On comparison of modified ADRCs for nonlinear uncertain systems with time delay

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Abstract To tackle systems with both uncertainties and time delays, several modified active disturbance rejection control (ADRC) methods, including delayed designed ADRC (DD-ADRC), polynomial based predictive ADRC (PP-ADRC), Smith predictor based ADRC (SP-ADRC) and predictor observer based ADRC (PO-ADRC), have been proposed in the past years. This paper is aimed at rigorously investigating the performance of these modified ADRCs, such that the improvements of each method can be demonstrated. The capability to tackle time delay, the necessity of stable open loop and the performance of rejecting uncertainties for these methods are fully studied and compared. It is proven that large time delay cannot be tolerated for the stability of the closed-loop systems based on DD-ADRC and PP-ADRC. Moreover, stable open loop is shown to be necessary for stabilizing the closed-loop systems based on SP-ADRC. Furthermore, the performance of rejecting the "total disturbance" at low frequency for these modified ADRCs is evaluated and quantitatively discussed. Finally, the simulations of a boiler turbine system illustrate the theoretical results.

Keywords extended state observer, active disturbance rejection control, uncertain system, time delay, predictor.

1 Introduction

Time delays, both in the input and the output of systems, are ubiquitous in engineering practice [1–4]. In the last decades, the control approaches featured with time delay compensation have been substantially developed, such as Smith predictor (SP), predictor observer (PO), model predictive control (MPC), etc [5–9]. These time delay compensation based control approaches have been rigorously proven to be effective for the systems with exactly known model information [6, 10, 11]. Nevertheless, uncertainties, which are commonly encountered in physical plants, can deteriorate the performance of tracking and even destroy the stability of the existing control systems with compensation schemes considering time delay only. Hence, designing the controller to handle both uncertainties and time delays is a fundamental problem in practical engineering.

In the last decades, uncertainty estimator/observer based control approaches have been widely employed in many industrial sectors due to their intuitive structure of two degree of freedom, i.e., one to achieve estimation and compensation for uncertainties, and the other to force the closed-loop system

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to have the desired performance. Inspired by such effective method for uncertainty rejection, lots of literatures have concerned with the control approaches with both time delay compensation and uncertainty cancellation. In [12], disturbance observer (DO) based control scheme for integral processes with time delay was firstly presented, and its performance of rejecting two typical disturbances (ramp disturbances and step disturbances) was rigorously studied. Moreover, the engineering applications of the DO based control approach for time delay systems were presented in [13, 14], where the stability conditions for this method are analyzed based on Popov criterion. The control scheme combining MPC and DO, proposed in [15], achieved setpoint tracking despite both model mismatches and external disturbances. Additionally, the control approach, which combines DO and the truncated predictor feedback method, was developed for time delay systems with uncertainty in [16] and the corresponding stability condition was studied via Krasovskii functionals. Moreover, several modifications of active disturbance rejection control (ADRC) based on different predictive methods were discussed in the literatures [17–22]. The conventional ADRC, proposed in [23], utilizes extended state observer (ESO) to timely estimate the "total disturbance", i.e., the total effect of both internal uncertainties and external disturbances. The strong robustness of ADRC against various uncertainties has been well proven for systems without delay [24–26]. Therefore, it is natural to develop modified ADRCs for uncertain systems with delays.

In [20], the delayed designed ADRC (DD-ADRC) was rigorously analyzed, where the control signal is delayed in ESO to match the time delay in the system plant. In the meanwhile, a linear matrix inequality (LMI) was presented as the stability condition for the closed-loop system based on DD-ADRC. Rational polynomial approximation is another widely used method for dealing with time delay in ADRC designs. The polynomial based predictive ADRC (PP-ADRC), which provides the prediction for control input or system output according to the prediction model calculated by rational polynomial approximation, was studied in [17, 18]. To obtain the predictions of the system state and the "total disturbance", the SP based ADRC (SP-ADRC) was proposed in [20], which designs ESO via the predictive output generated by SP. Since PO in terms of infinite-dimensional differential equations is capable of predicting the system state, [22] proposed the PO based ADRC (PO-ADRC) for a class of nonlinear uncertain systems with output delay. Actually, the modified designs of ADRC, including DD-ADRC, PP-ADRC, SP-ADRC, and PO-ADRC, have been an active issue in controlling uncertain systems with time delay. However, the performance analyses as well as the improvements of these modified ADRCs have not been fully studied and comprehensively compared. Thus, practitioners will have the headache of selecting the modification for their specific plants. In this paper, the performance of these modified ADRCs will be rigorously investigated in terms of the following.

(i) The capability to tackle time delay is discussed. It is proven that, to ensure the stability of closed-loop system, DD-ADRC and PP-ADRC have limitations with respect to the size of the time delay, whereas SP-ADRC and PO-ADRC can handle arbitrarily large time delay.

(ii) The necessity of stable open loop for the stability of the closed-loop system is studied. It is proved that stable open loop is necessary for stabilizing the corresponding closed-loop system based on SP-ADRC. Moreover, it is shown that the stability of the closed-loop systems based on DD-ADRC, PP-ADRC and PO-ADRC can be achieved despite unstable open loop.

(iii) The capability to reject uncertainties is analyzed. The capabilities of modified ADRCs to reject "total disturbance" at low frequency are explicitly discussed. Furthermore, the quantitative study for the first order system with time delay shows that by tuning the bandwidth of ESO, PP-ADRC can perform better uncertainty rejection than the other modifications.

Finally, the simulations of a boiler turbine system demonstrate the theoretical results in this paper.

The remainder of this paper is organized as follows. The system description is presented in Section 2. In Section 3, the detailed introductions for modified ADRCs, including DD-ADRC, PP-ADRC, SP-ADRC, and PO-ADRC are provided. In Section 4, the capability of modified ADRCs to tackle time delay is discussed. The necessity of stable open loop for the stability of the closed-loop systems based on modified ADRCs is demonstrated in Section 5. The discussions on the capability to reject uncertainty are shown in Section 6. The simulations of a boiler turbine system are shown in Section 7. Finally, the conclusion is given in Section 8.

2 System description

Consider the following class of n-th order nonlinear uncertain systems with input delay

$$\dot{x}(t) = Ax(t) + B(u(t-\tau) + \delta(x(t),t)), \quad y(t) = C^{\mathrm{T}}x(t), \quad t \geqslant t_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector, $y(t) \in \mathbb{R}$ is the measured output, $u(t) \in \mathbb{R}$ is the control input with the known input delay τ and $\delta(x(t),t) \in \mathbb{R}$ is an unknown and continuously differentiable function. In the frame of ADRC, $\delta(x(t),t)$ is referred to the "total disturbance", including both external disturbances and unmodeled dynamics. Moreover, the nominal model $(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1}, C \in \mathbb{R}^{n \times 1})$ is exactly known. Additionally, t_0 is the initial time. This paper is concerned with the performance of the system (1) in the time region $t \in [t_0, \infty)$. Hence, all time-varying variables are assumed to be zero before the initial time t_0.

The system (1) can model various practical systems in the fields of process control and chemical engineering, such as boiler turbine systems [27], temperature control systems [28], chemical reactor concentration systems [29, 30] and water tank systems [31]. Therefore, control design for the system (1) is an important and practical issue. Since both steady tracking error and transient tracking performance are crucial in the control process, this paper is aimed at achieving the ideal trajectory of the states for $t \in [t_0, \infty)$ against both uncertainties and input delay in (1). The dynamics of the ideal trajectory $x^*(t)$ is assumed to be

$$\dot{x}^*(t) = Ax^*(t) - BK^{\mathrm{T}}(x^*(t) - r(t)), \quad x^*(t_0) = x(t_0), \tag{2}$$

where the feedback gain $K \in \mathbb{R}^{n \times 1}$ is designed such that

$$A_K = A - BK^{\mathrm{T}} \tag{3}$$

has the desired eigenvalues and $x^*(t)$ satisfies the required transient performance, such as small overshoot and short rising time. Additionally, $r(t)$ is the reference signal of the system state vector $x(t)$, which is assumed to be bounded.

Assumption 1. There exists a positive $r_{\max}$ such that $\|r(t)\| \leqslant r_{\max}$ for $t \geqslant t_0$.

In addition, the controllability and the observability of the system (1), are assumed as follows.

Assumption 2. The system (1) is controllable. The system state $x(t)$ and the "total disturbance" $\delta(x(t),t)$ are observable.

The condition of the observability of the system state and the "total disturbance" for general nonlinear uncertain systems is referred to [32].

Remark 1. Although the system (1) is in the form of input delay, the control design and analysis in this paper can be applied to the system with both input and output delay since the latter one can be equivalently transformed into the former one. Consider the following uncertain system with both input and output delay.

$$\dot{x}_{\mathrm{IO}}(t) = Ax_{\mathrm{IO}}(t) + B(u(t-\tau_u) + \delta_{\mathrm{IO}}(x_{\mathrm{IO}}(t),t)), \quad y(t) = C^{\mathrm{T}}x_{\mathrm{IO}}(t-\tau_y), \quad t \geqslant t_{\mathrm{IO},0}, \tag{4}$$

where $x_{\mathrm{IO}}(t) \in \mathbb{R}^n$ is the system state vector, $\delta_{\mathrm{IO}} \in \mathbb{R}$ is the "total disturbance" and $t_{\mathrm{IO},0}$ is the initial time. In addition, τ_u and τ_y are the time delay for the input signal $u(t)$ and the output signal $y(t)$, respectively. By denoting

$$x(t) = x_{\mathrm{IO}}(t-\tau_y), \quad \delta(\cdot\,,t) = \delta_{\mathrm{IO}}(\cdot\,,t-\tau_y), \quad \tau = \tau_u + \tau_y, \quad t_0 = t_{\mathrm{IO},0} + \tau_y, \tag{5}$$

the uncertain system (4) becomes the system (1), which is in the form of only input delay.

