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Special Focus on Modeling, Analysis and Control of Stochastic Systems

Stabilization for multi-group coupled stochastic models by delay feedback control and nonlinear impulsive control

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Abstract Stabilization for multi-group coupled models stochastic by delay feedback control and nonlinear impulsive control are considered in this paper. Using graph theory and Lyapunov method, some sufficient conditions are acquired by some control methods. Those criteria are easier to verify and no need to solve any linear matrix inequalities. These results can be designed more easily in practice. At last, the effectiveness and advantage of the theoretical results are verified by an example.

Keywords stabilization, graph theory, stochastic, impulsive, delay

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1 Introduction

On a mathematical network model, directed graphs are usually composed of vertices and directed arcs [1–4]. At each vertex, the local dynamics are often represented by differential equations. It is called the vertex systems.

For example, the vertex systems

$$\begin{pmatrix} x_i' \\ y_i' \end{pmatrix} = A_i \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad i = 1, 2.$$

Let $A_i = \left(\begin{smallmatrix} -2 & 2 \\ -2 & 1 \end{smallmatrix}\right)$. The eigenvalues of A_i are $-\frac{1}{2} \pm \frac{\sqrt{7}}{2} i$. We know the solution of the vertex systems is globally asymptotically stable.

Linearly coupled systems are considered in the following:

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = A_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} 2x_2 - x_1 \\ 0 \end{pmatrix},$$

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$$\begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = A_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2x_1 - x_2 \\ 0 \end{pmatrix},$$

whose coefficient matrix

$$A = \begin{pmatrix} -3 & 2 & 2 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & -3 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

has a positive eigenvalue 0.2361, and then the coupled system is unstable.

In fact, real systems are often subject to some external disturbances. Those external disturbances may destroy or change system dynamic behavior. The stochastic perturbation is an important form in real world.

Then, a general multi-group coupled stochastic model is depicted in the following:

$$\begin{cases} \mathrm{d}x_k^{(i)}(t) = \left[f_k^{(i)}(t, x_k(t), x_k(t-\tau(t))) + \sum\limits_{h=1}^{l} H_{kh}^{(i)} \left(x_k^{(i)}(t), x_h^{(i)}(t) \right) \right] \mathrm{d}t \\ \qquad\qquad + g_k^{(i)} \left(t, x_k^{(i)}(t), x_k^{(i)}(t-\tau(t)) \right) \mathrm{d}\omega(t), \\ x_k^{(i)}(t) = \psi_k^{(i)}(s), \quad s \in [-\tau, 0]. \end{cases} \tag{1}$$

where $t \geqslant t_0$, $\tau(t) : [t_0, \infty) \to [0, \tau]$ is continuous, and $k \in \mathbb{L} = \{1, 2, \ldots, l\}$,

$$f_k^{(i)} \left(x_k^{(i)}(t), x_k^{(i)}(t-\tau(t)), t \right) : \mathbb{R}^{m_i} \times \mathbb{R}^{m_i} \times [t_0, \infty) \to \mathbb{R}^{m_i},$$
$$g_k^{(i)} \left(x_k^{(i)}(t), x_k^{(i)}(t-\tau(t)), t \right) : \mathbb{R}^{m_i} \times \mathbb{R}^{m_i} \times [t_0, \infty) \to \mathbb{R}^{m_i \times m},$$

$x_k^{(i)}(t) = ((x_k^{(1)}(t))^{\mathrm{T}}, (x_k^{(2)}(t))^{\mathrm{T}}, \ldots, (x_k^{(m_i)}(t))^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{m_i}$ represents performance and state of the k-th group, $\sum_{i=1}^{n} m_i = m$, $m_i \in \mathbb{Z}^+$. $H_{kh}^{(i)}(x_k^{(i)}(t), x_h^{(i)}(t)) : \mathbb{R}^{m_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$ is the dispersal of the i-th component from the h-th group to the k-th group, $\tau = \max\{\tau(t)\}$. It should be pointed out that $H_{kh}^{(i)} = 0$ when and only when there is no dispersal between the h-th group and the k-th group for the i-th component.

There are l vertices in every digraph. $(\mathcal{G}, A_i)$ stands for the i-th component in the k-th group vertex system. Then system (1) can be described by n digraphs.

Throughout this paper, we have assumed that $f_k^{(i)}(0, 0, t) \equiv 0$, $H_{kh}^{(i)}(0, 0) \equiv 0$, $g_k^{(i)}(0, 0, t) \equiv 0$.

Thus, to make multi-group stochastic models achieve stabilization has been an interesting and challenging issue. So far, various methods have been used to study the stabilization of systems. For example, adaptive control methods [5–8], active control methods under the compound systems [9, 10], variable structure system control methods [11], feedback control methods for a class of uncertain nonlinear systems [12–15], observer-based control methods [16], coupling control or nonlinear control methods [17–22], and impulsive control methods [23–28]. We discuss the stabilization of multi-group coupled stochastic models by delay feedback control and nonlinear impulsive control.

The whole paper is arranged as follows. In Section 2, we first introduce some basic concepts and lemmas, which are all needed later. In Section 3, some results are obtained for stabilization of linear multi-group models by delay feedback control and nonlinear impulsive control. In Section 4, an example is given to illustrate the correctness and feasibility of the obtained conclusions.

2 Preliminaries

At first, we give some basic notations. Define $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^n$ is n-dimensional Euclidean space, $\mathbb{R}^+ = [0, +\infty)$, and $\mathbb{Z}^+ = \{1, 2, \ldots, n, \ldots\}$. Define $\mathbb{L} = \{1, 2, \ldots, l\}$ and $\mathbb{N} = \{1, 2, \ldots, n\}$. The superscript "T" indicates the transpose. $\|\cdot\|$ is the Euclidean norm for n-dimensional vector. $\|\boldsymbol{A}\| = \sqrt{\mathrm{trace}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})}$

is its trace norm. $V(x,t) \in C^{2,1}(\mathbb{R}^m \times \mathbb{R}^+; \mathbb{R}^+)$ is continuously twice differentiable in x and once in t. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ is a complete probability space. Filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfies the usual conditions. $W(\cdot)$ is the appropriate-dimensional Brownian motion defined on the complete probability space. Let $\mathrm{E}(\cdot)$ be the mathematical expectation operator on probability measure.

