SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

July 2018, Vol. 61 070211:1–070211:13
https://doi.org/10.1007/s11432-017-9267-0

Special Focus on Modeling, Analysis and Control of Stochastic Systems

Stochastic maximum principle for partially observed forward-backward stochastic differential equations with jumps and regime switching

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Received 25 August 2017/Accepted 28 September 2017/Published online 18 May 2018

Abstract In this article, we consider the partially observed optimal control problem for forward-backward stochastic systems with Markovian regime switching. A stochastic maximum principle for optimal control is developed using a variational method and filtering technique. Our theoretical results are applied to the motivating example of the risk minimization for portfolio selection.

Keywords partial information, Markovian regime-switching, stochastic maximum principle, forward-backward stochastic differential equation (FBSDE)

Citation Zhang S Q, Xiong J, Liu X D. Stochastic maximum principle for partially observed forward-backward stochastic differential equations with jumps and regime switching. Sci China Inf Sci, 2018, 61(7): 070211, https://doi.org/10.1007/s11432-017-9267-0

1 Introduction

This paper considers the partially observed optimal control problem for forward-backward stochastic differential equations (FBSDEs) driven by Lévy processes with Markov regime-switching. One of the motivations of this study was the problem of finding risk-minimizing portfolios in finance, where the risk is represented in terms of a g-expectation. Suppose that an insurer invests his/her surplus in a financial market consisting of a risk-free asset (bond) and a risky asset (stock). Specifically, the price process of the risk-free asset is given by

$$\mathrm{d}S_0(t) = r_0(t)S_0(t)\mathrm{d}t, \quad r_0 > 0,$$

and the price process of the risky assets follows the following stochastic differential equation:

$$\mathrm{d}S_1(t) = \mu(t)S_1(t)\mathrm{d}t + \delta(t)S_1(t)\mathrm{d}W(t),$$

where $\mu(t)(> r_0(t))$ is the appreciation rate process, and $\delta(t)$ is the volatility coefficient, $W(t)$ is a standard Brownian motion with respect to $(\Omega, \mathcal{F}, \mathcal{F}_t, \bar{\mathbb{P}})$. In addition,

$$\mu(t) = \begin{cases} a, \quad \text{for} \quad \text{bull market;} \\ -b, \quad \text{for} \quad \text{bear market.} \end{cases}$$

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Strategy $v(t)$ represents the amount invested in the risky asset at time t. Naturally, it is assumed that $v(t)$ must be adapted to an observation generated sub-filtration $\mathcal{G}_t$. The dynamics of the controlled process $x(t)$ is given by

$$\mathrm{d}x(t) = (c + r_0(t)(x(t) - v(t)))\mathrm{d}t - \mathrm{d}C(t) + \mu(t)v(t)\mathrm{d}t + \delta(t)v(t)\mathrm{d}W(t), \tag{1}$$

where $C(t) = \sum_{i=1}^{N(t)} \zeta_i$ is a compound Poisson process. It represents the cumulative claims up to time t, independent of $W(t)$. $\{N(t)\}$ is a homogeneous Poisson process with intensity $\bar{\lambda}$, and the claim sizes $\{\zeta_i, i \geqslant 1\}$ are independent and identically distributed positive random variables. In addition, we assume that $N(t)$ is independent of the claim sizes $\zeta_i, i \geqslant 1$.

The policymaker can obtain information from the stock price,

$$\begin{cases} \mathrm{d}Y(t) = \dfrac{1}{\delta(t)}\left(\mu(t) - \dfrac{1}{2}\delta^2(t)\right)\mathrm{d}t + \mathrm{d}W(t), \\ Y(0) = 0, \end{cases}$$

where $Y(t) = \frac{1}{\delta(t)} \log S(t)$.

The objective is to find

$$J(v) = -\min \mathcal{E}_g(X(T)),$$

where $\mathcal{E}_g$ is the g-expectation related to g. The associated backward stochastic differential equation (BSDE) for the unknowns $(y(t), z(t), \xi(t,\zeta))$ is given by

$$\begin{cases} -\mathrm{d}y^v(t) = g(t, x^v(t), y^v(t), z^v(t), \xi^v(t), v(t), \alpha(t))\mathrm{d}t \\ \qquad\qquad -\bar{z}^v(t)\mathrm{d}Y(t) - \int_0^\infty \xi^v(t-,\zeta)\tilde{N}(d\zeta, \mathrm{d}t), \\ y^v(T) = \phi(x^v(T)). \end{cases} \tag{2}$$

Under some usual conditions, the forward-backward stochastic differential equation (FBSDE) (1) and (2) admits a unique solution.

The optimal control problem for a Markov regime-switching model has recently received much attention, e.g., see Donnelly [1], Donnelly and Heunis [2], and Zhang et al. [3]. Concerning optimal control problems with partial observation, we refer the readers to Zhang [4], Zhou [5], Tang [6], Wang and Wu [7], and Wang et al. [8] for more information. The work that was most similar to the current paper was that of [9], which presented various versions of the stochastic maximum principle. However, their model was different from ours. First, there was no Markovian switch in their model. More importantly, their information filtration was an abstract sub-filtration, which did not depend on the control itself. On the other hand, our information filtration, generated by the observation process, depends on the control in a coupled manner. Other studies with coupled information included that of Huang et al. [10], which dealt with the BSDE state equation, and those of Wang et al. [11,12], which considered the FBSDE state equation and LQ problem, respectively. Other recent studies include [7, 13–15]. We refer the reader to a recent monograph [16] for a more extensive reference on this subject. None of these studies involved jumps and a Markovian switch. Moreover, Huang and Zhang [17] investigated the near-optimal maximum principle of impulse control for a stochastic recursive system.

The rest of this paper is structured as follows. We formulate the partial information stochastic control problem in Section 2. In Section 3, the necessary condition is established for the optimal control of the FBSDE with Markovian switching under partial information. Section 4 discusses an example of a conditional mean field problem.

2 Model

Let $\left(\Omega, \mathcal{F}, \mathcal{F}_t, \bar{\mathbb{P}}\right)$ be a filtered probability space within which the independent real-valued standard Brownian motions $W(t)$ and $B(t)$ are defined. We consider a continuous-time, finite-state, time-homogeneous

Markov chain $\{\alpha(t) : t \geqslant 0\}$ with a finite state space $E = \{e_1, e_2, \ldots, e_D\}$, where $e_k \in \mathbb{R}^D$ and the lth component of e_k is the Kronecker delta δ_{kl}, for each $k = 1, 2, \ldots, D$. The generator of the Markov chain is $\Lambda = (\lambda_{kl})_{k,l=1,2,\ldots,D}$. It is assumed that $\lambda_{kl} \geqslant 0$ for $k \neq l$ and $\Sigma_{l=1}^{D} \lambda_{kl} = 0$, so $\lambda_{kk} \leqslant 0$. In what follows for each $k = 1, 2, \ldots, D$, we suppose that $\lambda_{kk} < 0$.