3 Modified ADRCs for time delay systems

In this section, four types of modified ADRCs for the uncertain system with input delay (1) are introduced in details. To distinguish the variables of the modified ADRCs, $(x_a, y_a, u_a, \delta_a)$ with $(a =$

DD, PP, PPo, SP, PO) are used for the system state, the system output, the control input and the "total disturbance" in the control system based on DD-ADRC, PP-ADRC of input type, PP-ADRC of output type, SP-ADRC and PO-ADRC, respectively. Thus, the system plant (1) is reformulated as

$$\dot{x}_a(t) = Ax_a(t) + B(u_a(t-\tau) + \delta_a(x_a(t), t)), \quad y_a(t) = C^{\mathrm{T}} x_a(t), \quad t \geqslant t_0, \quad a = \mathrm{DD, PP, PPo, SP, PO.} \quad (6)$$

In addition, for the sake of talking convenience, the following useful notations are introduced. Denote

$$A_e = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad C_e = \begin{bmatrix} C \\ 0 \end{bmatrix},$$

$A_{Le,a} = A_e - L_{e,a} C_e^{\mathrm{T}}$, $A_{L,a} = A - L_a C^{\mathrm{T}}$, $L_{e,a} = \begin{bmatrix} L_a \\ L_{\delta,a} \end{bmatrix}$, $L_a \in \mathbb{R}^{n \times 1}$, $L_{\delta,a} \in \mathbb{R}$, $a = \mathrm{DD, PP, PPo, SP, PO}$.

Additionally, the Laplace transforms of the reference signal, the control input, the system output and the "total disturbance" are denoted as follows:

$$R(s) = \mathcal{L}(r(t))(s), \quad U_a(s) = \mathcal{L}(u_a(t))(s), \quad Y_a(s) = \mathcal{L}(y_a(t))(s), \quad \Delta_a(s) = \mathcal{L}(\delta_a(x_a(t), t))(s), \quad (7)$$

where $(a = \mathrm{DD, PP, PPo, SP, PO})$ and $\mathcal{L}(\cdot)$ denotes the Laplace transform.

3.1 Delayed designed ADRC (DD-ADRC)

Since the input signal $u(t)$ is delayed in the system (6), it appears that this delay information could be considered in the design of ADRC. Ref. [20] proposes a simple and intuitive design that the control signal in ESO has the matched time delay, i.e.,

$$\begin{bmatrix} \dot{\hat{x}}_{\mathrm{DD}}(t) \\ \dot{\hat{\delta}}_{\mathrm{DD}}(t) \end{bmatrix} = A_e \begin{bmatrix} \hat{x}_{\mathrm{DD}}(t) \\ \hat{\delta}_{\mathrm{DD}}(t) \end{bmatrix} + L_{e,\mathrm{DD}}(y_{\mathrm{DD}}(t) - \hat{y}_{\mathrm{DD}}(t)) + B_e u_{\mathrm{DD}}(t-\tau), \quad \hat{y}_{\mathrm{DD}}(t) = C^{\mathrm{T}} \hat{x}_{\mathrm{DD}}(t), \quad (8)$$

where $u_{\mathrm{DD}}(t-\tau)$ is the delayed input signal, and the observer parameter vector $L_{e,\mathrm{DD}}$ is chosen such that $A_{Le,\mathrm{DD}}$ is Hurwitz. Additionally, $\hat{x}_{\mathrm{DD}}(t)$ and $\hat{\delta}_{\mathrm{DD}}(t)$ are the estimations of the system state $x_{\mathrm{DD}}(t)$ and the "total disturbance" $\delta_{\mathrm{DD}}(x_{\mathrm{DD}}(t), t)$, respectively. Therefore, with the aim of tracking the ideal trajectory $x^*(t)$ against the "total disturbance" $\delta_{\mathrm{DD}}(x_{\mathrm{DD}}(t), t)$, the control input is designed as follows:

$$u_{\mathrm{DD}}(t) = -K^{\mathrm{T}}(\hat{x}_{\mathrm{DD}}(t) - r(t+\tau)) - \hat{\delta}_{\mathrm{DD}}(t), \quad t \geqslant t_0. \quad (9)$$

Finally, the DD-ADRC based closed-loop system is obtained, i.e., (6), (8) and (9).

3.2 Polynomial based predictive ADRC (PP-ADRC)

PP-ADRC was first proposed in [17]. The core of this method is providing an approximation for the time delay system via rational polynomial approximation. Notice that the Laplace transform of $u(t-\tau)$ is

$$\mathcal{L}(u(t-\tau))(s) = \mathrm{e}^{-\tau s} \mathcal{L}(u(t))(s). \quad (10)$$

To deal with the non-rational function $\mathrm{e}^{-\tau s}$, rational polynomial approximation, such as Taylor expansion [17, 23] and Pade approximation [18], can be applied. The most representative and simplest one is the first-order Taylor expansion, which is presented as

$$\mathrm{e}^{\tau s} \approx \tau s + 1 \quad \text{or} \quad \mathrm{e}^{-\tau s} \approx \frac{1}{\tau s + 1}. \quad (11)$$

Based on the rational polynomial approximation (11), there are two designs of PP-ADRC that one is of input type and the other is of output type. First, the PP-ADRC of input type is introduced. Due to the Laplace transform (10) and the rational polynomial approximation (11), the approximation of $u(t-\tau)$, denoted as $u_c(t)$, satisfies

$$\mathcal{L}(u_c(t))(s) = \frac{1}{\tau s + 1} \mathcal{L}(u(t))(s). \quad (12)$$

According to (12) and the system plant (6), the rational polynomial approximation based system is obtained as follows:

$$\dot{\bar{x}}_{\mathrm{PP}}(t) = A\bar{x}_{\mathrm{PP}}(t) + B(u_c(t) + \bar{\delta}_{\mathrm{PP}}(\bar{x}_{\mathrm{PP}}(t), t)), \quad y_{\mathrm{PP}}(t) = C^{\mathrm{T}}\bar{x}_{\mathrm{PP}}(t), \tag{13}$$

$$\dot{u}_c(t) = -\frac{1}{\tau}u_c(t) + \frac{1}{\tau}u_{\mathrm{PP}}(t), \tag{14}$$

where $\bar{x}_{\mathrm{PP}}$ and $\bar{\delta}_{\mathrm{PP}}$ are the system state and the "total disturbance" in the rational polynomial approximation based system, respectively. For the rational polynomial approximation based system (13) and (14), there is an intuitional control approach, that is, applying ADRC to design the virtual control signal $u_c(t)$ based on (13) and solving the control input $u_{\mathrm{PP}}(t)$ from (14). In particular, the ESO for (13) is designed as follows:

$$\begin{bmatrix} \dot{\hat{x}}_{\mathrm{PP}}(t) \\ \dot{\hat{\delta}}_{\mathrm{PP}}(t) \end{bmatrix} = A_e \begin{bmatrix} \hat{x}_{\mathrm{PP}}(t) \\ \hat{\delta}_{\mathrm{PP}}(t) \end{bmatrix} + L_{e,\mathrm{PP}}(y_{\mathrm{PP}}(t) - \hat{y}_{\mathrm{PP}}(t)) + B_e u_c(t), \quad \hat{y}_{\mathrm{PP}}(t) = C^{\mathrm{T}}\hat{x}_{\mathrm{PP}}(t), \tag{15}$$

where the observer parameter $L_{e,\mathrm{PP}}$ is selected to let $A_{L_e,\mathrm{PP}}$ be Hurwitz. Moreover, $\hat{x}_{\mathrm{PP}}(t)$ and $\hat{\delta}_{\mathrm{PP}}(t)$ are the estimations of the system state and the "total disturbance", respectively. To achieve the ideal trajectory (2), the virtual control signal $u_c(t)$ is designed as

$$u_c(t) = -K^{\mathrm{T}}(\hat{x}_{\mathrm{PP}}(t) - r(t)) - \hat{\delta}_{\mathrm{PP}}(t), \quad t \geqslant t_0. \tag{16}$$

Then, the control input $u_{\mathrm{PP}}(t)$ is solved from (14) as follows:

$$u_{\mathrm{PP}}(t) = \tau\dot{u}_c(t) + u_c(t). \tag{17}$$

Since the control input $u_{\mathrm{PP}}(t)$ is obtained as a prediction via the rational polynomial approximation (11), the control approach (15)–(17) is named as PP-ADRC of input type. The corresponding closed-loop system is then obtained, i.e., (6) and (15)–(17).

Subsequently, the PP-ADRC of output type will be introduced. The key of the PP-ADRC of output type is to get the prediction for the output y_{PPo}. Then, by inputting the predictive output into ESO, the predictions for the system state x_{PPo} and the "total disturbance" δ_{PPo} can be acquired. Finally, by the predictive values for the system state and the "total disturbance", the control design is featured with two degree of freedom which is similar with (9) and (16).