For a real symmetric matrix, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues. If X and Y are symmetric matrices, $X \geqslant Y$ ($X > Y$) means that $X - Y$ is semi-positive definite (positive definite); $\mathrm{diag}(\cdot)$ expresses the diagonal matrix; I is the unit matrix of appropriate dimension. $R^{n\times m}$ denotes the set of all $n \times m$ real matrices.

Let $\mathcal{G} = (\mathbb{L}, \mathbb{E})$ be a digraph which contains a vertex set $\mathbb{L}$ and a edge set $\mathbb{E}$. The arc (k,h) indicates from initial vertex k to terminal vertex h. Each arc (h,k) is assigned a weight $a_{kh} > 0$ in a digraph $\mathcal{G}$. In $\mathcal{G}$, $a_{kh} > 0$ denotes there is only one arc from vertex h to vertex k. The weight $W(\mathcal{G})$ of $\mathcal{G}$ is the product of all arc weights. A directed path $\mathcal{P}$ has its distinct vertices $k_1, k_2, \ldots, k_s$ and arc set $\{(k_i, k_{i+1}) : i = 1, 2, \ldots, s-1\}$. If $k_s = k_1$, a directed path $\mathcal{C}$ is called a directed cycle. If a connected subgraph $\mathcal{T}$ contains no cycles, it is called a tree. For any pair of distinct vertices, if there is only one directed path from one to the other, a digraph $\mathcal{G}$ is strongly connected. $(\mathcal{G}, A)$ indicates the digraph with weight matrix A. If $W(\mathcal{C}) = W(-\mathcal{C})$ for all directed cycles $\mathcal{C}$, a weighted digraph $(\mathcal{G}, A)$ is balanced. $-\mathcal{C}$ expresses the reverse of $\mathcal{C}$, i.e., the direction of all arcs in $\mathcal{C}$ is reversed. If $\widetilde{\mathcal{Q}}$ expresses the reverse of $\mathcal{Q}$ and $(\mathcal{G}, A)$ is balanced, then $W(\mathcal{Q}) = W(\widetilde{\mathcal{Q}})$. The Laplacian matrix of $(\mathcal{G}, A)$ is often defined as

$$L = \begin{pmatrix} \sum\limits_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum\limits_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum\limits_{k\neq n} a_{nk} \end{pmatrix}.$$

Proposition 1 ([3, 4]). Suppose $n \geqslant 2$. Then

$$c_i = \sum_{\mathcal{T}\in\mathbb{T}_i} W(\mathcal{T}), \quad i = 1, 2, \ldots, n, \tag{2}$$

where c_i is a cofactor of the i-th element of $\mathrm{diag}(L)$. Here $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$. All spanning trees $\mathcal{T}$ are rooted at vertex i. In particular, if the digraph is strongly connected, it has $c_i > 0$ $(1 \leqslant i \leqslant n)$.

Lemma 1 ([3, 4]). Suppose $n \geqslant 2$. c_i is defined by Proposition 1. Then the following equation is established:

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r)\in E(\mathcal{C}_\mathcal{Q})}^{n} F_{r,s}(x_r, x_s), \tag{3}$$

where $F_{ij}(x_i, x_j)$ is arbitrary function, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$.

Lemma 2 ([3, 4]). Suppose $n \geqslant 2$. c_i is defined by Proposition 1. Then the following equation is established:

$$\sum_{i,j=1}^{n} c_i a_{ij} G_i(x_i) = \sum_{i,j=1}^{n} c_i a_{ij} G_j(x_j), \tag{4}$$

where $G_i(x_i), 1 \leqslant i \leqslant n$, are arbitrary functions.

Remark 1. The proof of Proposition 1, Lemmata 1 and 2 can be seen in [3, 4].

Definition 1. The trivial solution of (6) is mean square exponential stability if for any $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,

$$\lim_{t\to\infty} \sup \frac{1}{t} \log(\mathrm{E}|x(t,\xi)|^2) < 0 \quad \text{a.s.}$$

Definition 2. Let $V_k^{(i)} \triangleq V_k^{(i)}(x_k^{(i)}, t) \in C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^{m_i}; \mathbb{R}^+)$. $V_k^{(i)}(x_k^{(i)}, t)$ is twice continuously differentiable in $x_k^{(i)}$ and differentiable in t. an operator $\mathcal{L}V_k^{(i)}$ is defined by the following form:

$$\mathcal{L}V_k^{(i)} = \frac{\partial V_k^{(i)}}{\partial t} + \frac{\partial V_k^{(i)}}{\partial x_k^{(i)}} f(t,\cdot) + \frac{1}{2}\text{trace}\left[g^{\mathrm{T}}(t,\cdot)\frac{\partial^2 V_k^{(i)}}{(\partial x_k^{(i)})^2} g(t,\cdot)\right],$$

where $\frac{\partial V_k^{(i)}}{\partial x_k^{(i)}} = (\frac{\partial V_k^{(i)}}{\partial x_k^{(i_1)}}, \frac{\partial V_k^{(i)}}{\partial x_k^{(i_2)}}, \ldots, \frac{\partial V_k^{(i)}}{\partial x_k^{(i_{m_i})}})$, $\frac{\partial^2 V_k^{(i)}}{(\partial x_k^{(i)})^2} = (\frac{\partial^2 V_k^{(i)}}{\partial x_k^{(i_j)} \partial x_k^{(i_s)}})_{m_i \times m_i}$.

Lemma 3. For any $X, Y \in \mathbb{R}^n$ and $\epsilon > 0$, the following matrix inequality:

$$2X^{\mathrm{T}}Y \leqslant \epsilon X^{\mathrm{T}}QX + \epsilon^{-1}Y^{\mathrm{T}}Q^{-1}Y$$

holds, in which $Q \in \mathbb{R}^{n\times n}$ is any matrix with $Q > 0$.

