\hat{\eta}^k\otimes I_n))^{\mathrm{T}}(I_P\otimes I_n + (I_P\otimes\mathcal{C}_k)(\hat{\eta}^k\otimes I_n))\}$, $\delta$, $\varrho$, $\hat{\delta}$, $\hat{\varrho}$ are positive constants, and $\mu_M = \sup_{s\in\mathbb{T}}\mu(s)$.

Let $\mathbb{T} = \mathbb{R}$, $\tau(t) \equiv \tau$, $\delta(t) \equiv \delta_M$ and $H(s) = \mathrm{Dirac}(s-\tau_1)$, where $\mathrm{Dirac}(\cdot)$ is the Dirac function and $0 < \tau_1 < \delta_M$. Then system (1) automatically implies all the results for the systems that are considered in [5].

Compared with [5], our results are more general and avoid separated discussions for $\alpha < 0$ and $\alpha \geqslant 0$.

**Acknowledgements** This work was supported by National Natural Science Foundation of China (Grant Nos. 61573005, 11361010, 61573096, 61272530).

## References

1 Pecora L M, Carroll T L. Synchronization in chaotic system. Phys Rev Lett, 1990, 64: 821–824
2 Chen W H, Lu X M, Zheng W X. Impulsive stabilization and impulsive synchronization of discrete-time delayed neural networks. IEEE Trans Neural Netw Learn Syst, 2015, 26: 734–748
3 Zhou J, Wu Q J, Xiang L. Pinning complex delayed dynamical networks by a single impulsive controller. IEEE Trans Circ Syst I: Reg Papers, 2011, 58: 2882–2893
4 Lu J Q, Ho D W C, Cao J D. A unified synchronization criterion for impulsive dynamical networks. Automatica, 2010, 46: 1215–1221
5 Guan Z H, Liu Z W, Feng G, et al. Synchronization of complex dynamical networks with time-varying delays via impulsive distributed control. IEEE Trans Circ Syst I: Reg Papers, 2010, 57: 2182–2195
6 Liu Y R, Wang Z D, Liang J L, et al. Synchronization and state estimation for discrete-time complex networks with distributed delays. IEEE Trans Syst Man Cybern B, 2008, 38: 1314–1325
7 Adivar M, Bohner E A. Halanay type inequalities on time scales with applications. Nonlinear Anal: Theo Meth Appl, 2011, 74: 7519–7531
8 Bohner M, Peterson A. Dynamic Equations on Time Scales — An Introduction With Applications. Boston: Birkhäuser, 2003