Since

$$\mathcal{L}(y_{\mathrm{PPo}}(t + \tau))(s) = \mathrm{e}^{\tau s}Y_{\mathrm{PPo}}(s), \tag{18}$$

combined with the rational polynomial approximation (11), the predictive value of the output $y_{\mathrm{PPo}}(t+\tau)$ is obtained as

$$y_{p,\mathrm{PPo}}(t) = \tau\dot{y}_{\mathrm{PPo}}(t) + y_{\mathrm{PPo}}(t). \tag{19}$$

Based on the predictive output (19), the ESO is constructed as

$$\begin{bmatrix} \dot{\hat{x}}_{\mathrm{PPo}}(t + \tau) \\ \dot{\hat{\delta}}_{\mathrm{PPo}}(t + \tau) \end{bmatrix} = A_e \begin{bmatrix} \hat{x}_{\mathrm{PPo}}(t + \tau) \\ \hat{\delta}_{\mathrm{PPo}}(t + \tau) \end{bmatrix} + L_{e,\mathrm{PPo}}(y_{p,\mathrm{PPo}}(t) - \hat{y}_{\mathrm{PPo}}(t + \tau)) + B_e u_{\mathrm{PPo}}(t), \ \hat{y}_{\mathrm{PPo}}(t + \tau)$$
$$= C^{\mathrm{T}}\hat{x}_{\mathrm{PPo}}(t + \tau), \tag{20}$$

where the observer parameter $L_{e,\mathrm{PPo}}$ is designed to make $A_{L_e,\mathrm{PPo}}$ be Hurwitz. Moreover, $\hat{x}_{\mathrm{PPo}}(t + \tau)$ and $\hat{\delta}_{\mathrm{PPo}}(t + \tau)$ are the predictions for the system state and the "total disturbance", respectively. Then, aimed at achieving the ideal trajectory (2), the control input is designed as

$$u_{\mathrm{PPo}}(t) = -K^{\mathrm{T}}(\hat{x}_{\mathrm{PPo}}(t + \tau) - r(t + \tau)) - \hat{\delta}_{\mathrm{PPo}}(t + \tau), \quad t \geqslant t_0. \tag{21}$$

Finally, the PP-ADRC of output type based closed-loop system is obtained, i.e., (6) and (19)–(21).

Furthermore, the following lemma illuminates the relationship between the PP-ADRC of input type (15)–(17) and the PP-ADRC of output type (19)–(21).

Lemma 1. For the uncertain system with input delay (6), consider the PP-ADRC of input type (15)–(17) and the PP-ADRC of output type (19)–(21). Let the initial values of $(x_a, \hat{x}_a)$ be zero for $(a = \mathrm{PP}, \mathrm{PPo})$ and the observer parameters be the same, i.e., $L_{e,\mathrm{PP}} = L_{e,\mathrm{PPo}}$, then

$$U_{\mathrm{PP}}(s) = G_{uy,\mathrm{PP}}(s)Y_{\mathrm{PP}}(s) + G_{ur,\mathrm{PP}}(s)R(s), \quad U_{\mathrm{PPo}}(s) = G_{uy,\mathrm{PPo}}(s)Y_{\mathrm{PPo}}(s) + G_{ur,\mathrm{PPo}}(s)R(s), \quad (22)$$

where

$$G_{uy,\mathrm{PP}}(s) = G_{uy,\mathrm{PPo}}(s), \quad G_{ur,\mathrm{PP}}(s)\mathrm{e}^{\tau s} = G_{ur,\mathrm{PPo}}(s)(\tau s + 1). \quad (23)$$

According to (23), the transfer functions from the system output y to the control input u are the same for the PP-ADRC of input type and output type. Since the same system plant (6) is considered, the stability of the closed-loop systems based on the PP-ADRC of input type and output type is identical. Therefore, in the rest of this paper, only the analysis of the PP-ADRC of input type is presented to investigate the capability of PP-ADRC.

3.3 Smith predictor based ADRC (SP-ADRC)

Different from the prediction design via rational polynomial approximation of PP-ADRC, SP-ADRC is aimed at using SP to get the predictive value of the system output $y_{\mathrm{SP}}(t + \tau)$, which is the input of the ESO [19]. Since the input signal, the system output and the nominal model (A, B, C) are known, SP in the time domain has the following form:

$$\dot{\bar{x}}_{\mathrm{SP}}(t) = A\bar{x}_{\mathrm{SP}}(t) + Bu_{\mathrm{SP}}(t - \tau), \ \bar{y}_{\mathrm{SP}}(t) = C^{\mathrm{T}}\bar{x}_{\mathrm{SP}}(t), \ y_{p,\mathrm{SP}}(t) = y_{\mathrm{SP}}(t) - \bar{y}_{\mathrm{SP}}(t) + \bar{y}_{\mathrm{SP}}(t + \tau), \ t \geqslant t_0, \ (24)$$

where $\bar{x}_{\mathrm{SP}}(t)$ is the state of SP and $y_{p,\mathrm{SP}}(t)$ is the output of SP for predicting the system output $y_{\mathrm{SP}}(t+\tau)$. Then, to obtain the predictions for the system state x_{SP} and the "total disturbance" δ_{SP}, the prediction $y_{p,\mathrm{SP}}(t)$ is used in the design of ESO as follows:

$$\begin{bmatrix} \dot{\hat{x}}_{\mathrm{SP}}(t + \tau) \\ \dot{\hat{\delta}}_{\mathrm{SP}}(t + \tau) \end{bmatrix} = A_e \begin{bmatrix} \hat{x}_{\mathrm{SP}}(t + \tau) \\ \hat{\delta}_{\mathrm{SP}}(t + \tau) \end{bmatrix} + L_{e,\mathrm{SP}}(y_{p,\mathrm{SP}}(t) - \hat{y}_{\mathrm{SP}}(t+\tau)) + B_e u_{\mathrm{SP}}(t), \ \hat{y}_{\mathrm{SP}}(t+\tau) = C^{\mathrm{T}}\hat{x}_{\mathrm{SP}}(t+\tau), \ (25)$$

where $L_{e,\mathrm{SP}}$ is the designed parameter such that $A_{Le,\mathrm{SP}}$ is Hurwitz. Moreover, $\hat{x}_{\mathrm{SP}}(t + \tau)$ and $\hat{\delta}_{\mathrm{SP}}(t + \tau)$ are the predictions for the system state and the "total disturbance", respectively. To achieve the ideal trajectory (2), the control input is designed as

$$u_{\mathrm{SP}}(t) = -K^{\mathrm{T}}(\hat{x}_{\mathrm{SP}}(t + \tau) - r(t + \tau)) - \hat{\delta}_{\mathrm{SP}}(t + \tau), \quad t \geqslant t_0. \quad (26)$$

Finally, the SP-ADRC based closed-loop system is acquired, i.e., (6) and (24)–(26).

3.4 Predictor observer based ADRC (PO-ADRC)

Predictor observer (PO), essentially in terms of infinite-dimensional differential equations, is capable of predicting the system state under known model information. Therefore, it is natural to combine PO and ESO for uncertain systems with time delays. The extended state predictor observer is proposed in [22], which is constructed as follows:

$$\begin{cases} \begin{bmatrix} \dot{\hat{x}}_{\mathrm{PO}}(t + \tau) \\ \dot{\hat{\delta}}_{\mathrm{PO}}(t + \tau) \end{bmatrix} = A_e \begin{bmatrix} \hat{x}_{\mathrm{PO}}(t + \tau) \\ \hat{\delta}_{\mathrm{PO}}(t + \tau) \end{bmatrix} + \mathrm{e}^{A_e \tau} L_{e,\mathrm{PO}}(y_{\mathrm{PO}}(t) - \hat{y}_{\mathrm{PO}}(t + \tau)) + B_e u_{\mathrm{PO}}(t), \\ \hat{y}_{\mathrm{PO}}(t + \tau) = C^{\mathrm{T}}\hat{x}_{\mathrm{PO}}(t) + C_e^{\mathrm{T}}\int_0^\tau \mathrm{e}^{A_e s} L_{e,\mathrm{PO}}(y_{\mathrm{PO}}(t - s) - \hat{y}_{\mathrm{PO}}(t + \tau - s))\mathrm{d}s, \end{cases} \quad (27)$$

where the parameter vector $L_{e,\mathrm{PO}}$ is chosen such that $A_{Le,\mathrm{PO}}$ is Hurwitz. Additionally, $\hat{x}_{\mathrm{PO}}(t + \tau)$ and $\hat{\delta}_{\mathrm{PO}}(t + \tau)$ are the predictions for the system state x_{PO} and the "total disturbance" δ_{PO}, respectively. To track the ideal trajectory (2) against the "total disturbance", the control input is designed as follows:

$$u_{\mathrm{PO}}(t) = -K^{\mathrm{T}}(\hat{x}_{\mathrm{PO}}(t + \tau) - r(t + \tau)) - \hat{\delta}_{\mathrm{PO}}(t + \tau), \quad t \geqslant t_0. \quad (28)$$

Finally, the PO-ADRC based closed-loop system is obtained, i.e., (6), (27) and (28).

The ideas and the designs of modified ADRCs, including DD-ADRC, PP-ADRC, SP-ADRC and PO-ADRC, have been introduced in detail. In the next section, the capabilities of these modified ADRCs to tackle time delay will be presented.

4 Capability to tackle time delay

The following theorem shows that for DD-ADRC and PP-ADRC, the stability condition for the corresponding closed-loop systems depends on the size of time delay.

Theorem 1. Consider the system (6) with Assumptions 1 and 2, and $\delta = 0$. Then, for any nonzero controller parameters (K, $L_{e,\mathrm{DD}}$, $L_{e,\mathrm{PP}}$), there exists a positive τ^* such that the DD-ADRC (8) and (9), and the PP-ADRC (15)–(17) based closed-loop systems are both unstable for $\tau = \tau^*$.

The proof of Theorem 1 is presented in Appendix A. From Theorem 1, the stability of the closed-loop systems based on DD-ADRC and PP-ADRC has a restriction on the size of time delay.

Compared with DD-ADRC and PP-ADRC, the SP-ADRC and the PO-ADRC based closed-loop systems are able to be stable for arbitrarily large time delay, as indicated in the following theorem.