It follows from [18] that α admits the following semimartingale representation:

$$\alpha(t) = \alpha(0) + \int_0^t \Lambda^* \alpha(s) \mathrm{d}s + M(t),$$

where $\{M(t) | t \in [0, T]\}$ is an $\mathbb{R}^D$-valued $\{\mathcal{F}_t, \bar{\mathbb{P}}\}$ martingale. For each $i, j = 1, 2, \ldots, D$, with $i \neq j$, and $t \in [0, T]$, let $J_{ij}(t)$ be the number of jumps from state e_i to state e_j up to time t. Then,

$$J^{ij}(t) = \lambda_{ij} \int_0^t \langle \alpha(s), e_i \rangle \mathrm{d}s + m_{ij}(t),$$

where $m_{ij} := \{m_{ij}(t) | t \in T\}$, with $m_{ij}(t) := \int_0^t \langle \alpha(s-), e_i \rangle \mathrm{d} \langle M(s), e_j \rangle$, is an $\{\mathcal{F}_t, \bar{\mathbb{P}}\}$-martingale. Now, for each fixed $j = 1, 2, \ldots, D$, let $\Phi_j(t)$ be the number of jumps into state e_j up to time t. Then,

$$\begin{aligned} \Phi_j(t) &= \sum_{i=1}^{D} J^{ij}(t) = \sum_{i=1}^{D} \lambda_{ij} \int_0^t \langle \alpha(s-, e_i) \rangle \mathrm{d}s + \sum_{i=1}^{D} m_{ij}(t) \\ &\equiv \lambda_j(t) + \widetilde{\Phi}_j(t), \end{aligned}$$

where $\widetilde{\Phi}_j(t)$ is an $\{\mathcal{F}_t, \bar{\mathbb{P}}\}$-martingale.

Let $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ be a Polish space with the σ-finite measure ν. Suppose that $N(\mathrm{d}\zeta, \mathrm{d}t)$ is a Poisson random measures on $(\mathbb{R}^+ \times \mathcal{E}, \mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(\mathcal{E}))$ under $\bar{\mathbb{P}}$. Then, the compensated Poisson random measure is given by

$$\widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) = N(\mathrm{d}\zeta, \mathrm{d}t) - \nu(\mathrm{d}\zeta)\mathrm{d}t.$$

Consider the following forward-backward stochastic differential equations with partial information (FBSDEP):

$$\begin{cases} \mathrm{d}x^v(t) = & b(t, x^v(t), v(t), \alpha(t))\mathrm{d}t + \sigma(t, x^v(t), v(t), \alpha(t))\mathrm{d}B(t) \\ & + \delta(t, x^v(t), v(t), \alpha(t))\mathrm{d}W(t) \\ & + \int_{\mathcal{E}} \eta(t, x^v(t-), v(t-), \zeta, \alpha(t-))\widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) \\ & + \gamma(t, x^u(t-), v(t-), \alpha(t-)) \cdot \mathrm{d}\widetilde{\Phi}(t), \\ -\mathrm{d}y^v(t) = & g(t, x^v(t), y^v(t), z^v(t), \bar{z}^v(t), V^v(t), \xi^v(t), v(t), \alpha(t))\mathrm{d}t \\ & - z^v(t)\mathrm{d}B(t) - \bar{z}^v(t)\mathrm{d}Y(t) \\ & - \int_{\mathcal{E}} \xi^v(t, \zeta)\widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) - V^v(t-) \cdot \mathrm{d}\widetilde{\Phi}(t), \\ x^v(0) = & x_0, \quad y^v(T) = \phi(x^v(T), \alpha(T)), \quad 0 \leqslant t \leqslant T, \end{cases} \tag{3}$$

where $v(\cdot)$ is a control process taking values in a convex set $U \subseteq \mathbb{R}$; B and W are independent Brownian motions that are independent of $\widetilde{N}$ and $\widetilde{\Phi}$; $b : [0, T] \times \mathbb{R} \times U \times S \to \mathbb{R}$; $c : [0, T] \times \mathbb{R} \times U \times S \to \mathbb{R}$; $\sigma : [0, T] \times \mathbb{R} \times U \times S \to \mathbb{R}$; $\eta : [0, T] \times \mathbb{R} \times U \times S \to \mathbb{R}$; and $\gamma : [0, T] \times \mathbb{R} \times U \times S \to \mathbb{R}^D$.

Suppose that the observation equation is governed by

$$\begin{cases} \mathrm{d}Y(t) = & h(t, x^v(t))\mathrm{d}t + \mathrm{d}W(t), \\ Y(0) = & 0. \end{cases} \tag{4}$$

Inserting (4) into (3), we have

$$\begin{cases} \mathrm{d}x^v(t) = & \tilde{b}(t, x^v(t), v(t), \alpha(t))\mathrm{d}t + \sigma(t, x^v(t), v(t), \alpha(t))\mathrm{d}B(t) \\ & + \delta(t, x^v(t), v(t), \alpha(t))\mathrm{d}Y(t) \\ & + \int_{\mathcal{E}} \eta(t, x^v(t-), v(t-), \zeta, \alpha(t-))\widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) \\ & + \gamma(t, x^u(t-), v(t-), \alpha(t-)) \cdot \mathrm{d}\widetilde{\Phi}(t), \\ -\mathrm{d}y^v(t) = & g(t, x^v(t), y^v(t), z^v(t), \bar{z}^v(t), V^v(t), \xi^v(t), v(t), \alpha(t))\mathrm{d}t \\ & - z^v(t)\mathrm{d}B(t) - \bar{z}^v(t)\mathrm{d}Y(t) \\ & - \int_{\mathcal{E}} \xi^v(t, \zeta)\widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) - V^v(t-) \cdot \mathrm{d}\widetilde{\Phi}(t), \\ x^v(0) = & x_0, \quad y^v(T) = \phi(x^v(T)), \quad 0 \leqslant t \leqslant T, \end{cases} \tag{5}$$

where $\tilde{b} = b - \delta h$.

Consider the following cost functional:

$$J(v(\cdot)) = \bar{\mathrm{E}}\left[\int_0^T l\left(t, x^v(t), y^v(t), z^v(t), \bar{z}^v(t), \alpha(t), V^v(s), v(s)\right)\mathrm{d}t + \varphi(x^v(T)) + \psi(y^v(0))\right]. \tag{6}$$

Here, $\bar{\mathrm{E}}$ denotes the expectation with respect to the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \bar{\mathbb{P}})$.

Let

$$\mathcal{G}_t = \sigma\left(Y(s): \ 0 \leqslant s \leqslant t\right),$$

and let U be a non-empty convex subset of $\mathbb{R}$. Denote the set of admissible controls by

$$\mathcal{U}_{ad} = \left\{ v \ \middle| \ v(t) \text{ is } \mathcal{G}_t\text{-adapted } U\text{-valued process} \quad \text{s.t. } \bar{\mathrm{E}}\int_0^T v(t)^8 \mathrm{d}t < \infty \right\}.$$

The optimal control problem is to find the control v to minimize the cost functional (6) over $v(\cdot) \in \mathcal{U}_{ad}$, i.e.,

$$J(u) = \min_{v(\cdot)\in\mathcal{U}_{ad}} J(v(\cdot)).$$

For a function $g: \ L^2(\nu) \to \mathbb{R}$, we defined its Fréchet derivative at $\xi \in L^2(\nu)$ as $\nabla_\xi g \in L^2(\nu)$ such that

$$\langle \nabla_\xi g, \xi_1 \rangle_{L^2(\nu)} = \lim_{\epsilon\to 0} \epsilon^{-1}\left(g(\xi + \epsilon\xi_1) - g(\xi)\right), \qquad \forall \xi_1 \in L^2(\nu).$$

To characterize the optimal control, we need the following.