Lemma 4 ([29])**.** For Eq. (6), let $\lambda, p, c_1 > 0$, $c_2 > 0$ and $q > 1$. Suppose that there is a function $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^+)$ such that

$$c_1|x|^p \leqslant V(x,t) \leqslant c_2|x|^p, \quad \text{for all} \ \ (x,t) \in R^d \times [t_0 - \tau, \infty),$$

moreover,

$$\mathrm{E}\mathcal{L}V(\phi, t) \leqslant -\lambda \mathrm{E}V(\phi(0), t),$$

for all $t \geqslant t_0$, $\phi \in L^p_{\mathcal{F}_t}([-\tau, 0]; R^d)$ satisfying

$$\mathrm{E}V(\phi(\theta), t+\theta) < q\mathrm{E}V(\phi(0), t), \quad -\tau \leqslant \theta \leqslant 0.$$

Therefore for all $\xi \in L^p_{\mathcal{F}_t}([-\tau, 0]; R^d)$,

$$\mathrm{E}|x(\xi, t)|^p \leqslant \frac{c_2}{c_1}\mathrm{E}|\xi|^p \mathrm{e}^{-\gamma(t-t_0)}, \quad \text{on} \ \ t \geqslant t_0,$$

where $\gamma = \min\{\lambda, \log(q)/\tau\}$.

Remark 2. The proof of Lemma 4 can be seen in [29].

3 Main results

This section is concerned with stabilization of coupled stochastic system by delay feedback control and delay nonlinear impulsive control. Suppose that we are given an unstable linear stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}x_k^{(i)}(t) = \left[A_k^{(i)}x_k^{(i)}(t) + \sum\limits_{h=1}^{l} H_{kh}^{(i)}\left(x_k^{(i)}(t), x_h^{(i)}(t)\right)\right]\mathrm{d}t + \left(C_k^{(i)}x_k^{(i)}(t) + D_k^{(i)}x_k^{(i)}(t-\tau(t))\right)\mathrm{d}\omega(t), \\ x_k^{(i)}(t) = \psi_k^{(i)}(s), \quad s \in [-\tau, 0], \end{cases} \tag{5}$$

where $w(t) = (w_1(t), \ldots, w_m(t))$ is a Brownian motion with m-dimensional. It is required to find some controls $U_k^{(i)}(t, x_k(t-\tau(t)))$ in the drift part. Based on the past state, the controlled system can be written as follows:

$$\begin{cases} \mathrm{d}x_k^{(i)}(t) = \left[A_k^{(i)}x_k^{(i)}(t) + \sum\limits_{h=1}^{l} H_{kh}^{(i)}\left(x_k^{(i)}(t), x_h^{(i)}(t)\right) + U_k^{(i)}(t, x_k(t-\tau(t)))\right]\mathrm{d}t \\ \qquad\qquad + \left(C_k^{(i)}x_k^{(i)}(t) + D_k^{(i)}x_k^{(i)}(t-\tau(t))\right)\mathrm{d}\omega(t), \\ x_k^{(i)}(t) = \psi_k^{(i)}(s), \quad s \in [-\tau, 0]. \end{cases} \tag{6}$$

In the following, we discuss the stabilization of controlled system. Let us now begin to discuss the stabilization problem proposed in Section 1. At first, given an unstable SDE (5), we design a linear delay feedback controller to make the system (6) stability. Let $U_k^{(i)}(t, x_k^{(i)}(t-\tau(t))) = F_k^{(i)}x_k^{(i)}(t-\tau(t))$. The stabilization problem is to design $F_k^{(i)}$'s so that the controlled system (6) becomes stable.

3.1 Delay feedback control

Theorem 1. Assume that

(i) There exists a constant $a_{kh}^{(i)}$ such that

$$\sum_{h=1}^{l}\left(x_k^{(i)}\right)^{\mathrm{T}} H_{kh}^{(i)}\left(x_k^{(i)}, x_h^{(i)}\right) \leqslant \sum_{h=1}^{l} a_{kh}^{(i)}\left(\left\|x_k^{(i)}\right\|^2+\left\|x_h^{(i)}\right\|^2\right);$$

(ii) For every $i \in \mathbb{N}$, let digraph $(\mathcal{G}, A_i)$ be strongly connected and balanced, where matrix $A_i = (a_{kh}^{(i)})_{l\times l}$, and along each directed cycle $\mathcal{C}_{\mathcal{Q}_i}$ of digraph $(\mathcal{G}, A_i)$, where $a_{kh}^{(i)} \geqslant 0$, $c_k^{(i)} > 0$, $\lambda_1 = \min_{k,i}\{\lambda_{1k}^{(i)}\}$, $\lambda_2 = \max_{k,i}\{\lambda_{2k}^{(i)}\}$;

(iii) If $t \geqslant t_0$, $\phi_k^{(i)} \in L_{\mathcal{F}_t}^p([-\tau, 0]; \mathbb{R}^{m_i})$ satisfy

$$\mathrm{E}V\left(\phi_k^{(i)}(\theta), t+\theta\right) < q\mathrm{E}V\left(\phi_k^{(i)}(0), t\right) \quad \text{for all} \quad -\tau \leqslant \theta \leqslant 0,$$

for all initial data $\xi_k^{(i)} \in C_{\mathcal{F}_{t_0}}^b([-\tau, 0]; \mathbb{R}^{m_i})$;

(iv)

$$F_k^{(i)} = -\left(C_k^{(i)}\right)^{\mathrm{T}} D_k^{(i)}.$$

Then the controlled system SDE (6) is exponential stability in mean square and the Lyapunvov exponent is less than or equal to $-(\lambda_1 - q\lambda_2)$, where $q \in (1, \frac{\lambda_1}{\lambda_2})$ is the unique root of $\lambda_1 - q\lambda_2 = \log(q)/\tau$.

Proof.

$$V(x,t) = \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)} V_k^{(i)}\left(x_k^{(i)}, t\right) = \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\left\|x_k^{(i)}\right\|^2, \quad k \in \mathbb{L},\ i \in \mathbb{N}.$$