Theorem 2. Consider the system (6) with Assumptions 1 and 2, and $\delta = 0$. Let A_K, $A_{Le,\mathrm{SP}}$ and $A_{Le,\mathrm{PO}}$ be Hurwitz and the initial value of the SP (24) satisfy $\bar{x}_{\mathrm{SP}}(t_0) = x_{\mathrm{SP}}(t_0)$. Then, the closed-loop systems based on the SP-ADRC (24)–(26) and the PO-ADRC (27) and (28) are both stable for any $\tau \geqslant 0$.

The proof of Theorem 2 is given in Appendix A. From Theorem 2, the stability of the PO-ADRC and the SP-ADRC based closed-loop systems can be ensured for arbitrarily large time delay under the same condition for DD-ADRC and PP-ADRC.

In conclusion, SP-ADRC and PO-ADRC can handle arbitrarily large time delay for suitable controller parameter. However, DD-ADRC and PP-ADRC have limitations on the size of time delay.

5 Necessity of stable open loop

For the closed-loop systems based on modified ADRCs, the influence of the matrix A is studied in this section, which is started from the stability condition for the SP-ADRC based closed-loop system.

Theorem 3. Consider the SP-ADRC based closed-loop system (6) and (24)–(26) with Assumptions 1 and 2. Let the "total disturbance" be nonzero constant. Then, A being Hurwitz is necessary for the tracking error $\|x_{\mathrm{SP}}(t) - x^*(t)\|$ to be bounded for any $t \geqslant t_0$.

The proof of Theorem 3 is given in Appendix A. According to Theorem 3, even if the "total disturbance" is constant, the stability of the closed-loop system based on the SP-ADRC (24)–(26) requires the condition that the matrix A is Hurwitz, which implies that the corresponding open-loop system is stable. Actually, the requirement of stable open loop is implicated by that the dynamic equation of the prediction error $(y_{\mathrm{SP}}(t+\tau) - y_{p,\mathrm{SP}}(t))$ depends on the matrix A. Hence, the unstable matrix A results in the boundless estimation error of the ESO in SP-ADRC (25), as long as there exists uncertainty in the system (6).

Compared with SP-ADRC, for the DD-ADRC (8) and (9), and the PP-ADRC (15)–(17), the corresponding closed-loop systems can be stable for constant disturbance if the control parameters (K, $L_{e,\mathrm{DD}}$, $L_{e,\mathrm{PP}}$) are designed such that the matrices A_K, $A_{Le,\mathrm{DD}}$ and $A_{Le,\mathrm{PP}}$ are Hurwitz, which is illustrated in the following theorem.

Theorem 4. Consider the system (6) with Assumptions 1 and 2, and A being unstable. Let the "total disturbance" be nonzero constant and let A_K, $A_{Le,\mathrm{DD}}$ and $A_{Le,\mathrm{PP}}$ be Hurwitz. Then, for the DD-ADRC (8) and (9), and the PP-ADRC (15)–(17) based closed-loop systems, there exists a positive τ^{**} such that the tracking errors $\|x_{\mathrm{DD}}(t) - x^*(t)\|$ and $\|x_{\mathrm{PP}}(t) - x^*(t)\|$ are bounded for any $t \geqslant t_0$ and any $\tau < \tau^{**}$.

The proof of Theorem 4 is presented in Appendix A. From Theorem 4, the condition of stable open loop is no longer required for stabilizing the closed-loop systems based on DD-ADRC and PP-ADRC if the time delay is small.

For PO-ADRC, even if the open-loop system is unstable and the time delay is large, the corresponding closed-loop system is stable if the control parameters $(K, \ L_{e,\mathrm{PO}})$ are designed such that the matrices A_K and $A_{Le,\mathrm{PO}}$ are Hurwitz, as indicated in the following theorem.

Theorem 5. Consider the PO-ADRC based closed-loop system (6) and (27)–(28) with Assumptions 1 and 2. Let

$$|\delta_{\mathrm{PO}}(x_{\mathrm{PO}}, t)| \leqslant \alpha_x \|x_{\mathrm{PO}}\| + \alpha_d, \tag{29}$$

where α_x and α_d are positives. Let A_K and $A_{Le,\mathrm{PO}}$ be Hurwitz. Then, there exist α_x in (29) and a positive γ such that the tracking error satisfies

$$\|x_{\mathrm{PO}}(t) - x^*(t)\| \leqslant \gamma \left(\|x_{\mathrm{PO}}(t_0) - \hat{x}_{\mathrm{PO}}(t_0)\| + \alpha_d + \alpha_x(r_{\max} + \|x_{\mathrm{PO}}(t_0)\|) \right), \quad t \geqslant t_0. \tag{30}$$

The proof of Theorem 5 is presented in Appendix A. Theorem 5 illustrates that the PO-ADRC can handle the systems (6) with unstable open loop and any large time delay. Additionally, the boundary of the tracking error, which is related to the initial values, the size of uncertainty and the boundary of the reference signal, is explicitly displayed in (30).

6 Capability to reject uncertainty

For the closed-loop systems based on modified ADRCs, the influences on the tracking error and estimation error are from three aspects, i.e., the initial values of the system state and the states of observers, the reference signal, and the "total disturbance". In this section, the capabilities of modified ADRCs to handle uncertainty, which are the influences from the "total disturbance", are discussed. Hence, without loss of generality, it is assumed that the initial values and the reference signal are zero in this section.

Denote the estimation errors for the system state and the "total disturbance" as

$$x_{e,a} = x_a - \hat{x}_a, \quad \delta_{e,a} = \delta_a - \hat{\delta}_a, \quad a = \mathrm{DD}, \ \mathrm{PP}, \ \mathrm{SP}, \ \mathrm{PO}, \tag{31}$$

respectively. In addition, the corresponding Laplace transforms are denoted as

$$X_{e,a}(s) = \mathcal{L}(x_{e,a})(s), \quad \Delta_{e,a}(s) = \mathcal{L}(\delta_{e,a})(s), \quad a = \mathrm{DD}, \ \mathrm{PP}, \ \mathrm{SP}, \ \mathrm{PO}. \tag{32}$$

Moreover, the following notations are presented.

$$G_{\delta_e\delta,\mathrm{DD}}(s) = s(s + L_{\delta,\mathrm{DD}}C^{\mathrm{T}}(sI - A_{L,\mathrm{DD}})^{-1}B)^{-1}, \quad G_{x_e\delta,\mathrm{DD}}(s) = (sI - A_{L,\mathrm{DD}})^{-1}BG_{\delta_e\delta,\mathrm{DD}}(s), \tag{33}$$

$$\begin{cases} G_{\delta_e\delta,\mathrm{PP}}(s) = (s + (\mathrm{e}^{-\tau s}(1 + \tau s) - 1)L_{\delta,\mathrm{PP}}C^{\mathrm{T}}M_{\mathrm{PP}}B)(s + \mathrm{e}^{-\tau s}(1 + \tau s)L_{\delta,\mathrm{PP}}C^{\mathrm{T}}M_{\mathrm{PP}}B)^{-1}, \\ G_{x_e\delta,\mathrm{PP}}(s) = (1 - \mathrm{e}^{-\tau s}(1 + \tau s)(1 - G_{\delta_e\delta,\mathrm{PP}}(s)))M_{\mathrm{PP}}B, \\ M_{\mathrm{PP}} = (sI - A_{L,\mathrm{PP}} + (1 - \mathrm{e}^{-\tau s}(1 + \tau s))BK^{\mathrm{T}}(sI - A + \mathrm{e}^{-\tau s}(1 + \tau s)BK^{\mathrm{T}})^{-1}(sI - A))^{-1} \\ \quad \cdot (sI - A_K)(sI - A + \mathrm{e}^{-\tau s}(1 + \tau s)BK^{\mathrm{T}})^{-1}, \end{cases} \tag{34}$$

$$\begin{cases} G_{\delta_e\delta,\mathrm{SP}}(s) = (s + (1 - \mathrm{e}^{-\tau s})L_{\delta,\mathrm{SP}}C^{\mathrm{T}}(sI - A_{L,\mathrm{SP}})^{-1}B)(s + L_{\delta,\mathrm{SP}}C^{\mathrm{T}}(sI - A_{L,\mathrm{SP}})^{-1}B)^{-1}, \\ G_{x_e\delta,\mathrm{SP}}(s) = (sI - A_{L,\mathrm{SP}})^{-1}(-L_{\mathrm{SP}}C^{\mathrm{T}}(sI - A)^{-1}(\mathrm{e}^{-\tau s} - 1) + G_{\delta_e\delta,\mathrm{SP}}I)B, \end{cases} \tag{35}$$

$$\begin{cases} G_{\delta_e\delta,\mathrm{PO}}(s) = (s + (1 - \mathrm{e}^{-\tau s})L_{\delta,\mathrm{PO}}C^{\mathrm{T}}(sI - A_{L,\mathrm{PO}})^{-1}B)(s + L_{\delta,\mathrm{PO}}C^{\mathrm{T}}(sI - A_{L,\mathrm{PO}})^{-1}B)^{-1}, \\ G_{x_e\delta,\mathrm{PO}}(s) = \left(sI - A + \begin{bmatrix} I & 0 \end{bmatrix} \mathrm{e}^{A_e\tau}L_{e,\mathrm{PO}}E_s\right)^{-1}G_{\delta_e\delta,PO}B, \\ E_s = \left(1 + C_{\mathrm{e}}^{\mathrm{T}} \int_0^\tau \mathrm{e}^{-(sI - A_e)\xi}\mathrm{d}\xi L_{e,\mathrm{PO}}\right)^{-1}C^{\mathrm{T}}\mathrm{e}^{-\tau s}, \end{cases} \tag{36}$$