Hypothesis 1. The functions b, σ, δ, γ, ξ, ϕ, g, and h are continuously differentiable with respect to (x, v), (x, y, z, ξ, V, v), and x. They are bounded by $k(1 + |x| + |v|)$. In addition,

$$\int_{\mathcal{E}} \eta^2(t, x, v, \zeta, \alpha)\nu(\mathrm{d}\zeta) \leqslant k(1 + |x|^2 + |v|^2).$$

The derivatives b_x, b_v, σ_x, σ_v, δ_x, δ_v, η_x, η_v, γ_x, γ_v, ϕ_x, g_x, g_y, g_z, g_V, $\nabla_\xi g$ (Fréchet derivative), g_v, and h_x are uniformly bounded and Lipschitz continuous. Moreover, there is a constant C such that $|h(t, x)| \leqslant C$ for any $(t, x) \in ([0, 1] \times \mathbb{R})$.

Let $\mathbb{P}$ be a probability measure defined as $\mathrm{d}\bar{\mathbb{P}} = Z^v(t)\mathrm{d}\mathbb{P}$, where

$$\begin{cases} Z^v(t) = \exp\left\{\int_0^t h(s, x^v(s))\mathrm{d}Y(s) - \frac{1}{2}\int_0^t |h(s, x^v(s))|^2 \mathrm{d}s\right\}, \\ Z^v(0) = 1, \end{cases}$$

i.e.,

$$\begin{cases} \mathrm{d}Z^v(t) = Z^v(t)h(t, x^v(t))\mathrm{d}Y(t), \\ Z^v(0) = 1. \end{cases} \tag{7}$$

Then, the cost functional (6) can be rewritten as

$$\begin{aligned} J(v(\cdot)) = \mathrm{E}\bigg[\int_0^T & Z^v(t)\ l\left(t, x^v(t), y^v(t), z^v(t), \bar{z}^v(t), V^v(s), v(s)\right) \mathrm{d}t \\ & + Z^v(T)\ \varphi(x^v(T)) + \psi(y^v(0))\bigg]. \end{aligned} \tag{8}$$

Thus, the original optimization problem is equivalent to minimizing the cost functional (8) over $v(\cdot) \in \mathcal{U}_{ad}$ subject to (5) and (7).

3 Stochastic maximum principle

In this section, we will derive the stochastic maximum principle for the cost function (8) subject to (5) and (7).

For any $\epsilon \in (0,1)$ and $v+u \in \mathcal{U}_{ad}$, by convexity, $u+\varepsilon v \in \mathcal{U}_{ad}$. Let $(x^{u+\varepsilon v}(\cdot), y^{u+\varepsilon v}(\cdot), z^{u+\varepsilon v}(\cdot), \xi^{u+\varepsilon v}(\cdot,\cdot), V^{u+\varepsilon v}(\cdot))$ be the solutions of (5) and (7) with v replaced by $u+\varepsilon v$. From the Burkholder-Davis-Gundy inequality and Gronwall's inequality, we have the following estimates (Lemmas 1 and 2). Because the proofs are routine, we omit them.

Lemma 1. Let Hypothesis 1 hold

$$\sup_{0\leqslant t\leqslant T} \mathrm{E}|x^v(t)|^8 \leqslant C\left(1 + \mathrm{E}\int_0^T |v(t)|^8 \mathrm{d}t\right),$$

$$\sup_{0\leqslant t\leqslant T} \mathrm{E}|y^v(t)|^2 \leqslant C\left(1 + \mathrm{E}\int_0^T |v(t)|^2 \mathrm{d}t\right),$$

$$\begin{aligned} \mathrm{E}\bigg(\int_0^T |z^v(t)|^2 \mathrm{d}t + \int_0^T |\bar{z}^v(t)|^2 \mathrm{d}t + \int_0^T \int_{\mathcal{E}} |\xi^v(t,\zeta)|^2 \nu(\mathrm{d}\zeta) \mathrm{d}t \\ + \int_0^T |V^v(t)|^2 \mathrm{d}t \bigg) \leqslant C\left(1 + \mathrm{E}\int_0^T |v(t)|^2 \mathrm{d}t\right), \end{aligned}$$

$$\sup_{0\leqslant t\leqslant T} \mathrm{E}|Z^v(t)|^2 < \infty.$$

Lemma 2. Let Hypothesis 1 hold. Then, there exists a constant C such that

$$\sup_{0\leqslant t\leqslant T} \mathrm{E}|x^{u+\varepsilon v}(t) - x^u(t)|^8 \leqslant C\varepsilon^8,$$

$$\sup_{0\leqslant t\leqslant T} \mathrm{E}|y^{u+\varepsilon v}(t) - y^u(t)|^2 \leqslant C\varepsilon^2,$$

$$\mathrm{E}\int_0^T |z^{u+\varepsilon v}(t) - z^u(t)|^2 \mathrm{d}t \leqslant C\varepsilon^2,$$

$$\mathrm{E}\int_0^T |\bar{z}^{u+\varepsilon v}(t) - \bar{z}^u(t)|^2 \mathrm{d}t \leqslant C\varepsilon^2,$$

$$\mathrm{E}\int_0^T \int_{\mathcal{E}} |\xi^{u+\varepsilon v}(t,\zeta) - \xi^u(t,\zeta)|^2 \nu(\mathrm{d}\zeta) \mathrm{d}t \leqslant C\varepsilon^2,$$

$$\mathrm{E}\int_0^T |V^{u+\epsilon v}(t) - V^u(t)|^2 \mathrm{d}t \leqslant C\varepsilon^2,$$

and

$$\sup_{0\leqslant t\leqslant T} \mathrm{E}\int_0^T |Z^{u+\varepsilon v}(t) - Z^u(t)|^2 \mathrm{d}t \leqslant C\varepsilon^2.$$

For simplicity of notation, we write

$$h(t) = h(t, x(t)) \quad \text{and} \quad h_x(t) = h_x(t, x(t)).$$

With obvious modification, we can introduce notations with h replaced by $\tilde{b}$, σ, δ, η, γ, and g.

We introduce the following variational equations:

$$\begin{cases} \mathrm{d}Z^1(t) = \Big[Z^1(t)h(t) + Z(t)h_x(t)x^1(t)\Big]\mathrm{d}Y(t), \\ Z^1(0) = \ 0, \end{cases} \tag{9}$$

and

$$\begin{cases} \mathrm{d}x^1(t) = \big\{\tilde{b}_x(t)x^1(t) + \tilde{b}_v(t)v(t)\big\}\mathrm{d}t + [\sigma_x(t)x^1(t) + \sigma_v(t)v(t)]\mathrm{d}B(t) \\ \qquad\qquad + [\delta_x(t)x^1(t) + \delta_v(t)v(t)]\mathrm{d}Y(t) \\ \qquad\qquad + \int_{\mathcal{E}} \big[\eta_x(t-,\zeta)x^1(t-) + \eta_v(t-)v(t-)\big]\widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) \\ \qquad\qquad + \big[\gamma_x(t-)x^1(t-) + \gamma_v(t-)v(t-)\big] \cdot \mathrm{d}\widetilde{\Phi}(t), \\ -\mathrm{d}y^1(t) = \Big[g_x(t)x^1(t) + g_y(t)y^1(t) + g_z(t)z^1(t) + g_{\bar{z}}(t)\bar{z}^1(t) \\ \qquad\qquad + \sum_{j=1}^{D} g_{V_j}(t)V^{1j}(t)\lambda_j + \int_{\mathcal{E}} \big\langle \nabla_\xi g(t,\zeta), \xi^1\big\rangle_{L^2(\nu)} \nu(\mathrm{d}\zeta) \\ \qquad\qquad + g_v(t)v(t)\Big]\mathrm{d}t - z^1(t)\mathrm{d}B(t) - \bar{z}^1(t)\mathrm{d}Y(t) \\ \qquad\qquad - \int_{\mathcal{E}} \xi^1(t,\zeta)\widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) - V^1(t) \cdot \mathrm{d}\widetilde{\Phi}(t), \\ x^1(0) = \ 0, \quad y^1(T) = \phi_x(x(T), \alpha(T))x^1(T), \end{cases} \tag{10}$$

For any $v(\cdot) + u(\cdot) \in \mathcal{U}_{ad}$, it is easy to see that (10) and (9) admit unique solutions under Hypothesis 1. Furthermore, we have the following lemma.