$$\begin{aligned}
\mathcal{L}V_k^{(i)}\left(x_k^{(i)}, t\right) =& 2\left(x_k^{(i)}\right)^{\mathrm{T}}\left[A_k^{(i)} x_k^{(i)}(t) + \sum_{h=1}^{l} H_{kh}^{(i)}\left(x_k^{(i)}(t), x_h^{(i)}(t)\right) + F_k^{(i)} x_k^{(i)}(t-\tau(t))\right] \\
&+ \operatorname{trace}\left[\left(C_k^{(i)} x_k^{(i)}(t) + D_k^{(i)} x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}}\left(C_k^{(i)} x_k^{(i)}(t) + D_k^{(i)} x_k^{(i)}(t-\tau(t))\right)\right] \\
=& 2\left(x_k^{(i)}(t)\right)^{\mathrm{T}} A_k^{(i)} x_k^{(i)}(t) + 2\sum_{h=1}^{l} a_{kh}^{(i)}\left(\left\|x_k^{(i)}\right\|^2 + \left\|x_h^{(i)}\right\|^2\right) + 2\left(x_k^{(i)}(t)\right)^{\mathrm{T}} F_k^{(i)} x_k^{(i)}(t-\tau(t)) \\
&+ \left(\left(C_k^{(i)} x_k^{(i)}(t) + D_k^{(i)} x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}}\left(C_k^{(i)} x_k^{(i)}(t) + D_k^{(i)} x_k^{(i)}(t-\tau(t))\right)\right) \\
\leqslant& \left(x_k^{(i)}(t)\right)^{\mathrm{T}}\left(2A_k^{(i)} + \left(C_k^{(i)}\right)^{\mathrm{T}} C_k^{(i)}\right) x_k^{(i)}(t) + 2\left(x_k^{(i)}(t)\right)^{\mathrm{T}}\left(F_k^{(i)} + \left(C_k^{(i)}\right)^{\mathrm{T}} D_k^{(i)}\right) x_k^{(i)}(t-\tau(t)) \\
&+ \left(x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}}\left(D_k^{(i)}\right)^{\mathrm{T}} D_k^{(i)} x_k^{(i)}(t-\tau(t)) + 2\sum_{h=1}^{l} a_{kh}^{(i)}\left(\left\|x_k^{(i)}\right\|^2 + \left\|x_h^{(i)}\right\|^2\right) \\
=& \left(x_k^{(i)}(t)\right)^{\mathrm{T}}\left(2A_k^{(i)} + \left(C_k^{(i)}\right)^{\mathrm{T}} C_k^{(i)} + 4\sum_{h=1}^{l} a_{kh}^{(i)} I\right) x_k^{(i)}(t) \\
&+ \left(x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}}\left(D_k^{(i)}\right)^{\mathrm{T}} D_k^{(i)} x_k^{(i)}(t-\tau(t)) + 2\sum_{h=1}^{l} a_{kh}^{(i)}\left(\left\|x_h^{(i)}\right\|^2 - \left\|x_k^{(i)}\right\|^2\right) \\
=& -\left(x_k^{(i)}(t)\right)^{\mathrm{T}} \Omega_{11} x_k^{(i)}(t) + \left(x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}} \Omega_{22} x_k^{(i)}(t-\tau(t)) \\
&+ 2\sum_{h=1}^{l} a_{kh}^{(i)}\left(\left\|x_h^{(i)}\right\|^2 - \left\|x_k^{(i)}\right\|^2\right) \\
\leqslant& -\lambda_{1k}^{(i)}\left\|x_k^{(i)}\right\|^2 + \lambda_{2k}^{(i)}\left\|x_k^{(i)}(t-\tau(t))\right\|^2 + 2\sum_{h=1}^{l} a_{kh}^{(i)}\left(\left\|x_h^{(i)}\right\|^2 - \left\|x_k^{(i)}\right\|^2\right),
\end{aligned}$$

where $\Omega_{11} = -2A_k^{(i)} - (C_k^{(i)})^{\mathrm{T}} C_k^{(i)} - 4\sum_{h=1}^{l} a_{kh}^{(i)} I$, $\Omega_{22} = (D_k^{(i)})^{\mathrm{T}} D_k^{(i)}$, $\lambda_{1k}^{(i)} = \lambda_{\min}(\Omega_{11})$, $\lambda_{2k}^{(i)} = \lambda_{\max}(\Omega_{22})$.

So, we can compute

$$\begin{aligned}
\mathcal{L}V(x,t) =& \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)} \mathcal{L} \left\| x_k^{(i)} \right\|^2 \\
\leqslant & -\sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)} \lambda_{1k}^{(i)} \left\| x_k^{(i)} \right\|^2 + \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)} \lambda_{2k}^{(i)} \left\| x_k^{(i)}(t-\tau(t)) \right\|^2 \\
& + 2\sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)} \sum_{h=1}^{l} a_{kh}^{(i)} \left(\left\| x_h^{(i)} \right\|^2 - \left\| x_k^{(i)} \right\|^2 \right) \\
\leqslant & -\lambda_1 V(x,t) + \lambda_2 V(y, t-\tau(t)) + \sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{h=1}^{l} c_k^{(i)} a_{kh}^{(i)} \left(\left\| x_h^{(i)} \right\|^2 - \left\| x_k^{(i)} \right\|^2 \right) \\
= & -\lambda_1 V(x,t) + \lambda_2 V(y, t-\tau(t)) + \sum_{i=1}^{n} \sum_{\mathcal{Q}_i \in \mathbb{Q}} W(\mathcal{Q}_i) \sum_{(k,h)\in E(\mathcal{C}_{\mathcal{Q}_i})} \left(\left\| x_h^{(i)} \right\|^2 - \left\| x_k^{(i)} \right\|^2 \right).
\end{aligned}$$

By condition (i), the weighted digraph $(\mathcal{G}, A_i)$ is balanced. Without loss of generality,

$$E(\mathcal{C}_{\mathcal{Q}_i}) = \{(i_1,i_2),(i_2,i_3),\ldots,(i_{l-1},i_l),(i_l,i_1)\},$$

so we have

$$\begin{aligned}
& \sum_{(k,h)\in E(\mathcal{C}_{\mathcal{Q}_i})}^{l} \left(Q_k^{(i)}\left(x_k^{(i)}\right) - Q_h^{(i)}\left(x_h^{(i)}\right) \right) \\
&= \left(Q_{i_2}^{(i)}\left(x_{i_2}^{(i)}\right) - Q_{i_1}^{(i)}\left(x_{i_1}^{(i)}\right) \right) + \left(Q_{i_3}^{(i)}\left(x_{i_3}^{(i)}\right) - Q_{i_2}^{(i)}\left(x_{i_1}^{(i)}\right) \right) \\
&\quad + \cdots + \left(Q_{i_l}^{(i)}\left(x_{i_l}^{(i)}\right) - Q_{i_{l-1}}^{(i)}\left(x_{i_{l-1}}^{(i)}\right) \right) + \left(Q_{i_1}^{(i)}\left(x_{i_1}^{(i)}\right) - Q_{i_l}^{(i)}\left(x_{i_l}^{(i)}\right) \right) \\
&= 0.
\end{aligned}$$

From the Lemmata 1 and 2, we obtain

$$\mathcal{L}V(x,t) \leqslant -(\lambda_1 - q\lambda_2)V(t,x).$$

By the Lemma 4, the controlled system SDE (6) is mean square exponential stability and the Lyapunvov exponent is less than or equal to $-(\lambda_1 - q\lambda_2)$.