$$\begin{cases} G_{y\delta,\mathrm{DD}}(s) = C^{\mathrm{T}}(sI - A + \mathrm{e}^{-\tau s}BK^T)^{-1}(\mathrm{e}^{-\tau s}BK^{\mathrm{T}}G_{x_e\delta,\mathrm{DD}}(s) + B - \mathrm{e}^{-\tau s}(1 - G_{\delta_e\delta,\mathrm{DD}})B), \\ G_{y\delta,\mathrm{PP}}(s) = C^{\mathrm{T}}(sI - A + \mathrm{e}^{-\tau s}(1 + \tau s)BK^T)^{-1}(\mathrm{e}^{-\tau s}(1 + \tau s)(BK^{\mathrm{T}}G_{x_e\delta,\mathrm{PP}} - (1 - G_{\delta_e\delta,\mathrm{PP}}(s))B) + B), \\ G_{y\delta,\mathrm{SP}}(s) = C^{\mathrm{T}}(sI - A + BK^T)^{-1}(BK^{\mathrm{T}}G_{x_e\delta,\mathrm{SP}} + BG_{\delta_e\delta,\mathrm{SP}}), \\ G_{y\delta,\mathrm{PO}}(s) = C^{\mathrm{T}}(sI - A + BK^T)^{-1}(BK^{\mathrm{T}}G_{x_e\delta,\mathrm{PO}} + BG_{\delta_e\delta,\mathrm{PO}}). \end{cases} \tag{37}$$

Table 1 Comparison of the capabilities of modified ADRCs to reject uncertainty

	$\lim\limits_{\omega\to 0}\dfrac{G_{y\delta,a}(j\omega)}{G_{y\delta,b}(j\omega)}, \quad a,b = \mathrm{DD, PP, SP, PO}$	The main part
$\dfrac{G_{y\delta,\mathrm{PP}}(j\omega)}{G_{y\delta,\mathrm{DD}}(j\omega)}$	$\dfrac{2\omega_o+KB}{\tau\omega_o^2+2\omega_0+KB}$	$O(\frac{1}{\omega_o})$
$\dfrac{G_{y\delta,\mathrm{PP}}(j\omega)}{G_{y\delta,\mathrm{SP}}(j\omega)}$	$\dfrac{A(2\omega_o+KB)}{\tau A_K\omega_o^2+2A\omega_o+2AK}$	$O(\frac{1}{\omega_o})$
$\dfrac{G_{y\delta,\mathrm{PP}}(j\omega)}{G_{y\delta,\mathrm{PO}}(j\omega)}$	$\dfrac{A^2(2\omega_o+KB)}{(KB(\mathrm{e}^{A\tau}-1)+AA_K\tau)\omega_o^2+2A(BK\mathrm{e}^{A\tau}+A_K)\omega_o+A^2BK\mathrm{e}^{A\tau}}$	$O(\frac{1}{\omega_o})$
$\dfrac{G_{y\delta,\mathrm{DD}}(j\omega)}{G_{y\delta,\mathrm{SP}}(j\omega)}$	$\dfrac{A(\tau\omega_o^2+2\omega_o+BK)}{\tau A_K\omega_o^2+2A\omega_o+2AK}$	$\dfrac{A}{A_K}+O(\frac{1}{\omega_o})$
$\dfrac{G_{y\delta,\mathrm{DD}}(j\omega)}{G_{y\delta,\mathrm{PO}}(j\omega)}$	$\dfrac{A^2(\tau\omega_o^2+2\omega_o+BK)}{(KB(\mathrm{e}^{A\tau}-1)+AA_K\tau)\omega_o^2+2A(BK\mathrm{e}^{A\tau}+A_K)\omega_o+A^2BK\mathrm{e}^{A\tau}}$	$\dfrac{A^2\tau}{KB(\mathrm{e}^{A\tau}-1)+AA_K\tau}+O(\frac{1}{\omega_o})$
$\dfrac{G_{y\delta,\mathrm{SP}}(j\omega)}{G_{y\delta,\mathrm{PO}}(j\omega)}$	$\dfrac{A(\tau A_K\omega_o^2+2A\omega_o+2AK)}{(KB(\mathrm{e}^{A\tau}-1)+AA_K\tau)\omega_o^2+2A(BK\mathrm{e}^{A\tau}+A_K)\omega_o+A^2BK\mathrm{e}^{A\tau}}$	$\dfrac{AA_K\tau}{KB(\mathrm{e}^{A\tau}-1)+AA_K\tau}+O(\frac{1}{\omega_o})$

Then, the following proposition presents the transfer functions from the "total disturbance" to the estimation error and the system output.

Proposition 1. Consider the system plant (6) with the DD-ADRC (8) and (9), the PP-ADRC (15)–(17), the SP-ADRC (24)–(26), and the PO-ADRC (27)–(28). Assume that the reference signal $r(t)$ and the initial value of the system state and the state of observers (x_a, $\hat{x}_a$, $\hat{\delta}_a$, $\bar{x}_{\mathrm{SP}}$) for ($a = $ DD, PP, SP, PO) are zero, then

$$X_{e,a}(s) = G_{x_e\delta,a}(s)\Delta_a(s), \quad \Delta_{e,a}(s) = G_{\delta_e\delta,a}(s)\Delta_a(s), \quad Y_a(s) = G_{y\delta,a}(s)\Delta_a(s). \tag{38}$$

From Proposition 1, how the "total disturbance" affects the ESO's estimation error as well as the system output is explicitly shown in the form of the transfer function (38), where the definition of $(G_{x_e\delta,a}, G_{\delta_e\delta,a}, G_{y\delta,a})$ are given in (33)–(37). To further analyze the capabilities of modified ADRCs to reject uncertainty, by denoting j as imaginary unit, the following proposition is presented.

Proposition 2. Consider the system plant (6) with the DD-ADRC (8) and (9), the PP-ADRC (15)–(17), the SP-ADRC (24)–(26), and the PO-ADRC (27) and (28). Assume that all conditions in Proposition 1 are satisfied, A is Hurwitz and Assumption 2 is satisfied. Let A_K and $A_{Le,a}$ be Hurwitz for ($a = $ DD, PP, SP, PO). Then, the following equations are hold.

$$\lim_{\omega\to 0} G_{y\delta,a}(j\omega) = 0, \quad a = \mathrm{DD, \ PP, \ SP, \ PO}. \tag{39}$$

Proposition 2 indicates the influence of the "total disturbance" on the system output tends to zero as the frequency becomes lower owing to (39). Therefore, these four types of modified ADRCs have strong capability to mitigate uncertainty at low frequency.

Furthermore, the smaller $|G_{y\delta,a}(j\omega)|$ implies the stronger capability to rejecte the "total disturbance". Consequently, the comparison of the capabilities of modified ADRCs to reject uncertainty is demonstrated. Since $|G_{y\delta,a}(j\omega)|$ are complex polynomials for high order systems, it is impossible to determine which $|G_{y\delta,a}(j\omega)|$ is smaller for general n-th order systems. The following discussion is limited to the first order uncertain systems with time delay, which are widely used for describing the physical plants in engineering practice [2, 33, 34].

Theorem 6. Consider the system plant (6) with the system order $n = 1$ and $C = 1$. Assume that all the conditions in Proposition 2 are satisfied. Let the parameters of ESOs be

$$L_{e,a} = \begin{bmatrix} 2\omega_o + A \\ \omega_o^2/B \end{bmatrix}, \quad a = \mathrm{DD, \ PP, \ SP, \ PO}, \tag{40}$$

where $\omega_o > 0$. Then, the comparison of the capabilities of modified ADRCs to reject uncertainty is shown in Table 1.

The proof of Theorem 6 is given in Appendix A. In Theorem 6, all the eigenvalues of $A_{Le,a}$ for ($a = $ DD, PP, SP, PO) are placed at $(-\omega_o, 0)$ in the complex plane by (40), which implies that the ESOs have the same bandwidth ω_o [35]. The comparison of the capabilities of modified ADRCs to reject
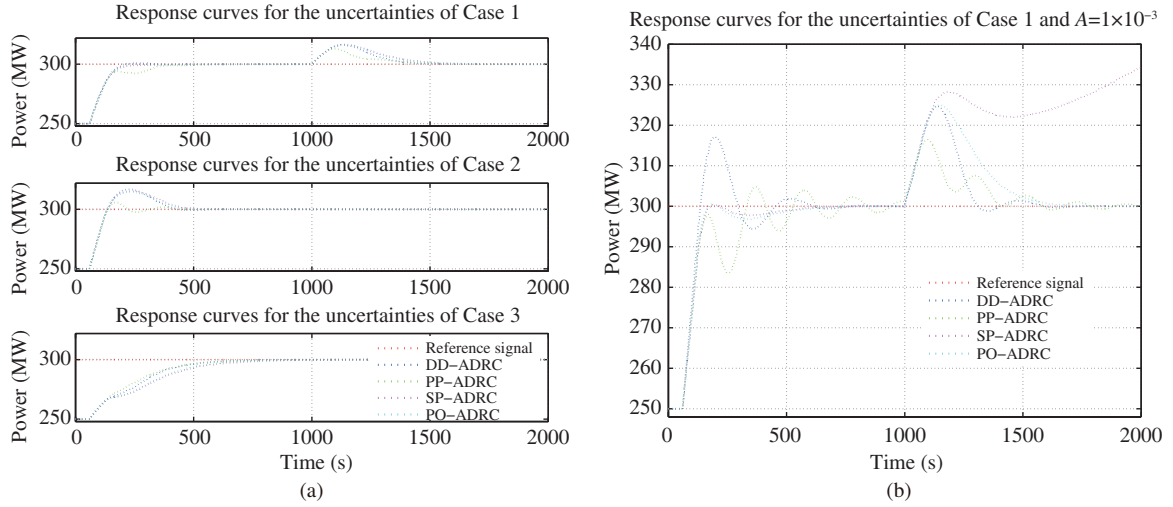
Figure 1 (Color online) (a) The simulation results of four modified ADRCs for three cases of uncertainties; (b) the simulation results of four modified ADRCs for the uncertainties of Case 1 and $A = 1 \times 10^{-3}$.

uncertainty is quantitatively discussed in Table 1, where the expressions of the comparison between each two different methods at low frequency are explicitly presented in the 2nd column. Furthermore, to simplify such complicated expressions, the corresponding magnitudes of the observer bandwidth ω_o are calculated as shown in the 3rd column in Table 1. As a result, PP-ADRC shows the stronger capability to reject the "total disturbance" than the other modified ADRCs at low frequency by tuning the bandwidth of ESO to be larger. For DD-ADRC, SP-ADRC and PO-ADRC, the capabilities to reject uncertainty are on the same level, since the transfer functions have the same magnitudes of the observer bandwidth. Moreover, if the feedback gain K is selected such that the real parts of A_K's eigenvalues become smaller, DD-ADRC is more capable of cancelling the "total disturbance" than SP-ADRC and PO-ADRC.

