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Special Focus on Learning and Real-time Optimization of Automotive Powertrain Systems

Constrained control of free piston engine generator based on implicit reference governor

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Abstract The free piston engine generator (FPEG) is a novel power plant concept for series hybrid electric vehicles (HEV) that requires reliable control to regulate piston motion and guarantee safe operation during load transitions. This paper focuses on the control and constraint enforcement in a FPEG using a reference governor. A discrete, implicit, control oriented model describing the piston motion in a two-stroke two-cylinder FPEG at the turnaround point is derived based on energy balance and a feedback controller is designed to track the desired turnaround position by regulating fuel. An implicit reference governor is developed to guarantee safe piston motion by managing the load transitions. The reference governor utilizes Newton's method applied to an implicit nonlinear model for response prediction and a bisection search algorithm to enforce the constraints for all the future time instants by adjusting the reference command. Additionally, the error in applying one iteration of Newton's method in predicting the response of the implicit nonlinear system is estimated and accounted for in constraint tightening to guarantee that constraints are robustly enforced. The simulation results show that the feedback control scheme incorporating the developed implicit reference governor can effectively enforce the prescribed constraints during load transition.

Keywords free piston engine generator, implicit system, reference governor, Newton's method, error bound

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1 Introduction

The increasingly stricter regulations have promoted the development of novel technologies for high efficiency vehicles. A free piston engine (FPE) is a new kind of an internal combustion engine without crank shaft which is expected to become an alternative auxiliary power unit for the next generation of series hybrid vehicles [1]. Rather than using a crank shaft to transfer power to a load, a free piston engine relies on an alternative mechanism such as electric or hydraulic power takeoff [1]. Compared with a conventional internal combustion engine, the FPE which has no crank shaft, provides some unique advantages. For instance, it has less moving parts which translates to lower friction. More interestingly, the piston motion is no longer constrained so that the displacement can vary from stroke to stroke. In such a case, the compression ratio can be optimized very quickly for a variety of combustion strategies or fuel types leading to a significant improvement in thermal efficiency. However, the added flexibility

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also brings challenges: the mechanical constraint for the piston motion disappears without the crank shaft. Excessive fuel may cause the piston collision with cylinder head, and insufficient fuel results in combustion instabilities or misfire.

In recent years, the literature on the topic of FPE and its control has grown. Early on, Mikalsen and Roskilly [2, 3] explored engine dynamics and piston motion control using simulations of a single-cylinder, diesel free piston engine generator (FPEG) with an air bounce chamber for piston rebounding. Using a similar structure to Mikalsen and Roskilly, Toyota developed a prototype, multi-fuel FPEG and a corresponding control system. The Toyota researchers exploited both piston position and velocity for feedback and used the electrical load coefficient as the control actuator to maintain robust operation under unexpected combustion disturbances such as misfire [4, 5]. In [6], a model-based controller for an opposed-piston, opposed-cylinder, hydraulic FPE was designed where the controller acts as a virtual crankshaft that guides the piston to follow a reference trajectory. Lin et al. [7] proposed an optimal control strategy for a four-stroke free piston engine using top dead center (TDC) and bottom dead center (BDC) piston position tracking as control objectives. Start control [8] and air path control [9] of FPE have also been investigated. Most of the previous studies have focused on piston motion tracking but a few of them have also addressed constraint enforcement. Notably, due to the 'free' nature of FPE, the operational safety constraints should be considered during load transitions. Motivated by the need to develop control solutions that enforce constraints, Gong et al. [10] presented a model predictive control (MPC) design for the piston motion control and constraints enforcement. As an alternative, Zaseck et al. [11] employed a state-feedback controller augmented by a robust reference governor to enforce system constraints in a four-stroke, hydraulic free piston engine during a transient load change. Then, a similar idea was applied to a free piston engine used for electricity generation [12].

This paper is distinguished from [11, 12] by exploiting an implicit reference governor for FPEG control and constraint enforcement which accounts for the errors in predicting the system response based on the implicit model. We start by deriving a control-oriented model for a two cylinder two-stroke FPEG based on the energy balance in a discrete time implicit form, and proceed by designing a feedback controller to regulate fuel to track the desired turnaround position. To ensure the safe operation during load transitions in hybrid electric vehicles (HEV) configuration, an implicit reference governor is developed to manage load and to guarantee that the piston motion remains within safe boundaries during required load changes. As the model is given in implicit form, nonlinear equations need to be solved in order to predict system response over the future horizon. In this paper, we treat the problem of constraint enforcement by the reference governor wherein we account for the error arising from the application of Newton's method to the response prediction. Specifically, the reference governor with one Newton's iteration is designed to enforce the tightened constraints based on the error bound. Our simulation results demonstrate that the proposed reference governor is able to effectively and robustly enforce constraints on the piston motion.

The outline of the paper is as follows. In Section 2, the piston motion control problem for FPEG is formulated and a corresponding control oriented model is derived. Then, a control scheme for position set-point tracking and constraints enforcement for FPEG is introduced by Section 3. Section 4 gives the details of the reference governor development based on models in implicit form as motivated by our FPEG application. In Section 5, the error bound on the response prediction is analyzed and a one step iterative reference governor design that exploits this error bound is proposed. Section 6 reports simulation results, and Section 7 summarizes the primary findings of this paper.