3.2 Delay nonlinear impulsive control

Design a delay nonlinear impulsive controller

$$U\left(t, x_k^{(i)}\right) = \sum_{l=1}^{\infty} R_l\left(x_k^{(i)}(t), x_k^{(i)}(t-\tau(t))\right)\delta(t-t_l),$$

where

$$\delta_a(t) = \delta(t-a) = 0, \quad (t \neq a), \quad \int_{-\infty}^{+\infty} \delta_a(t)\mathrm{d}t = 1,$$

$$x_k^{(i)}(t_l^+) = \tilde{Q}_l\left(x_k^{(i)}(t_l), x_k^{(i)}(t_l - \tau(t_l))\right),$$

so that $U(t, x_k^{(i)})$ is an impulsive controller.

In the following, Eq. (6) can be written the impulsive system

$$\begin{cases} \mathrm{d}x_k^{(i)}(t) = \left[A_k^{(i)}x_k^{(i)}(t) + \sum\limits_{h=1}^{l} H_{kh}^{(i)}\left(x_k^{(i)}(t), x_h^{(i)}(t)\right)\right]\mathrm{d}t + \left[C_k^{(i)}x_k^{(i)}(t) + D_k^{(i)}x_k^{(i)}(t-\tau(t))\right]\mathrm{d}\omega(t), \\ x_k^{(i)}(t_l^+) = \tilde{Q}_l\left(x_k^{(i)}(t_l), x_k^{(i)}(t_l-\tau(t_l))\right), \\ x_k^{(i)}(t) = \psi_k^{(i)}(s), \quad s \in [-\tau, 0], \end{cases} \tag{7}$$

where the impulse times t_l satisfy $0 = t_0 < t_1 < \cdots < t_l < \cdots$, $t_l \to +\infty$, as $l \to \infty$.

Lemma 5. Suppose $x_i \geqslant 0$, $a_i \geqslant 0$, $i = 1, 2, \ldots, n$, and p is a positive integral number. The inequality holds as follows:

$$\left(\sum_{i=1}^{n} a_i x_i\right)^p \leqslant \left(\sum_{i=1}^{n} a_i\right)^{p-1}\left(\sum_{i=1}^{n} a_i x_i^p\right).$$

Lemma 6 ([30]). Let us consider the impulsive differential inequalities as follows:

$$\begin{cases} D^+v(t) \leqslant av(t) + b_1[v(t)]_{\tau_1} + b_2[v(t)]_{\tau_2} + \cdots + b_m[v(t)]_{\tau_m}, \quad t \neq t_l,\ t \geqslant 0, \\ v(t_l) \leqslant p_l v(t_l^-) + q_{1l}[v(t_l^-)]_{\sigma_1} + q_{2l}[v(t_l^-)]_{\sigma_2} + \cdots + q_{rl}[v(t_l^-)]_{\sigma_r}, \quad t = t_l,\ l \in \mathbb{N}_+, \\ v(t) = \phi(t), \quad t \in [t_0 - \tau, t_0], \end{cases} \tag{8}$$

where a, b_i, p_l, q_{il}, t_i are constants and $b_i \geqslant 0$, $p_l \geqslant 0$, $q_{il} \geqslant 0$, $\tau_i \geqslant 0$, $i = 1, 2, \ldots, m$, $v(t) \geqslant 0$, $[v(t)]_{\tau_i} = \sup_{t-\tau_i(t)\leqslant s\leqslant t} v(s)$, $[v(t_l^-)]_{\sigma_i} = \sup_{t_l-\sigma_i(t_l)\leqslant s\leqslant t_l} v(s)$, $\phi(t)$ is continuous on $[t_0 - \tau, t_0]$, and $v(t)$ is continuous except t_l, $l \in \mathbb{N}_+$. The consequence $\{t_l\}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_l < t_{l+1}, \ldots,$ and $\lim_{l\to+\infty} t_l = +\infty$. Suppose

$$\begin{aligned} &p_l + \sum_{j=1}^{r} q_{jl} < 1, \\ &a + \frac{\sum_{i=1}^{m} b_i}{p_l + \sum_{j=1}^{r} q_{jl}} + \frac{\ln(p_l + \sum_{j=1}^{r} q_{jl})}{t_{l+1} - t_l} < 0. \end{aligned}$$

Therefore there are some constants $\tilde{\beta} > 1$ and $\lambda > 0$ such that

$$v(t) \leqslant \|\phi\|_\tau \tilde{\beta} \mathrm{e}^{-\lambda(t-t_0)}, \quad t \geqslant t_0,$$

where $\|\phi\|_\tau = \sup_{t_0-\tau\leqslant s\leqslant t_0} \|\phi(s)\|$, $\tau = \max\{\tau_i,\ \sigma_j,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, r\}$.

The proof of Lemme 6 is given in [30].

Theorem 2. Assume that

(i) There exists a constant $a_{kh}^{(i)}$ such that

$$\sum_{h=1}^{l}\left(x_k^{(i)}\right)^{\mathrm{T}} H_{kh}^{(i)}\left(x_k^{(i)}, x_h^{(i)}\right) \leqslant \sum_{h=1}^{l} a_{kh}^{(i)}\left(\left\|x_k^{(i)}\right\|^2 + \left\|x_h^{(i)}\right\|^2\right);$$

(ii) For every $i \in \mathbb{N}$, let digraph $(\mathcal{G}, A_i)$ be strongly connected and balanced, where matrix $A_i = (a_{kh}^{(i)})_{l\times l}$, and directed cycle $\mathcal{C}_{\mathcal{Q}_i}$ of digraph $(\mathcal{G}, A_i)$, where $a_{kh}^{(i)} \geqslant 0$, $c_k^{(i)} > 0$ $\tilde{\lambda}_1 = \max_{k,i}\{\tilde{\lambda}_{1k}^{(i)}\}$, $\tilde{\lambda}_2 = \max_{k,i}\{\tilde{\lambda}_{2k}^{(i)}\}$;

(iii) There exist nonnegative constants α_s $(s = 0, 1)$ such that

$$\|\tilde{Q}_l\left(v_0, v_1\right)\| \leqslant \alpha_0\|v_0\| + \alpha_1\|v_1\|;$$

(iv) $\beta_0 + \beta_1 < 1$, and $\tilde{\lambda}_1 + \frac{\tilde{\lambda}_2}{\beta_0+\beta_1} + \frac{\ln(\beta_0+\beta_1)}{t_{l+1}-t_l} < 0$, where $\beta_0 = \alpha_0(\alpha_0 + \alpha_1)$, $\beta_1 = \alpha_1(\alpha_0 + \alpha_1)$.