7 Simulation

In this section, the simulations of the boiler turbine unit modeled in [27] are presented to illustrate the theoretical results in this paper. The dynamics of the boiler turbine unit has the same form as the system (1) where the system parameters satisfy ($A = -6.9 \times 10^{-3}$, $B = 4.35 \times 10^{-2}$, $C = 1$, $\tau = 60$, $t_0 = 0$). Moreover, the control input is the fuel rate fed into the furnace and the system output is the power generated by burning the fuel. The control objective for the boiler turbine system is letting the output track the reference signal $r = 300$ (MW) despite various uncertainties. To demonstrate the effectiveness of the modified ADRCs, the following three typical cases of uncertainties are considered, including external disturbance (Case 1), unmodeled nonlinear dynamics (Case 2) and parameter perturbation (Case 3).

$$\text{Case 1: } \delta = \begin{cases} 0, & 0 \leqslant t < 1000, \\ 5, & t \geqslant 1000, \end{cases} \quad \text{Case 2: } \delta = (Ax(t))^2 + \sin\left(\frac{x(t)}{2\pi}\right) + e^{A(x(t)-300)}, \quad \text{Case 3: } \delta = \frac{0.2Ax(t)}{B}.$$

The four main modifications of ADRC, i.e., the DD-ADRC (8) and (9), the PP-ADRC (15)–(17), the SP-ADRC (24)–(26), and the PO-ADRC (27) and (28), are applied to the boiler turbine system. With the same observer bandwidth ($\omega_o = 0.015$) and the same feedback gain ($K = 0.013$), the simulation results of the modified ADRCs for the three cases of uncertainties are shown in Figure 1(a). From Figure 1(a), the response curves of the system output for four modified ADRCs and three cases of uncertainties demonstrate the capabilities of modified ADRCs to deal with both time delay and uncertainties. Additionally, Figure 1(a) demonstrates that PP-ADRC is more capable of rejecting disturbance than the other modified ADRCs, since the corresponding closed-loop system has faster tracking performance despite the uncertainties occurring in $t \geqslant 1000$ s in Case 1 and $t \geqslant 0$ s in Cases 2 and 3.

To illustrate the necessity of stable open loop for SP-ADRC to handle uncertainties, the stable open loop of the boiler turbine system is changed into an unstable one by letting $A = 1 \times 10^{-3}$. With the same control parameters, Figure 1(b) shows the simulation results of the four modified ADRCs for the uncertainty of Case 1 and $(A = 1 \times 10^{-3})$. Since there is no uncertainty in the boiler turbine system for $0 \leqslant t < 1000$ (in s), the closed-loop systems based on modified ADRCs are stable. Since the external disturbance affects the physical plant for $t \geqslant 1000$ s, the SP-ADRC based closed-loop system becomes unstable for $t \geqslant 1000$ s, which illustrates that stable open loop is necessary for stabilizing the closed-loop system based on SP-ADRC. Additionally, the other three modified ADRCs are capable of handling time delay systems with unstable open loop as shown in Figure 1(b).

8 Conclusion

In this paper, the modified ADRCs for time delay systems, namely, DD-ADRC, PP-ADRC, SP-ADRC and PO-ADRC, are rigorously studied. Firstly, the capabilities of these modified ADRC to deal with time delay are discussed. It is shown that the stability of the DD-ADRC and the PP-ADRC based closed-loop system has the restriction of the size of time delay. Nevertheless, SP-ADRC and PO-ADRC can handle arbitrarily large time delay when there is no uncertainty. Then, in the discussion of necessity of stable open loop, the stability of the closed-loop system based on SP-ADRC requires stable open loop, whereas DD-ADRC, PP-ADRC and PO-ADRC can stabilize uncertain systems even without this condition. Thirdly, for the capability to reject uncertainty, these modified ADRCs are capable of uncertainty rejection at low frequency. Furthermore, the quantitative comparison of the capabilities to reject uncertainty between each two control approaches is explicitly presented. PP-ADRC are more capable of uncertainty rejection than other modified ADRCs by tuning ESO's bandwidth. Finally, the simulations of a boiler turbine system illustrate the theoretical results.

The paper presents a comprehensively comparison of the four main modified ADRCs by rigorous analysis. It is expected that the results will greatly help practitioners to design the modification of ADRC in dealing with systems with both uncertainties and time delays.

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Appendix A

Proof of Theorem 1.   For the DD-ADRC based closed-loop system (6), (8) and (9), the characteristic function is obtained as follows:

$$F_{c,\mathrm{DD}}(s,\tau) = (sI - A + BK^{\mathrm{T}}\mathrm{e}^{-\tau s})(sI - A_{Le,\mathrm{DD}}). \tag{A1}$$

Since the control parameters $(K, L_{e,\mathrm{DD}})$ are nonzero, the characteristic function (A1) is related to the time delay τ. Therefore, there exists $\tilde{\tau}_{\mathrm{DD}} > 0$ such that the equation $F_{c,\mathrm{DD}}(j\omega, \tilde{\tau}_{\mathrm{DD}}) = 0$ has solution ω^*. Hence, there exists $\tau_{\mathrm{DD}}^* \geqslant \tilde{\tau}_{\mathrm{DD}}$ such that the characteristic equation $(F_{c,\mathrm{DD}}(s,\tau^*) = 0)$ has solutions in the right-half plane. The closed-loop system based on the DD-ADRC (8) and (9) is stable if and only if all the solutions of the characteristic equation $(F_{c,\mathrm{DD}}(s,\tau) = 0)$ are in the left plane[1]. Hence, the closed-loop system based on the DD-ADRC (8) and (9) is unstable if $\tau = \tau_{\mathrm{DD}}^*$.

Since the characteristic function of the PP-ADRC based closed-loop system (6) and (15)–(17) is

$$F_{c,\mathrm{PP}} = \det\left(\begin{bmatrix} sI - A & (1+\tau s)\mathrm{e}^{-\tau s}BK^{\mathrm{T}} & (1+\tau s)\mathrm{e}^{-\tau s}B \\ -L_{\mathrm{PP}}C^{\mathrm{T}} & sI - A_{L,\mathrm{PP}} & 0 \\ -L_{\delta,\mathrm{PP}}C^{\mathrm{T}} & L_{\delta,\mathrm{PP}}C^{\mathrm{T}} & s \end{bmatrix} \right), \tag{A2}$$

which is also related to time delay, the closed-loop system is unstable for the time delay τ_{PP}^* due to the same proof as in the case of DD-ADRC.

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Proof of Theorem 2. It will be first proved that the closed-loop system based on the SP-ADRC (24)–(26) is stable for arbitrarily large time delay. Owing to the system plant (6) and the SP in time domain (24), since the "total disturbance" $\delta_{\rm SP} = 0$, the following equations are obtained.

$$x_{\rm SP}(t) = {\rm e}^{A(t-t_0)}x_{\rm SP}(t_0) + \int_{t_0}^{t} {\rm e}^{A(t-\theta)}Bu_{\rm SP}(\theta-\tau){\rm d}\theta, \quad \bar{x}_{\rm SP}(t) = {\rm e}^{A(t-t_0)}\bar{x}_{\rm SP}(t_0) + \int_{t_0}^{t} {\rm e}^{A(t-\theta)}Bu_{\rm SP}(\theta-\tau){\rm d}\theta. \quad (A3)$$

Since the initial values are the same, i.e., $x_{\rm SP}(t_0) = \bar{x}_{\rm SP}(t_0)$, combined with (24) and (A3), it yields that $y_{p,\rm SP}(t) = y_{\rm SP}(t+\tau)$. By (25) and (26), the closed-loop system (6) and (24)–(26) is stable due to the stable matrices A_K and $A_{Le,\rm SP}$.

Then, consider the PO-ADRC based closed-loop system (6), (27) and (28). Since the "total disturbance" $\delta_{\rm PO} = 0$, the system (6) is reformulated into the following linear system with time delays.

$$\dot{\bar{x}}_{\rm PO}(t) = A_e\bar{x}_{\rm PO}(t) + B_e u(t-\tau), \quad y_{\rm PO}(t) = C_e^{\rm T}\bar{x}_{\rm PO}(t), \quad t \geqslant t_0, \quad (A4)$$

where the extended state $\bar{x}_{\rm PO}(t) = [x_{\rm PO}^{\rm T}(t)\ 0]^{\rm T}$. According to the stability analysis of the PO based linear time delay system[2], since the matrices A_K and $A_{Le,\rm PO}$ are Hurwitz, the closed-loop system (A4), (27) and (28) is stable which implies that the closed-loop system (6), (27) and (28) is stable.