Lemma 3. If Hypothesis 1 holds, then

$$\mathrm{E}|x^1(t)|^8 < \infty, \quad \mathrm{E}|Z^1(t)|^4 < \infty. \tag{11}$$

Lemma 4. If Hypothesis 1 holds and

$$\tilde{\vartheta}^\epsilon(t) = \frac{\vartheta^{u+\epsilon v}(t) - \vartheta^u(t)}{\epsilon} - \vartheta^1(t) \quad \text{with} \quad \vartheta = x, y, z, \bar{z}, V, \xi, Z,$$

then,

$$\lim_{\epsilon\to 0} \sup_{0\leqslant t\leqslant T} \mathrm{E}|\tilde{x}^\epsilon(t)|^2 = 0, \tag{12}$$

$$\lim_{\epsilon\to 0} \sup_{0\leqslant t\leqslant T} \mathrm{E}|\tilde{Z}^\epsilon(t)|^2 = 0, \tag{13}$$

and

$$\begin{aligned} &\lim_{\epsilon\to 0} \sup_{0\leqslant t\leqslant T} \mathrm{E}|\tilde{y}^\epsilon(t)|^2 = 0, \\ &\lim_{\epsilon\to 0} \mathrm{E}\int_0^T |\tilde{z}^\epsilon(t)|^2\mathrm{d}t = 0, \quad \lim_{\epsilon\to 0} \mathrm{E}\int_0^T |\tilde{\bar{z}}^\epsilon(t)|^2\mathrm{d}t = 0, \\ &\lim_{\epsilon\to 0} \mathrm{E}\int_0^T \int_{\mathcal{E}} |\tilde{\xi}^\epsilon(t,\zeta)|^2\nu(\mathrm{d}\zeta)\mathrm{d}t = 0, \\ &\lim_{\epsilon\to 0} \mathrm{E}\sum_{j=1}^{D}\int_0^T |\tilde{V}^{j,\epsilon}(t)|^2\lambda_j\mathrm{d}t = 0. \end{aligned} \tag{14}$$

Proof. For simplicity of notation, we denote

$$b_x^\epsilon(t) = \int_0^1 b_x(t, x(t) + \kappa\epsilon(x^1(t) + \tilde{x}^\epsilon(t)), u(t) + \kappa\epsilon v(t), \alpha(t))\mathrm{d}\kappa.$$

The notation is modified in an obvious manner when b is replaced by other mappings.

It follows from (3) and (10) that $\tilde{x}^\epsilon(0) = 0$ and

$$\begin{aligned}\mathrm{d}\tilde{x}^\epsilon(t) = &\left[\tilde{x}^\epsilon(t)\tilde{b}_x^\epsilon(t) + (\tilde{b}_x^\epsilon(t) - \tilde{b}_x(t))x^1(t) + (\tilde{b}_v^\epsilon(t) - \tilde{b}_v(t))v(t)\right]\mathrm{d}t\\ &+ \left[\tilde{x}^\epsilon(t)\sigma_x^\epsilon(t) + (\sigma_x^\epsilon(t) - \sigma_x(t))x^1(t) + (\sigma_v^\epsilon(t) - \sigma_v(t))v(t)\right]\mathrm{d}B(t)\\ &+ \left[\tilde{x}^\epsilon(t)\delta_x^\epsilon(t) + (\delta_x^\epsilon(t) - \delta_x(t))x^1(t) + (\delta_v^\epsilon(t) - \delta_v(t))v(t)\right]\mathrm{d}Y(t)\\ &+ \int_{\mathcal{E}}\left[\tilde{x}^\epsilon(t)\eta_x^\epsilon(t,\zeta) + (\eta_x^\epsilon(t,\zeta) - \eta_x(t,\zeta))x^1(t)\right.\\ &\left.+ (\eta_v^\epsilon(t,\zeta) - \eta_v(t,\zeta))v(t)\right]\widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t)\\ &+ \left[\tilde{x}^\epsilon(t)\gamma_x^\epsilon(t) + (\gamma_x^\epsilon(t) - \gamma_x(t))x^1(t) + (\gamma_v^\epsilon(t) - \gamma_v(t))v(t)\right]\mathrm{d}\widetilde{\Phi}(t).\end{aligned}$$

Based on Hypothesis 1, the Hölder inequality, the Burkholder-Davis-Gundy inequality, and an elementary inequality, we obtain

$$\begin{aligned}f(s) \equiv\ & \mathrm{E}\sup_{s\leqslant t}|\tilde{x}^\epsilon(s)|^4\\ \leqslant\ & C\mathrm{E}\int_0^t |\tilde{x}^\epsilon(t)|^4\mathrm{d}s\\ &+C\left(\mathrm{E}\int_0^t |x^1(s)||\tilde{b}_x^\epsilon(s) - \tilde{b}_x(s)|\mathrm{d}s\right)^4\\ &+C\left(\mathrm{E}\int_0^t |v(s)||\tilde{b}_v^\epsilon(s) - \tilde{b}_v(s)|\mathrm{d}s\right)^4\\ &+C\mathrm{E}\left(\int_0^t |x^1(s)|^2|\sigma_x^\epsilon(s) - \sigma_x(s)|^2\mathrm{d}s\right)^2\\ &+C\mathrm{E}\left(\int_0^t |v(s)|^2|\sigma_v^\epsilon(s) - \sigma_v(s)|^2\mathrm{d}s\right)^2 + \cdots,\end{aligned}$$

where the omitted terms are similar to the last two terms. Applying the Lipschitz continuity of b_x, b_v, σ_x, σ_v, η_x, η_v, γ_x,γ_v, and Lemma 3, we arrive at

$$f(t) \leqslant C_1\int_0^t f(s)\mathrm{d}s + \epsilon^4 C_2.$$

Then, Gronwall's inequality implies

$$f(t) \leqslant \epsilon^4 C_2 \mathrm{e}^{C_1 t} \leqslant C_3\epsilon^4.$$

Eq. (13) follows from the same arguments used above. Finally, we prove (14). Let

$$\begin{aligned}g_x^\epsilon =& \int_0^1 g_x(x + \kappa\epsilon(x^1 + \tilde{x}^\epsilon), y + \kappa\epsilon(y^1 + \tilde{y}^\epsilon), z + \kappa\epsilon(z^1 + \tilde{z}^\epsilon),\\ &\bar{z} + \kappa\epsilon(\bar{z}^1 + \tilde{\bar{z}}^\epsilon), V_j + \kappa\epsilon(V^{1j} + \tilde{V}^{\epsilon j}), \xi + \kappa\epsilon(\xi^1 + \tilde{\xi}^\epsilon), u + \kappa\epsilon v)\mathrm{d}\kappa.\end{aligned}$$