Therefore, the controlled system SDE (7) is mean square exponential stability and the Lyapunvov exponent is not greater than $-\lambda$.

Proof. Let us consider the following Lyapunov function:

$$V(x,t)=\sum_{k=1}^{l}\sum_{i=1}^{n}c_k^{(i)}V_k^{(i)}\left(x_k^{(i)},t\right)=\sum_{k=1}^{l}\sum_{i=1}^{n}c_k^{(i)}\left\|x_k^{(i)}\right\|^2,\quad k\in\mathbb{L},\ i\in\mathbb{N}.$$

The operator $\mathcal{L}V_k^{(i)}$ along the trajectory of impulsive system (7) yields, and by Lemma 3, for $t\neq t_l$,

$$\begin{aligned}
\mathcal{L}V_k^{(i)}\left(x_k^{(i)},t\right)=&2\left(x_k^{(i)}\right)^{\mathrm{T}}\left[A_k^{(i)}x_k^{(i)}(t)+\sum_{h=1}^{l}H_{kh}^{(i)}\left(x_k^{(i)}(t),x_h^{(i)}(t)\right)\right]\\
&+\operatorname{trace}\left[\left(C_k^{(i)}x_k^{(i)}(t)+D_k^{(i)}x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}}\left(C_k^{(i)}x_k^{(i)}(t)+D_k^{(i)}x_k^{(i)}(t-\tau(t))\right)\right]\\
\leqslant&2\left(x_k^{(i)}(t)\right)^{\mathrm{T}}A_k^{(i)}x_k^{(i)}(t)+2\sum_{h=1}^{l}a_{kh}^{(i)}\left(\left\|x_k^{(i)}\right\|^2+\left\|x_h^{(i)}\right\|^2\right)\\
&+\left(\left(C_k^{(i)}x_k^{(i)}(t)+D_k^{(i)}x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}}\left(C_k^{(i)}x_k^{(i)}(t)+D_k^{(i)}x_k^{(i)}(t-\tau(t))\right)\right)\\
\leqslant&\left(x_k^{(i)}(t)\right)^{\mathrm{T}}\left(2A_k^{(i)}+2\left(C_k^{(i)}\right)^{\mathrm{T}}C_k^{(i)}\right)x_k^{(i)}(t)+2\sum_{h=1}^{l}a_{kh}^{(i)}\left(\left\|x_k^{(i)}\right\|^2+\left\|x_h^{(i)}\right\|^2\right)\\
&+\left(x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}}\left(2\left(D_k^{(i)}\right)^{\mathrm{T}}D_k^{(i)}\right)x_k^{(i)}(t-\tau(t))\\
=&\left(x_k^{(i)}(t)\right)^{\mathrm{T}}\left(2A_k^{(i)}+2\left(C_k^{(i)}\right)^{\mathrm{T}}C_k^{(i)}+4\sum_{h=1}^{l}a_{kh}^{(i)}I\right)x_k^{(i)}(t)+2\sum_{h=1}^{l}a_{kh}^{(i)}\left(\left\|x_h^{(i)}\right\|^2-\left\|x_k^{(i)}\right\|^2\right)\\
&+\left(x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}}\left(2\left(D_k^{(i)}\right)^{\mathrm{T}}D_k^{(i)}\right)x_k^{(i)}(t-\tau(t))\\
=&\left(x_k^{(i)}(t)\right)^{\mathrm{T}}\tilde{\Omega}_{11}x_k^{(i)}(t)+\left(x_k^{(i)}(t-\tau(t))\right)^{\mathrm{T}}\tilde{\Omega}_{22}x_k^{(i)}(t-\tau(t))\\
&+2\sum_{h=1}^{l}a_{kh}^{(i)}\left(\left\|x_h^{(i)}\right\|^2-\left\|x_k^{(i)}\right\|^2\right)\\
\leqslant&\tilde{\lambda}_{1k}^{(i)}\left\|x_k^{(i)}\right\|^2+\tilde{\lambda}_{2k}^{(i)}\left\|x_k^{(i)}(t-\tau(t))\right\|^2+2\sum_{h=1}^{l}a_{kh}^{(i)}\left(\left\|x_h^{(i)}\right\|^2-\left\|x_k^{(i)}\right\|^2\right),
\end{aligned}$$

where $\widetilde{\Omega}_{11}=2A_k^{(i)}+2(C_k^{(i)})^{\mathrm{T}}C_k^{(i)}+4\sum_{h=1}^{l}a_{kh}^{(i)}I$, $\widetilde{\Omega}_{22}=2(D_k^{(i)})^{\mathrm{T}}D_k^{(i)}$, $\tilde{\lambda}_{1k}^{(i)}=\lambda_{\max}(\tilde{\Omega}_{11})$, $\tilde{\lambda}_{2k}^{(i)}=\lambda_{\max}(\tilde{\Omega}_{22})$.