Proof of Theorem 3. It will be prove that the matrix A being stable is a necessary condition for bounded tracking error $\|x_{\rm SP}(t) - x^*(t)\|$ by contradiction. Assume that the matrix A is unstable. Based on the system (6) and the SP in time domain (24), the following formulas are obtained.

$$\begin{cases} x_{\rm SP}(t) = {\rm e}^{A(t-t_0)}x_{\rm SP}(t_0) + \int_{t_0}^{t} {\rm e}^{A(t-\theta)}B(u_{\rm SP}(\theta-\tau) + \delta_{\rm SP}(x_{\rm SP}(\theta),\theta)){\rm d}\theta, \\ \bar{x}_{\rm SP}(t) = {\rm e}^{A(t-t_0)}\bar{x}_{\rm SP}(t_0) + \int_{t_0}^{t} {\rm e}^{A(t-\theta)}Bu_{\rm SP}(\theta-\tau){\rm d}\theta. \end{cases} \quad (A5)$$

According to (A5), the prediction error of $y_{\rm SP}(t+\tau)$, denoted as $e_{y,\rm SP}(t)$, is presented as

$$e_{y,\rm SP}(t) = y_{p,\rm SP}(t) - y_{\rm SP}(t+\tau)$$
$$= C^{\rm T}\left(\begin{array}{l} \int_{t_0}^{t} {\rm e}^{A(t-\theta)}B\delta_{\rm SP}(x_{\rm SP}(\theta),\theta){\rm d}\theta - \int_{t_0}^{t+\tau} {\rm e}^{A(t+\tau-\theta)}B\delta_{\rm SP}(x_{\rm SP}(\theta),\theta){\rm d}\theta \\ + {\rm e}^{At}({\rm e}^{-At_0} - {\rm e}^{-A(t_0-\tau)})(x_{\rm SP}(t_0) - \bar{x}_{\rm SP}(t_0)) \end{array}\right). \quad (A6)$$

Since the matrix A is unstable and $\delta_{\rm SP}$ is a nonzero constant, the prediction error $e_{y,\rm SP}(t)$ will tend to infinity as t tends to infinity.

On the other hand, Assumption 1 implies that the reference signal $r(t)$ is bounded for $t \geqslant t_0$. Combined with the ideal trajectory (2), it yields that $x^*(t)$ is bounded for $t \geqslant t_0$. Moreover, since $\|x_{\rm SP}(t) - x^*(t)\|$ is bounded for $t \geqslant t_0$, it is deduced that $y_{\rm SP}(t)$ is bounded for $t \geqslant t_0$. Consequently, by the boundless prediction error $e_{y,\rm SP}(t)$ and the inequality $(|y_{p,\rm SP}(t)| \geqslant |e_{y,\rm SP}(t)| - |y_{\rm SP}(t+\tau)|)$, we have that $y_{p,\rm SP}(t)$ is boundless when t tends to infinity.

Since the boundless signal $y_{p,\rm SP}(t)$ is the input of the ESO (25), the estimations $\hat{x}_{\rm SP}(t)$ and $\hat{\delta}_{\rm SP}(t)$ are boundless which leads to an infinity control input $u_{\rm SP}(t)$ due to (26). Hence, the system state $x_{\rm SP}(t)$ is boundless by (6), which is a contradiction.

Proof of Theorem 4. Since the "total disturbance" is assumed to be zero, according to (A1), the DD-ADRC based closed-loop system (6), (8) and (9) is stable if and only if all the solutions of the characteristic equation $(F_{c,\rm DD}(s,\tau) = 0)$ are in the left plane[1]. Since A_k and $A_{Le,\rm DD}$ are Hurwitz, all the solutions of the characteristic equation $(F_{c,\rm DD}(s,0) = 0)$ are in the left plane. It should be noted that the solutions of the characteristic equation $(F_{c,\rm DD}(s,\tau) = 0)$ continuously changes as time delay τ varies. Hence, there exists a region of time delay $\Omega_{\tau,\rm DD} = \{\tau|\tau < \tau_{\rm DD}^{**}\}$ such that all the solutions of the characteristic equation $(F_{c,\rm DD}(s,\tau) = 0)$ are in the left plane for $\tau \in \Omega_{\tau,\rm DD}$.

Owing to a similar derivation, there exists a region of time delay $\Omega_{\tau,\rm PP} = \{\tau|\tau < \tau_{\rm PP}^{**}\}$ such that all the solutions of the characteristic equation $(F_{c,\rm PP}(s,\tau) = 0)$ are in the left plane for $\tau \in \Omega_{\tau,\rm PP}$. By denoting $\tau^{**} = \min\{\tau_{\rm DD}^{**}, \tau_{\rm PP}^{**}\}$, the proof of Theorem 4 is completed.

Proof of Theorem 5. The estimation error and the tracking error of the system state are defined as

$$x_{e,\rm PO}(t) = x_{\rm PO}(t) - \hat{x}_{\rm PO}(t), \quad e_{\rm PO}(t) = x_{\rm PO}(t) - x^*(t), \quad (A7)$$

respectively. Accordingly, the PO-ADRC based closed-loop system (6), (27) and (28) becomes

$$\begin{cases} \dot{e}_{\rm PO}(t) = A_K e_{\rm PO}(t) + B(\delta(x_{\rm PO}(t),t) - \hat{\delta}_{\rm PO}(t) + K^{\rm T}x_{e,\rm PO}(t)), \\ \begin{bmatrix} \dot{x}_{e,\rm PO}(t) \\ -\dot{\hat{\delta}}_{\rm PO}(t) \end{bmatrix} = A_e \begin{bmatrix} x_{e,\rm PO}(t) \\ -\hat{\delta}_{\rm PO}(t) \end{bmatrix} - {\rm e}^{A_e\tau}L_{e,\rm PO}(y_{\rm PO}(t-\tau) - \hat{y}_{\rm PO}(t)) + B_e\delta_{\rm PO}(x(t),t), \\ \hat{y}_{\rm PO}(t) = C_e^{\rm T}\begin{bmatrix} \hat{x}_{\rm PO}(t-\tau) \\ -\hat{\delta}_{\rm PO}(t-\tau) \end{bmatrix} + C_e^{\rm T}\int_0^{\tau} {\rm e}^{A_e s}L_{e,\rm PO}(y_{\rm PO}(t-\tau-s) - \hat{y}_{\rm PO}(t-s)){\rm d}s. \end{cases} \quad (A8)$$

It can be verified that the following function:

$$\tilde{h}(\sigma,t) = y_{\rm PO}(t+\sigma-\tau) - \hat{y}_{\rm PO}(t+\sigma) - C_e^{\rm T}{\rm e}^{A_e(\sigma-\tau)}\begin{bmatrix} x_{e,\rm PO}(t) \\ -\hat{\delta}_{\rm PO}(t) \end{bmatrix}$$
$$+ \int_0^{\sigma} C_e^{\rm T}{\rm e}^{A_e\xi}L_{e,\rm PO}(y_{\rm PO}(t+\sigma-\xi-\tau) - \hat{y}_{\rm PO}(t+\sigma-\xi)){\rm d}\xi \quad (A9)$$

2) Krstic M, Kanellakopoulos I, Kokotovic P V. Nonlinear and Adaptive Control Design. Boston: Wiley, 1995.

satisfies

$$
\begin{aligned}
\frac{\partial \tilde{h}}{\partial t} &= \frac{\partial \tilde{h}}{\partial \sigma} - C_e^{\mathrm{T}} \mathrm{e}^{A_e(\sigma-\tau)} B_e \delta_{\mathrm{PO}}(x_{\mathrm{PO}}(t), t), \\
\tilde{h}(0, t) &= y_{\mathrm{PO}}(t - \tau) - \hat{y}_{\mathrm{PO}}(t) - C_e^{\mathrm{T}} \mathrm{e}^{-A_e \tau} \begin{bmatrix} x_{e,\mathrm{PO}}(t) \\ -\hat{\delta}_{\mathrm{PO}}(t) \end{bmatrix}, \\
\tilde{h}(\tau, t) &= 0,
\end{aligned}
\tag{A10}
$$

where $\sigma \in [0, \tau]$ and $t \in [t_0, \infty)$. Substituting (A10) into (A8), it yields that

$$
\begin{cases}
\dot{e}_{\mathrm{PO}}(t) = A_K e_{\mathrm{PO}}(t) + B(\delta_{\mathrm{PO}}(x_{\mathrm{PO}}(t), t) - \hat{\delta}_{\mathrm{PO}}(t) + K^{\mathrm{T}} x_{e,\mathrm{PO}}(t)), \\
\begin{bmatrix} \dot{x}_{e,\mathrm{PO}}(t) \\ -\dot{\hat{\delta}}_{\mathrm{PO}}(t) \end{bmatrix} = \mathrm{e}^{A_e \tau} A_{Le,\mathrm{PO}} \mathrm{e}^{-A_e \tau} \begin{bmatrix} x_{e,\mathrm{PO}}(t) \\ -\hat{\delta}_{\mathrm{PO}}(t) \end{bmatrix} - \mathrm{e}^{A_e \tau} L_{e,\mathrm{PO}} \tilde{h}(0, t) + B_e \delta_{\mathrm{PO}}(x_{\mathrm{PO}}(t), t), \\
\frac{\partial \tilde{h}}{\partial t} = \frac{\partial \tilde{h}}{\partial \sigma} - C_e^{\mathrm{T}} \mathrm{e}^{A_e(\sigma-\tau)} B_e \delta_{\mathrm{PO}}(x_{\mathrm{PO}}(t), t), \quad \tilde{h}(\tau, t) = 0,
\end{cases}
\tag{A11}
$$