The other notations used below are defined similarly. Then,

$$\begin{cases} -\mathrm{d}\tilde{y}^{\epsilon}(t) = \big[g_x^{\epsilon}(t)\tilde{x}^{\epsilon}(t) + g_y^{\epsilon}(t)\tilde{y}^{\epsilon}(t) + g_z^{\epsilon}(t)\tilde{z}^{\epsilon}(t) + g_{\tilde{z}}^{\epsilon}(t)\tilde{\tilde{z}}^{\epsilon}(t) \\ \qquad + \sum_{j=1}^{D} g_{V_j}^{\epsilon}(t)\tilde{V}^{\epsilon j}(t)\lambda_j + \int_{\mathcal{E}} \nabla_{\xi} g^{\epsilon}(t)\nabla_{\xi} g(t)\tilde{\xi}^{\epsilon}\nu(\mathrm{d}\zeta) \\ \qquad + g_v^{\epsilon}(t) + b_0^{\epsilon}(t)\big]\mathrm{d}t - \tilde{z}^{\epsilon}(t)\mathrm{d}B(t) - \tilde{\tilde{z}}^{\epsilon}(t)\mathrm{d}Y(t) \\ \qquad - \int_{\mathcal{E}} \tilde{\xi}^{\epsilon}(t,\zeta)\widetilde{N}(\mathrm{d}\zeta\mathrm{d}t) - \tilde{V}^{\epsilon}(t)\cdot \mathrm{d}\widetilde{\Phi}(t), \\ \tilde{y}^{\epsilon}(T) = \ \epsilon^{-1}[\phi(x_{\epsilon}(T),\alpha(T)) - \phi(x(T),\alpha(T))] \\ \qquad - \phi_x(x(T),\alpha(T))x^1(T,\alpha(T)), \end{cases}$$

where

$$\begin{aligned} b_0^{\epsilon} &= [g_x^{\epsilon} - g_x]x^1 + [g_y^{\epsilon} - g_y]y^1 + [g_z^{\epsilon} - g_z]z^1 + [g_{\tilde{z}}^{\epsilon} - g_{\tilde{z}}]\tilde{z}^1 \\ &\quad + \sum_{j=1}^{D}[g_{V_j}^{\epsilon} - g_{V_j}]V^{1j}\lambda_j + \int_{\mathcal{E}}[\nabla_{\xi} g^{\epsilon} - \nabla_{\xi} g]\xi^1\nu(\mathrm{d}\zeta). \end{aligned}$$

Applying the Itô formula to $|\tilde{y}^{\epsilon}(t)|^2$, noting the assumption, we have

$$\begin{aligned} &\mathrm{E}|\tilde{y}^{\epsilon}(t)|^2 + \mathrm{E}\int_t^1 |\tilde{z}^{\epsilon}(s)|^2\mathrm{d}s + \mathrm{E}\int_t^1 |\tilde{\tilde{z}}^{\epsilon}(s)|^2\mathrm{d}s + \mathrm{E}\int_t^1\int_{\mathcal{E}} |\tilde{\xi}^{\epsilon}(s,\zeta)|^2\nu(\mathrm{d}\zeta)\mathrm{d}s \\ &\quad + \mathrm{E}\sum_{j=1}^{D}\int_t^1 |\tilde{V}^{\epsilon j}(t)|^2\lambda_j\mathrm{d}s \\ &= \ \mathrm{E}\int_t^T \Big[2\tilde{y}^{\epsilon}(s)\big(b_1^{\epsilon}(t)\tilde{x}^{\epsilon}(t) + b_2^{\epsilon}(t)\tilde{y}^{\epsilon}(t) + b_3^{\epsilon}(t)\tilde{z}^{\epsilon}(t) + b_4^{\epsilon}(t)\tilde{\tilde{z}}^{\epsilon}(t) \\ &\quad + \sum_{j=1}^{D} b_5^{\epsilon}(t)\tilde{V}^{\epsilon j}(t)\lambda_j + \int_{\mathcal{E}} b_6^{\epsilon}(t)\nabla_{\xi} g(t)\tilde{\xi}^{\epsilon}\nu(d\zeta) + b_7^{\epsilon}(t)\big)\Big]\mathrm{d}s \\ &\quad + \mathrm{E}\Big\{\epsilon^{-1}[\phi(x^{\epsilon}(T),\alpha(T)) - \phi(x(T),\alpha(T))] - \phi_x(x(T),\alpha(T))x^1(T)\Big\}^2 \\ &\leqslant \mathrm{E}\int_t^1 K\left(|\tilde{x}^{\epsilon}(s)|^2 + |\tilde{y}^{\epsilon}(s)|^2\right)\mathrm{d}s + \frac{1}{2}\mathrm{E}\int_t^1 |\tilde{z}^{\epsilon}(s)|^2\mathrm{d}s + \frac{1}{2}\mathrm{E}\int_t^1 |\tilde{\tilde{z}}^{\epsilon}(s)|^2\mathrm{d}s \\ &\quad + \frac{1}{2}\mathrm{E}\int_t^1\sum_{j=1}^{D}|\tilde{V}^{\epsilon j}(t)|^2\lambda_j\mathrm{d}s + \frac{1}{2}\int_t^1\int_{\mathcal{E}}|\tilde{\xi}^{\epsilon}(s,\zeta)|^2\nu(\mathrm{d}\zeta) + o(1). \end{aligned}$$

Combining with the first assertion, we see that

$$\begin{aligned} &\mathrm{E}|\tilde{y}^{\epsilon}(t)|^2 + \frac{1}{2}\mathrm{E}\int_t^1 |\tilde{z}^{\epsilon}(s)|^2\mathrm{d}s + \frac{1}{2}\mathrm{E}\int_t^1 |\tilde{\tilde{z}}^{\epsilon}(s)|^2\mathrm{d}s \\ &\quad + \mathrm{E}\int_t^1\int_{\mathcal{E}}|\tilde{\xi}^{\epsilon}(s,\zeta)|^2\nu(\mathrm{d}\zeta)\mathrm{d}s) + \frac{1}{2}\mathrm{E}\int_t^1\sum_{j=1}^{D}|\tilde{V}^{\epsilon j}(t)|^2\lambda\mathrm{d}s \\ &\leqslant \mathrm{E}\int_t^T K|\tilde{y}^{\epsilon}(s)|^2\mathrm{d}s + o(1). \end{aligned} \tag{15}$$

Dropping the last four terms on the left hand side of (15), by Gronwall's inequality, we get the second statement of the theorem. Finally, we get the last four convergence results of (14) by dropping the first term on the left hand side of (15).

Hypothesis 2. Now, we suppose that for any t, τ, such that $t+\tau \in [0,T]$, and bounded random variable ς, we can formulate the control process $v(s) \in U$, with

$$v(s) = \varsigma I_{[t,t+\tau]}(s), \quad s \in [0,T],$$

where $I_{[t,t+\tau]}(s)$ is the indicator function. For any $v(s) \in \mathcal{G}_s$, with $v(s)$ bounded and $u + v \in \mathcal{U}_{ad}$, we have such that $u(\cdot) + \epsilon v(\cdot) \in \mathcal{U}_{ad}$ for $\epsilon \in (0, 1)$.