So, we can compute

$$\begin{aligned}
\mathcal{L}V(x,t)=&\sum_{k=1}^{l}\sum_{i=1}^{n}c_k^{(i)}\mathcal{L}\left\|x_k^{(i)}\right\|^2\\
\leqslant&-\sum_{k=1}^{l}\sum_{i=1}^{n}c_k^{(i)}\tilde{\lambda}_{1k}^{(i)}\left\|x_k^{(i)}\right\|^2+\sum_{k=1}^{l}\sum_{i=1}^{n}c_k^{(i)}\tilde{\lambda}_{2k}^{(i)}\left\|x_k^{(i)}(t-\tau(t))\right\|^2\\
&+2\sum_{k=1}^{l}\sum_{i=1}^{n}c_k^{(i)}\sum_{h=1}^{l}\sum_{h=1}^{l}\left(Q_k^{(i)}\left(x_k^{(i)}\right)-Q_h^{(i)}\left(x_h^{(i)}\right)\right)\\
\leqslant&\tilde{\lambda}_1V(x,t)+\tilde{\lambda}_2V(y,t-\tau(t))+\sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{h=1}^{l}c_k^{(i)}a_{kh}^{(i)}\left(Q_k^{(i)}\left(x_k^{(i)}\right)-Q_h^{(i)}\left(x_h^{(i)}\right)\right)\\
=&\tilde{\lambda}_1V(x,t)+\tilde{\lambda}_2V(y,t-\tau(t))+\sum_{i=1}^{n}\sum_{\mathcal{Q}_i\in\mathbb{Q}}W(\mathcal{Q}_i)\sum_{(k,h)\in E(\mathcal{C}_{\mathcal{Q}_i})}\left(Q_k^{(i)}\left(x_k^{(i)}\right)-Q_h^{(i)}\left(x_h^{(i)}\right)\right).
\end{aligned}$$

By condition (i), the weighted digraph $(\mathcal{G},A_i)$ is balanced. Without loss of generality,

$$E(\mathcal{C}_{\mathcal{Q}_i})=\{(i_1,i_2),(i_2,i_3),\ldots,(i_{l-1},i_l),(i_l,i_1)\},$$

so we have

$$\begin{aligned}
&\sum_{(k,h)\in E(\mathcal{C}_{\mathcal{Q}_i})}^{l} \left(Q_k^{(i)}\left(x_k^{(i)}\right) - Q_h^{(i)}\left(x_h^{(i)}\right)\right) \\
&= \left(Q_{i_2}^{(i)}\left(x_{i_2}^{(i)}\right) - Q_{i_1}^{(i)}\left(x_{i_1}^{(i)}\right)\right) + \left(Q_{i_3}^{(i)}\left(x_{i_3}^{(i)}\right) - Q_{i_2}^{(i)}\left(x_{i_1}^{(i)}\right)\right) \\
&\quad + \cdots + \left(Q_{i_l}^{(i)}\left(x_{i_l}^{(i)}\right) - Q_{i_{l-1}}^{(i)}\left(x_{i_{l-1}}^{(i)}\right)\right) + \left(Q_{i_1}^{(i)}\left(x_{i_1}^{(i)}\right) - Q_{i_l}^{(i)}\left(x_{i_l}^{(i)}\right)\right) \\
&= 0.
\end{aligned}$$

From Lemmata 1 and 2, we have

$$\mathcal{L}V(x,t) = \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\mathcal{L}\left\|x_k^{(i)}\right\|^2 \leqslant \tilde{\lambda}_1 V(x,t) + \tilde{\lambda}_2 V(y, t-\tau(t)).$$

For $t = t_l$, by Lemma 5, we can obtain

$$\begin{aligned}
V(x(t_l^+), t_l^+) &= \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)} V_k^{(i)}\left(x_k^{(i)}(t_l^+), t_l^+\right) \\
&= \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\left\|x_k^{(i)}(t_l^+)\right\|^2 \\
&= \sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\left\|\tilde{Q}_l\left(x_k^{(i)}(t_l), x_k^{(i)}(t_l-\tau(t_l))\right)\right\|^2 \\
&\leqslant \beta_0\sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\left\|x_k^{(i)}(t_l)\right\|^2 + \beta_1\sum_{k=1}^{l}\sum_{i=1}^{n} c_k^{(i)}\left\|x_k^{(i)}(t_l-\tau(t_l))\right\|^2 \\
&= \beta_0 V(t_l) + \beta_1[V(t_l)]_\tau.
\end{aligned} \tag{9}$$

By Lemma 6, there are some constants $\tilde{\beta} > 1$ and $\lambda > 0$ such that

$$V(x,t) \leqslant \|\phi\|_\tau \tilde{\beta}\mathrm{e}^{-\lambda(t-t_0)}, \quad t \geqslant t_0.$$

The controlled system SDE (7) is mean square exponential stability and the Lyapunvov exponent is not greater than $-\lambda$.

4 An example

At last, an example is given to illustrate the effectiveness of the conclusions in the paper. To simplify calculations, we only consider $w(t)$ as a scalar Brown motion.

$$\begin{cases}
\mathrm{d}x_k^{(i)}(t) = \left(A_k^{(i)}x_k^{(i)}(t) + \sum_{h=1}^{2} H_{kh}^{(i)}\left(x_k^{(i)}(t), x_h^{(i)}(t)\right)\right)\mathrm{d}t + \left(C_k^{(i)}x_k^{(i)}(t) + D_k^{(i)}x_k^{(i)}(t-\tau)\right)\mathrm{d}\omega(t), \\
x_k^{(i)}(t) = \psi_k^{(i)}(s), \quad s \in [-\tau, 0],
\end{cases} \tag{10}$$

where $x_k^{(i)}(t) = (x_k^{(i_1)}, x_k^{(i_2)})^{\mathrm{T}} \in R^2$, $x_k^{(i)}(t-\tau(t)) = (x_k^{(i_1)}(t-\tau(t)),\ x_k^{(i_2)}(t-\tau(t)))^{\mathrm{T}} \in \mathbb{R}^2$, $\tau = 1$, $c_k^{(1)} = 2\times 10^{-6}$, $c_k^{(2)} = 2\times 10^{-6}$, $\mathbb{N} = \{1,2\}$, $i = 1,2$, $\mathbb{L} = \{1,2\}$, $k = 1,2$, $m_i = 2$, $\sum_{h=1}^{2} a_{kh} = 1$.

Let $A_k^{(1)} = A_k^{(2)} = \begin{pmatrix} -8 & -2 \\ 2 & -5 \end{pmatrix}$, $C_k^{(1)} = C_k^{(2)} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}$, $D_k^{(1)} = D_k^{(2)} = \begin{pmatrix} 0.5 & 1 \\ 0 & 0.5 \end{pmatrix}$, $H_{kh}^{(1)} = H_{kh}^{(2)} = \frac{1}{3}a_{kh}^{(i)}(x_k^{(i)} - x_h^{(i)})$.

This stochastic differential equation (SDE) is unstable. Please refer to [29].