which implies

$$
\dot{\eta}(t) = A_c \eta(t) + H_c \tilde{h}(0, t) + B_c \delta(x(t), t), \quad \frac{\partial \tilde{h}}{\partial t} = \frac{\partial \tilde{h}}{\partial \sigma} - C_e^{\mathrm{T}} \mathrm{e}^{A_e(\sigma-\tau)} B_e \delta(x(t), t), \quad \tilde{h}(\tau, t) = 0,
\tag{A12}
$$

where

$$
\eta(t) = \begin{bmatrix} e_{\mathrm{PO}}(t) \\ \begin{bmatrix} x_{e,\mathrm{PO}}(t) \\ -\hat{\delta}_{\mathrm{PO}}(t) \end{bmatrix} \end{bmatrix}, \quad A_c = \begin{bmatrix} A_K & \begin{bmatrix} BK^{\mathrm{T}} & B \end{bmatrix} \\ 0 & \mathrm{e}^{A_e \tau} A_{Le,\mathrm{PO}} \mathrm{e}^{-A_e \tau} \end{bmatrix}, \quad H_c = \begin{bmatrix} 0 \\ -\mathrm{e}^{A_e \tau} L_{e,\mathrm{PO}} \end{bmatrix}, \quad B_c = \begin{bmatrix} B \\ B_e \end{bmatrix}.
\tag{A13}
$$

Since A_K and $A_{Le,\mathrm{PO}}$ are Hurwitz, A_c is Hurwitz. Moreover, there exists a positive definite matrix P such that $PA_c + A_c^{\mathrm{T}} P = -I$. Define the following Lyapunov-Krasovskii function:

$$
V(t) = \eta(t)^{\mathrm{T}} P \eta(t) + a_1 \int_0^{\tau} (1 + \xi) \tilde{h}(\xi, t)^2 \mathrm{d}\xi,
\tag{A14}
$$

where $a_1 > 0$. With (A12), we have

$$
\begin{aligned}
\dot{V}(t) = {}& -\|\eta(t)\|^2 + 2\eta(t)^{\mathrm{T}} P H_c \tilde{h}(0, t) + 2\eta(t)^{\mathrm{T}} P B_c \delta_{\mathrm{PO}}(x_{\mathrm{PO}}(t), t) + 2a_1 \int_0^{\tau} (1 + \xi) \tilde{h}(\xi, t) \frac{\partial \tilde{h}}{\partial \xi} \mathrm{d}\xi \\
& - 2a_1 \int_0^{\tau} (1 + \xi) \delta_{\mathrm{PO}}(x_{\mathrm{PO}}(t), t) \tilde{h}(\xi, t) C_e^{\mathrm{T}} \mathrm{e}^{A_e(\xi-\tau)} B_e \mathrm{d}\xi \\
\leqslant {}& -\|\eta(t)\|^2 + 2\|\eta(t)\| \|P H_c\| |\tilde{h}(0, t)| + 2\|\eta(t)\| \|P B_c\| |\delta_{\mathrm{PO}}(x_{\mathrm{PO}}(t), t)| - a_1 \tilde{h}(0, t)^2 - a_1 \int_0^{\tau} \tilde{h}(\xi, t)^2 \mathrm{d}\xi \\
& + a_1 \frac{1}{\theta} \delta_{\mathrm{PO}}(x_{\mathrm{PO}}(t), t)^2 \int_0^{\tau} (1 + \xi) |C_e^{\mathrm{T}} \mathrm{e}^{A_e(\xi-\tau)} B_e| \mathrm{d}\xi + a_1 \theta \int_0^{\tau} (1 + \xi) \tilde{h}(\xi, t)^2 |C_e^{\mathrm{T}} \mathrm{e}^{A_e(\xi-\tau)} B_e| \mathrm{d}\xi
\end{aligned}
\tag{A15}
$$

for any $\theta > 0$. Therefore, the positives a_1 and θ can be chosen such that there exist $\mu_1 > 0$ and $\mu_2 > 0$ to ensure

$$
\dot{V}(t) \leqslant -\mu_1 V(t) + \mu_2 |\delta_{\mathrm{PO}}(x_{\mathrm{PO}}(t), t)|^2.
\tag{A16}
$$

According to (29) and the exact form of V (A14), it follows that

$$
\|e_{\mathrm{PO}}(t)\| \leqslant \sqrt{V(t)}/\sqrt{\lambda_p}, \quad |\delta_{\mathrm{PO}}(x_{\mathrm{PO}}, t)| \leqslant \alpha_x \|x_{\mathrm{PO}}\| + \alpha_d \leqslant \alpha_x \|x^*\| + \alpha_x \sqrt{V(t)}/\sqrt{\lambda_p} + \alpha_d,
\tag{A17}
$$

where λ_p is the minimal eigenvalue of P. Hence, for small enough α_x, there exist $\mu_1^* > 0$ and $\mu_2^* > 0$ such that

$$
\dot{V}(t) \leqslant -\mu_1^* V(t) + \mu_2^* (\alpha_x \|x^*(t)\|^2 + \alpha_d^2),
\tag{A18}
$$

which implies

$$
V(t) \leqslant V(t_0) + \frac{\mu_2^*}{\mu_1^*} \left(\alpha_x \sup_{t \in [t_0, \infty)} \|x^*(t)\|^2 + \alpha_d^2 \right).
\tag{A19}
$$

Combined with Assumption 1, the ideal trajectory (2) shows that there exists a positive μ_3^* such that

$$
\sup_{t \in [t_0, \infty)} \|x^*(t)\| \leqslant \mu_3^* (\|x_{\mathrm{PO}}(t_0)\| + r_{\max}).
\tag{A20}
$$

Moreover, the definitions of $\tilde{h}$ (A9) and V (A14) imply that there exists a positive μ_4^* such that

$$
\|V(t_0)\| \leqslant \mu_4^* \|x_{e,\mathrm{PO}}(t_0)\|^2.
\tag{A21}
$$

By (A19)–(A21), $e_{\mathrm{PO}}(t_0) = 0$ and (A17), the boundary of tracking error (30) is obtained.

Proof of Theorem 6. The notation $(A(s) \underset{s \to 0}{\sim} B(s))$ represents $\lim_{s \to 0} \frac{A(s)}{B(s)} = 1$, which implies that $A(s)$ behaves asymptotically like $B(s)$ as s tends to zero[3]. Since (33) and (40) are satisfied and $C = 1$, by using the notation $(A(s) \underset{s \to 0}{\sim} B(s))$, it is deduced that

$$
G_{\delta_e \delta, \mathrm{DD}}(s) \underset{s \to 0}{\sim} -\frac{1}{L_{\delta, \mathrm{DD}} C^{\mathrm{T}} A_{L, \mathrm{DD}}^{-1} B} s \underset{s \to 0}{\sim} \frac{2}{\omega_o} s, \quad G_{x_e \delta, \mathrm{DD}}(s) \underset{s \to 0}{\sim} \frac{A_{L, \mathrm{DD}}^{-1} B}{L_{\delta, \mathrm{DD}} C^{\mathrm{T}} A_{L, \mathrm{DD}}^{-1} B} s \underset{s \to 0}{\sim} \frac{B}{\omega_o^2} s.
\tag{A22}
$$

---

3) Zorich V Z, Cooke R. Mathematical Analysis I. Berlin: Springer-Verlag, 2004.

With (37), it follows that

$$G_{y\delta,\mathrm{DD}}(s) \underset{s\to 0}{\sim} -C^T A_K^{-1} B \left(\frac{K^T A_{L,\mathrm{DD}}^{-1} B - 1}{L_{\delta,\mathrm{DD}} C^T A_{L,\mathrm{DD}}^{-1} B} + \tau \right) s \underset{s\to 0}{\sim} -\frac{B(\tau\omega_o^2 + 2\omega_o + BK)}{A_K \omega_o^2} s. \tag{A23}$$

Next, for PP-ADRC and SP-ADRC, from (34), (35) and (37), the following equations are obtained owing to the same derivations (A22) and (A23).

$$G_{y\delta,\mathrm{PP}}(s) \underset{s\to 0}{\sim} -\frac{B(2\omega_o + KB)}{A_K \omega_o^2} s, \quad G_{y\delta,\mathrm{SP}}(s) \underset{s\to 0}{\sim} -\frac{B(\tau A_K \omega_o^2 + 2A\omega_o + 2AK)}{A_K A \omega_o^2} s. \tag{A24}$$

It should be noted that

$$\mathrm{e}^{A_e \xi} = \begin{bmatrix} 1 & -\frac{B}{A} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{A\xi} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{B}{A} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{A\xi} & \frac{B}{A}(\mathrm{e}^{A\xi} - 1) \\ 0 & 1 \end{bmatrix}. \tag{A25}$$

For PO-ADRC, combining (A25), (36), (37) and the derivations (A22) and (A23) yields that

$$G_{y\delta,\mathrm{PO}}(s) \underset{s\to 0}{\sim} -\frac{B(KB(\mathrm{e}^{A\tau} - 1) + AA_K\tau)\omega_o^2 + 2AB(BK\mathrm{e}^{A\tau} + A_K)\omega_o + A^2 B^2 K \mathrm{e}^{A\tau}}{A_K A^2 \omega_o^2} s. \tag{A26}$$

By substituting ($s = j\omega$) into (A23), (A24) and (A26), the 2nd column of Table 1 is obtained. Furthermore, the 3rd column of Table 1 is obtained directly from the second column.