Now, we define

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} J(\epsilon)\bigg|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{J(u(\cdot) + \epsilon v(\cdot)) - J(u(\cdot))}{\epsilon}.$$

It follows from Lemmas 2 and 4 that

$$\begin{aligned}
0 =& \frac{\mathrm{d}}{\mathrm{d}\epsilon} J(\epsilon)\bigg|_{\epsilon=0} \\
=& \lim_{\epsilon \to 0} \frac{J(u(\cdot) + \epsilon v(\cdot)) - J(u(\cdot))}{\epsilon} \\
=& \mathrm{E} \int_0^T \Big\{ Z^1(t) l(t) + Z(t) \left[l_x(t) x^1(t) + l_y(t) y^1(t) + l_z(t) z^1(t) \right. \\
& \left. + l_{\bar{z}}(t) \bar{z}^1(t) + \sum_{j=1}^{D} l_{V_j} V^{1j}(t) \lambda_j + l_v(t) v(t) \right] \Big\} \mathrm{d}t + \mathrm{E}[Z(T) \varphi_x(x(T)) x^1(T)] \\
& + \psi_y(y(0)) y^1(0) + \mathrm{E}[Z^1(T) \varphi(x(T))].
\end{aligned}$$

We define the Hamiltonian $\mathcal{H}(t, x, y, z, \bar{z}, \xi, v, p, q, r, \bar{r}, \theta, s)$ by

$$\begin{aligned}
& \mathcal{H}(t, x, y, z, \bar{z}, V, v, p, q, r, \bar{r}, \theta, s) \\
& \doteq \tilde{b}(x, v) q + \sigma(x, v) r + \delta(x, v) \bar{r} + g(t, x, y, z, \bar{z}, V, v) p \\
& \quad + Z(t) l(t, x, y, z, V, v) + \int_{\mathcal{E}} \eta(x, v, \alpha) \theta(\zeta) \nu(\mathrm{d}\zeta) + \sum_{j=1}^{D} \gamma^j(t, x, \alpha, v) s^j \lambda_j,
\end{aligned} \tag{16}$$

and the adjoint equation as follows:

$$\begin{cases}
\mathrm{d}p(t) = [g_y(t) p(t) + Z(t) l_y(t)] \, \mathrm{d}t + [g_z(t) p(t) + Z(t) l_z(t)] \, \mathrm{d}B(t) \\
\qquad\qquad + [g_{\bar{z}}(t) p(t) + Z(t) l_{\bar{z}}(t)] \, \mathrm{d}Y(t) + \displaystyle\int_{\mathcal{E}} \nabla_\xi g(t, \zeta) p(t) \widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) \\
\qquad\qquad + \displaystyle\sum_{j=1}^{D} \left(g_{V_j}(t) p(t) + Z(t) l_{V_j}(t) \right) \mathrm{d}\widetilde{\Phi}_j(t), \\
-\mathrm{d}q(t) = \Big[\tilde{b}_x(t) q(t) + \sigma_x(t) r(t) + \delta_x(t) \bar{r} + g_x(t) p(t) + Z(t) l_x(t) \\
\qquad\qquad + \beta(t) Z(t) h_x(t) + \displaystyle\int_{\mathcal{E}} \eta_x(t, e) \theta(t, \zeta) \nu(\mathrm{d}\zeta) \\
\qquad\qquad + \displaystyle\sum_{j=1}^{D} \gamma_x^j(t, x, e_i, u) s^j \lambda_j \Big] \mathrm{d}t - r(t) \mathrm{d}B(t) - \bar{r}(t) \mathrm{d}Y(t) \\
\qquad\qquad - \displaystyle\int_{\mathcal{E}} \theta(t-, \zeta) \widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) - \sum_{j=1}^{D} s^j(t) \mathrm{d}\widetilde{\Phi}_j(t), \\
p(0) = \psi_y(y(0)), \quad q(T) = \phi_x(x(T)) p(T) + Z(T) \varphi_x(x(T)).
\end{cases} \tag{17}$$

To deal with the additional term $Z^1(\cdot)$, we introduce an auxiliary BSDE:

$$\begin{cases}
-\mathrm{d}P(t) = (\beta(t) h(t, x) + l(t, x(t), y(t), z(t), V(t), v(t)) \mathrm{d}t - \beta(t) \mathrm{d}Y(t), \\
P(T) = \varphi(x(T)).
\end{cases} \tag{18}$$

A unique solution to (18) exists.

It is easy to show that the forward-backward stochastic differential equation (17) has a unique solution.

Applying Itô's formula,

$$\begin{aligned}\mathrm{E}[x^1(T)q(T)] &= \mathrm{E}[\phi_x(x(T))p(T)x^1(T) + Z(T)\varphi_x(x(T))x^1(T)]\\ &= -\mathrm{E}\int_0^1 x^1(t)(Z(t)l_x(t) + p(t)g_x(t) + \beta(z)Z(t)h_x(t))\mathrm{d}t\\ &\quad +\mathrm{E}\int_0^1 \Big[\tilde{b}_v(t)q(t) + \sigma_v(t)r(t) + \delta_v(t)\bar{r}(t)\\ &\quad + \int_{\mathcal{E}} \eta_v(t)\theta(t)\nu(\mathrm{d}\zeta) + \sum_{j=1}^{D} \gamma_v^j(t)s^j\lambda_j\Big]v(t)\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\mathrm{E}[y^1(0)p(0)] &= \mathrm{E}[\phi_x(x(T))x^1(T)p(T)] - \mathrm{E}\int_0^T \big[Z(t)\big(l_y(t)y^1(t)\\ &\quad +l_z(t)z^1(t) + l_{\bar{z}}(t)\bar{z}^1(t) + \sum_{j=1}^{D} l_{V_j}(t)V^{1j}(t)\lambda_j\big)\big]\mathrm{d}t\\ &\quad +\mathrm{E}\int_0^T p(t)\Big[g_x(t)x^1(t) + g_v(t)v(t)\Big]\mathrm{d}t,\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}[Z^1(T)P(T)] &= \mathrm{E}[Z^1(T)\varphi(x(T))]\\ &= -\mathrm{E}\int_0^T Z^1(t)l(t)\mathrm{d}t + \mathrm{E}\int_0^T Z(t)\beta(t)h_x(t)x^1(t)\mathrm{d}t.\end{aligned}$$

Combining the above three equalities,

$$\begin{aligned}&\mathrm{E}[Z(T)\varphi_x(x(T))x^1(T)] + \psi_y(y(0))y^1(0) + \mathrm{E}[Z^1(T)\varphi(x(T))]\\ &= \mathrm{E}\int_0^T \mathcal{H}_v(t)v(t)\mathrm{d}t - \mathrm{E}\int_0^T \Big\{Z^1(t)l(t) + Z(T)\big(l_x(t)x^1(t) + l_y(t)y^1(t)\\ &\quad +l_z(t)z^1(t) + l_{\bar{z}}(t)\bar{z}^1(t) + \sum_{j=1}^{D} l_{V_j}V^{1j}(t)\lambda_j + l_v(t)v(t)\big)\Big\}\mathrm{d}t.\end{aligned}$$

Therefore, we have

$$\mathrm{E}\left(\mathcal{H}_v v(t)|\mathcal{G}_t\right) = 0.$$

Thus, we have the following theorem.

Theorem 1. Assume that the Hypotheses 1 and 2 hold. Let $v(\cdot)$ be an optimal control and $(x, y, z, \bar{z}, \xi, V, v)$ be the corresponding solution of FBSDEP. Then, the maximum principle

$$\mathrm{E}\left[\mathcal{H}_v(t, x, y, z, \bar{z}, V, v, p, q, r, \bar{r}, \theta, s)\Big|\mathcal{G}_t\right] = 0$$

holds.

The following remark will be useful in the application presented in the next section. The observation functions h are linear and hence not bounded. Based on a careful check of the proofs in this section, we note that the boundedness of h is only needed to derive a certain integrability of Z and Z^1.