Then the delay feedback control system can be depicted as follows:

$$\begin{cases} \mathrm{d}x_k^{(i)}(t) = \left[A_k^{(i)} x_k^{(i)}(t) + \sum_{h=1}^{2} H_{kh}^{(i)}\left(x_k^{(i)}(t), x_h^{(i)}(t)\right) + U_k^{(i)}(t, x_k(t-\tau))\right]\mathrm{d}t \\ \qquad + \left(C_k^{(i)} x_k^{(i)}(t) + D_k^{(i)} x_k^{(i)}(t-\tau)\right)\mathrm{d}\omega(t), \\ x_k^{(i)}(t) = \psi_k^{(i)}(s), \quad s \in [-\tau, 0]. \end{cases} \tag{11}$$

Case I. Delay feedback control. Let $U_k^{(i)}(t, x_k(t-1) = F_k^{(i)} x_k^{(i)}(t-1)$, where $F_k^{(i)} = -(C_k^{(i)})^{\mathrm{T}} D_k^{(i)}$. We can compute $\Omega_{11} = -2A_k^{(i)} - (C_k^{(i)})^{\mathrm{T}} C_k^{(i)} - 4\sum_{h=1}^{2} a_{kh}^{(i)} I = \left(\begin{smallmatrix} 10 & 7 \\ -1 & 1 \end{smallmatrix}\right)$, $\Omega_{22} = (D_k^{(i)})^{\mathrm{T}} D_k^{(i)} = \left(\begin{smallmatrix} 0.25 & 0.5 \\ 0.5 & 1.25 \end{smallmatrix}\right)$, $\lambda_{1k}^{(i)} = \lambda_{\min}(\Omega_{11}) = 1.8599$, $\lambda_{2k}^{(i)} = \lambda_{\max}(\Omega_{22}) = 1.4571$, and

$$\begin{aligned} \sum_{h=1}^{l} \left(x_k^{(i)}\right)^{\mathrm{T}} H_{kh}^{(i)}\left(x_k^{(i)}, x_h^{(i)}\right) &\leqslant \frac{1}{3}\sum_{h=1}^{l} a_{kh}^{(i)} \left(x_k^{(i)}\right)^{\mathrm{T}} \left(x_k^{(i)} - x_h^{(i)}\right) \\ &= \sum_{h=1}^{l} \frac{1}{3} a_{kh}^{(i)} \left(\left\|x_k^{(i)}\right\|^2 - \left(x_k^{(i)}\right)^{\mathrm{T}} x_h^{(i)}\right) \\ &\leqslant \sum_{h=1}^{l} a_{kh}^{(i)} \left(\frac{1}{2}\left\|x_k^{(i)}\right\|^2 + \frac{1}{6}\left\|x_h^{(i)}\right\|^2\right) \\ &\leqslant \sum_{h=1}^{l} a_{kh}^{(i)} \left(\left\|x_k^{(i)}\right\|^2 + \left\|x_h^{(i)}\right\|^2\right). \end{aligned}$$

Let $\lambda_1 = \min_{k,i}\{\lambda_{1k}^{(i)}\}$, $\lambda_2 = \max_{k,i}\{\lambda_{2k}^{(i)}\}$.

So, the conditions of Theorem 1 are satisfied. Then the controlled system (11) is exponential stability in mean square and the Lyapunvov exponent is not greater than $-(\lambda_1 - q\lambda_2)$.

Case II. Delay nonlinear impulsive control. $U(t, x_k^{(i)}) = \sum_{l=1}^{\infty} R_l(x_k^{(i)}(t), x_k^{(i)}(t-\tau))\delta(t-t_l)$, $i = 1, 2$. Then the system (11) can be rewritten as follows:

$$\begin{cases} \mathrm{d}x_k^{(i)}(t) = \left[A_k^{(i)} x_k^{(i)}(t) + \sum_{h=1}^{l} H_{kh}^{(i)}\left(x_k^{(i)}(t), x_h^{(i)}(t)\right)\right]\mathrm{d}t \\ \qquad + \left[C_k^{(i)} x_k^{(i)}(t) + D_k^{(i)} x_k^{(i)}(t-\tau)\right]\mathrm{d}\omega(t), \\ x_k^{(i)}(t_l^+) = \tilde{Q}_l\left(x_k^{(i)}(t_l), x_k^{(i)}(t_l-\tau)\right), \\ x_k^{(i)}(t) = \psi_k^{(i)}(s), \quad s \in [-\tau, 0]. \end{cases} \tag{12}$$

Let $t_{l+1} - t_l = 0.05$, $\beta_0 = 0.1$, $\beta_1 = 0.1$.

We can also compute $\widetilde{\Omega}_{11} = 2A_k^{(i)} + 2(C_k^{(i)})^{\mathrm{T}} C_k^{(i)} + 4\sum_{h=1}^{l} a_{kh}^{(i)} I = \left(\begin{smallmatrix} -8 & -10 \\ -2 & 4 \end{smallmatrix}\right)$, $\widetilde{\Omega}_{22} = 2(D_k^{(i)})^{\mathrm{T}} D_k^{(i)} = \left(\begin{smallmatrix} 0.5 & 1 \\ 1 & 2.5 \end{smallmatrix}\right)$, $\tilde{\lambda}_{1k}^{(i)} = \lambda_{\max}(\tilde{\Omega}_{11}) = 5.4833$, $\tilde{\lambda}_{2k}^{(i)} = \lambda_{\max}(\tilde{\Omega}_{22}) = 2.9142$, $\tilde{\lambda}_1 = \max_{k,i}\{\tilde{\lambda}_{1k}^{(i)}\} = 5.4833$, $\tilde{\lambda}_2 = \max_{k,i}\{\tilde{\lambda}_{2k}^{(i)}\} = 2.9142$.

Let $\beta_0 = 0.1$, $\beta_1 = 0.1$, $\beta_0 + \beta_1 < 1$, it is easy to verify $\tilde{\lambda}_1 + \frac{\tilde{\lambda}_2}{\beta_0+\beta_1} + \frac{\ln(\beta_0+\beta_1)}{t_{l+1}-t_l} < 0$.

So, the conditions of Theorem 2 are satisfied. Then the controlled system (12) is exponential stability in mean square and the Lyapunvov exponent is not greater than $-\lambda$.

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