Remark 1. The conclusion remains true when the h bounded in Hypothesis 1 is replaced by

$$\mathrm{E}\left(|Z(t)|^2 + |Z^1(t)|^4\right) < \infty.$$

Remark 2. If $h = 0$, then $Z = 1$. Therefore, in this special case,

$$\begin{aligned}&\mathcal{H}(t,x,y,z,\bar{z},V,v,p,q,r,\bar{r},\theta,s)\\&\doteq b(x,v)q+\sigma(x,v)r+\delta(x,v)\bar{r}+g(t,x,y,z,\bar{z},V,v)p\\&\quad+l(t,x,y,z,V,v)+\int_{\mathcal{E}}\eta(x,v,\alpha)\theta(\zeta)\nu(\mathrm{d}\zeta)+\sum_{j=1}^{D}\gamma^j(t,x,\alpha,v)s^j\lambda_j,\end{aligned}$$

and the adjoint equation is as follows:

$$\begin{cases}\mathrm{d}p(t)=[g_y(t)p(t)+l_y(t)]\,\mathrm{d}t+[g_z(t)p(t)+l_z(t)]\,\mathrm{d}B(t)\\ \qquad\qquad+[g_{\bar{z}}(t)p(t)+l_{\bar{z}}(t)]\,\mathrm{d}Y(t)+\displaystyle\int_{\mathcal{E}}\nabla_{\xi}g(t,\zeta)p(t)\widetilde{N}(\mathrm{d}\zeta,\mathrm{d}t)\\ \qquad\qquad+\displaystyle\sum_{j=1}^{D}\left(g_{V_j}(t)p(t)+l_{V_j}(t)\right)\mathrm{d}\widetilde{\Phi}_j(t),\\ -\mathrm{d}q(t)=\Big[b_x(t)q(t)+\sigma_x(t)r(t)+\delta_x(t)\bar{r}+g_x(t)p(t)+l_x(t)\\ \qquad\qquad+\displaystyle\int_{\mathcal{E}}\eta_x(t,e)\theta(t,\zeta)\nu(\mathrm{d}\zeta)\\ \qquad\qquad+\displaystyle\sum_{j=1}^{D}\gamma_x^j(t,x,e_i,u)s^j\lambda_j\Big]\mathrm{d}t-r(t)\mathrm{d}B(t)-\bar{r}(t)\mathrm{d}Y(t)\\ \qquad\qquad-\displaystyle\int_{\mathcal{E}}\theta(t-,\zeta)\widetilde{N}(\mathrm{d}\zeta,\mathrm{d}t)-\sum_{j=1}^{D}s^j(t)\mathrm{d}\widetilde{\Phi}_j(t),\\ p(0)=\psi_y(y(0)),\quad q(T)=\phi_x(x(T))p(T)+\varphi_x(x(T)).\end{cases}$$

Theorem 2. Assume that Hypothesis 2 holds. Let $v(\cdot)$ be an optimal control and $(x,y,z,\bar{z},\xi,V,v)$ be the corresponding solutions of the FBSDEs. Then, the maximum principle

$$\bar{\mathrm{E}}\left[\mathcal{H}_v(t,x,y,z,\bar{z},V,v,p,q,r,\bar{r},\theta,s)\Big|\mathcal{G}_t\right]=0 \tag{19}$$

is true.

4 Example

In this section, we solve the problem proposed in Section 1. First, $x(t)$ can be rewritten as the following form:

$$\begin{aligned}\mathrm{d}x(t)&=(c+r_0(t)(x(t)-v(t)))\mathrm{d}t-\mathrm{d}C(t)+\mu(t)v(t)\mathrm{d}t+\delta(t)v(t)\mathrm{d}W(t)\\&\quad+1\cdot\mathrm{d}\widetilde{\Phi}(t)\\&=(c+r_0(t)(x(t)-v(t)))\mathrm{d}t-\int_0^{\infty}\zeta(t)N(\mathrm{d}\zeta,\mathrm{d}t)+\mu(t)v(t)\mathrm{d}t\\&\quad+\delta(t)v(t)\mathrm{d}W(t)+1\cdot\mathrm{d}\widetilde{\Phi}(t)\\&=(r_0(t)\left(x(t)-v(t)\right)+\bar{c})\,\mathrm{d}t+\delta(t)v(t)\mathrm{d}W(t)+\mu(t)v(t)\mathrm{d}t\\&\quad-\int_0^{\infty}\zeta(t)\widetilde{N}(\mathrm{d}\zeta,\mathrm{d}t)+1\cdot\mathrm{d}\widetilde{\Phi}(t),\end{aligned}$$

where $\bar{c}=c-\bar{\lambda}\bar{\mathrm{E}}\zeta$. Similar to Theorem 3.1 in [5], using the separation principle,

$$\begin{cases}\mathrm{d}x(t)=(r_0(t)(x(t)-v(t))+\bar{c})\mathrm{d}t-\displaystyle\int_0^{\infty}\zeta(t)\widetilde{N}(\mathrm{d}\zeta,\mathrm{d}t)+\hat{\mu}(t)v(t)\mathrm{d}t\\ \qquad\qquad+\delta(t)v(t)\mathrm{d}\nu(t)+1\cdot\mathrm{d}\widetilde{\Phi}(t),\\ x(0)=x_0,\end{cases}$$

where $\hat{\mu}(t) = \mathrm{E}[\mu(t)|\mathcal{G}_t]$, and the innovation process is as follows:

$$\mathrm{d}\nu(t) = \frac{1}{\delta(t)}\mathrm{d}\log S(t) - \frac{1}{\delta(t)}\left(\hat{\mu}(t) - \frac{1}{2}\delta^2(t)\right)\mathrm{d}t,$$

and

$$\begin{cases} \mathrm{d}Y(t) = \dfrac{1}{\delta(t)}\left(\mu(t) - \dfrac{1}{2}\delta^2(t)\right)\mathrm{d}t + \mathrm{d}W(t), \\ Y(0) = 0, \end{cases}$$

where $Y(t) = \frac{1}{\delta(t)}\log S(t)$.

Let

$$\begin{cases} -\mathrm{d}y(t) = \left(g_1(t)y(t) + g_2(t)v(t)\right)\mathrm{d}t - \bar{z}(t)\mathrm{d}Y(t) \\ \qquad\qquad - \displaystyle\int_0^{\infty}\xi(t)\widetilde{N}(\mathrm{d}\zeta,\mathrm{d}t) - 1\cdot\mathrm{d}\widetilde{\Phi}(t), \\ y(T) = x^v(T), \end{cases}$$

i.e.,

$$\begin{cases} -\mathrm{d}y(t) = \left(g_1(t)y(t) + g_2(t)v(t) - \dfrac{1}{\delta}\left(\hat{\mu} - \dfrac{1}{2}\delta^2(t)\right)\bar{z}(t)\right)\mathrm{d}t - \bar{z}(t)\mathrm{d}\nu(t) \\ \qquad\qquad - \displaystyle\int_0^{\infty}\xi(t)\widetilde{N}(\mathrm{d}\zeta,\mathrm{d}t) - 1\cdot\mathrm{d}\widetilde{\Phi}(t), \\ y(T) = x^v(T), \end{cases}$$

where $g_1(t)$ and $g_2(t)$ are bounded. The objective is to minimize

$$J(v(\cdot)) = \frac{1}{2}\mathrm{E}\left[\int_0^T (v(t) - a(t))^2\mathrm{d}t + x^2(T) - 2y^v(0)\right].$$

Proposition 1. The optimal control is given by

$$v(t) = \left(r_0(t) - \hat{\mu}(t)\right)\hat{q}(t) - \delta(t)\hat{\bar{r}} - g_2(t)\hat{p}(t) + a(t).$$

Proof. In this setting, from (16) and (17), the Hamiltonian is

$$\begin{aligned} &\mathcal{H}(t,x,y,z,\bar{z},V,v,p,q,r,\bar{r},\theta,s) \\ &\quad = \left(r_0(t)\left(x(t) - v(t)\right) + \hat{\mu}(t)v(t) + \bar{c}\right)q(t) + \delta(t)v(t)\bar{r} \\ &\qquad + \left(g_1(t)y(t) + g_2(t)v(t) - \frac{1}{\delta}\left(\hat{\mu} - \frac{1}{2}\delta^2(t)\right)\bar{z}(t)\right)p(t) + \frac{1}{2}\left(v(t) - a(t)\right)^2 \\ &\qquad + \int_{\mathcal{E}}\zeta(t)\theta(\zeta)\nu(\mathrm{d}\zeta) + \sum_{j=1}^{2}s^j\lambda_j. \end{aligned}$$

and the adjoint equations are given by

$$\begin{cases} \mathrm{d}p(t) = g_1(t)p(t)\mathrm{d}t - \dfrac{1}{\delta}\left(\hat{\mu}(t) - \dfrac{1}{2}\delta^2(t)\right)p(t)\mathrm{d}\nu, \\ -\mathrm{d}q(t) = r_0(t)q(t)\mathrm{d}t - \bar{r}(t)\mathrm{d}\nu(t) \\ \qquad\qquad - \displaystyle\int_{\mathcal{E}}\theta(t-,\zeta)\widetilde{N}(\mathrm{d}\zeta,\mathrm{d}t) - \sum_{j=1}^{D}s^j(t)\mathrm{d}\widetilde{\Phi}_j(t), \\ p(0) = -1, \quad q(T) = p(T) + x(T). \end{cases}$$

Applying the stochastic maximum principle (19), we have

$$v(t) = \left(r_0(t) - \hat{\mu}(t)\right)\hat{q}(t) - \delta(t)\hat{\bar{r}} - g_2(t)\hat{p}(t) + a(t),$$

where $\hat{p}(t)$, $\hat{q}(t)$, and $\hat{\hat{r}}$ are the unique solutions of

$$\begin{cases} \mathrm{d}\hat{p}(t) = g_1(t)\hat{p}(t)\mathrm{d}t - \dfrac{1}{\delta}\left(\hat{\mu}(t) - \dfrac{1}{2}\delta^2(t)\right)\hat{p}(t)\mathrm{d}\nu, \\ -\mathrm{d}\hat{q}(t) = r_0(t)\hat{q}(t)\mathrm{d}t - \hat{\hat{r}}(t)\mathrm{d}\nu(t), \\ \hat{p}(0) = -1, \quad \hat{q}(T) = \hat{p}(T) + \hat{x}(T). \end{cases}$$

$$\begin{cases} \mathrm{d}\hat{x}(t) = & (r_0(t)(\hat{x}(t) - (r_0(t) - \hat{\mu}(t))\,\hat{q}(t) - \delta(t)\hat{\hat{r}} - g_2(t)\hat{p}(t) + a(t)) + \bar{c})\mathrm{d}t \\ & - \displaystyle\int_0^\infty \zeta(t)\widetilde{N}(\mathrm{d}\zeta, \mathrm{d}t) + \hat{\mu}(t)v(t)\mathrm{d}t + \delta(t)v(t)\mathrm{d}\nu(t) + 1\cdot\mathrm{d}\widetilde{\Phi}(t), \\ \hat{x}(0) = & x_0, \end{cases}$$

The existence of a solution to the above system of equations is a kind of conditional mean field problem. The result in [19] can be extended to the conditional case. Under the condition that all the coefficients are Lipschitz continuous, a solution exists.

Acknowledgements ZHANG's research was supported in part by National Natural Science Foundation of China (Grant Nos. 11501129, 71571053, 71771058) and Natural Science Foundation of Hebei Province (Grant No. A2014202202). XIONG's research was supported by Macao Science and Technology Fund FDCT (Grant No. FDCT025/2016/A1).

References

1 Donnelly C. Sufficient stochastic maximum principle in a regime-switching diffusion model. Appl Math Optim, 2011, 64: 155–169
2 Donnelly C, Heunis A J. Quadratic risk minimization in a regime-switching model with portfolio constraints. SIAM J Control Optim, 2012, 50: 2431–2461
3 Zhang X, Elliott R J, Siu T K. A stochastic maximum principle for a markov regime-switching jump-diffusion model and its application to finance. SIAM J Control Optim, 2012, 50: 964–990
4 Zhang Q. Controlled partially observed diffusions with correlated noise. Appl Math Optim, 1990, 22: 265–285
5 Xiong J, Zhou X Y. Mean-variance portfolio selection under partial information. SIAM J Control Optim, 2007, 46: 156–175
6 Tang S. The maximum principle for partially observed optimal control of stochastic differential equations. SIAM J Control Optim, 1998, 36: 1596–1617
7 Wang G C, Wu Z. The maximum principles for stochastic recursive optimal control problems under partial information. IEEE Trans Automat Contr, 2009, 54: 1230–1242
8 Wang G C, Zhang C, Zhang W. Stochastic maximum principle for mean-field type optimal control under partial information. IEEE Trans Automat Contr, 2014, 59: 522–528
9 Øksendal B, Sulem A. Maximum principles for optimal control of forward-backward stochastic differential equations with jumps. SIAM J Control Optim, 2009, 48: 2945–2976
10 Huang J, Wang G, Xiong J. A Maximum principle for partial information backward stochastic control problems with applications. SIAM J Control Optim, 2009, 48: 2106–2117
11 Wang G C, Wu Z, Xiong J. Maximum principles for forward-backward stochastic control systems with correlated state and observation noises. SIAM J Control Optim, 2013, 51: 491–524
12 Wang G C, Wu Z, Xiong J. A linear-quadratic optimal control problem of forward-backward stochastic differential equations with partial information. IEEE Trans Automat Contr, 2015, 60: 2904–2916
13 Wang G C, Wu Z. Kalman-Bucy filtering equations of forward and backward stochastic systems and applications to recursive optimal control problems. J Math Anal Appl, 2008, 342: 1280–1296
14 Wang G C, Wu Z. General maximum principles for partially observed risk-sensitive optimal control problems and applications to finance. J Optim Theor Appl, 2009, 141: 677–700
15 Wu Z. A maximum principle for partially observed optimal control of forward-backward stochastic control systems. Sci China Inf Sci, 2010, 53: 2205–2214
16 Wang G C, Wu Z, Xiong J. An introduction to optimal control of FBSDE with incomplete information. In: SpringerBriefs in Mathematics. Berlin: Springer, 2018
17 Huang J, Zhang D. The near-optimal maximum principle of impulse control for stochastic recursive system. Sci China Inf Sci, 2016, 59: 112206
18 Elliott R J, Aggoun L, Moore J B. Hidden Markov Models: Estimation and Control. New York: Springer, 1994. 176–182
19 Buckdahn R, Li J, Peng S. Mean-field backward stochastic differential equations and related partial differential equations. Stochastic Process Appl, 2009, 119: 3133–3154