Notation. We use standard notations: $\mathbb{Z}^+$ is the set of nonnegative integers; $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote sets of real vectors and matrices. In this paper, we often assume that a set is compact, convex, and contains 0 in its interior; we refer to such sets as sets satisfying the Minkowski assumptions. For two sets $A, B \subset \mathbb{R}^n$, the Minkowski sum is defined as $A \bigoplus B = \{s \in \mathbb{R}^n : s = a + b,\ a \in A,\ b \in B\}$, and the Pontryagin difference is defined as $A \sim B = \{d \in \mathbb{R}^n : d + b \in A,\ \forall b \in B\}$. For a vector $x \in \mathbb{R}^n$, we define its infinity norm as $\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, and for a matrix $X \in \mathbb{R}^{m\times n}$, we define the norm $\|x\|_\infty = \max_{1\leqslant i\leqslant m} \sum_{j=1}^{n} |x_{ij}|$. For a continuously differentiable function $F(x) \in \mathbb{R}^n$, we define the Jacobian of $F(x)$ as $\nabla_J F(x) \in \mathbb{R}^{n\times n}$.

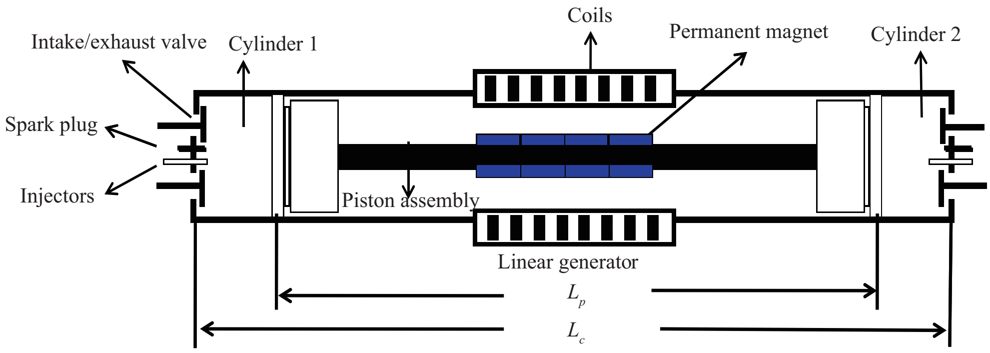

Figure 1 (Color online) Schematics of the proposed FPEG.

2 Problem statement and control oriented model

The FPEG considered in this paper is a dual-piston, two stroke, direct injection gasoline engine integrated with a linear generator as shown in Figure 1. The linear generator is composed of a permanent magnet embedded in the piston assembly and coils arranged in the stator. Combustion occurs sequentially in each cylinder, forcing the piston assembly to move back and forth to produce electricity. See [10] for the discussion of physics-based modeling and dynamics of the FPEG.

2.1 Problem statement and control requirement

Rather than using a crankshaft to transfer power to a load, the FPEG supports electrical power takeoff. As the FPEG does not posses a crank to constrain piston movement, the stroke length is determined by the balance of energy entering the system through combustion and energy leaving the system through either losses or useful work. Excessive fuel may cause piston collision with the cylinder head and damage the engine. Insufficient fuel may result in misfire, undesirable compression ratio or even engine stall. Motion fluctuation must also be constrained during load transitions in HEV configuration.

In this paper, the piston motion control objective is to track a desired clearance height reference x_r at TDC while enforcing clearance constraints during load transitions. As a simplification for the piston motion, we consider piston stroke as a discrete event, where fuel injection occurs only at the beginning of the stroke and injected fuel volume remains fixed for the duration of the stroke. Figure 2 depicts the sampling of state for a dual-piston assembly over an entire cycle. In such a case, the clearance height x_k at the turnaround points is chosen as the state. The injection fuel mass of each cycle is selected as a control input $u_{1,k}$. Assuming the FPEG is integrated into a hybrid vehicle configuration, the electrical power demand $u_{2,k}$ is another adjustable input realized through regulating pulse-width modulation (PWM) duty cycle in the DC-DC converter which should follow the desired demand $u_{2,r}$ [4, 10]. Note that using the turnaround position as the state variable decouples the discrete model from time and enables the model to capture any FPEG operating frequency, because sample period automatically changes with the engine speed.

2.2 Control oriented model based on energy balance

For the considered FPEG, the control-oriented model is built based on energy balance which requires that the change in kinetic energy between two subsequent turnaround points in one cycle is necessarily zero [10, 13]. Consequently, the net work from cycle to cycle can be expressed as

$$f(x_{k+1}, x_k, x_{k-1}, u_{1,k}, u_{2,k}) = W_{\text{comp}} + W_{\text{exp}} + W_{\text{exh}} + W_{\text{int}} + W_{\text{mag}} + W_d = 0, \tag{1}$$

where W_{exp} is the expansion work, W_{comp} is the compression work, W_{mag} is the magnetic work output, W_{int} is the intake work, W_{exh} is the exhaust work, and the term W_d represents the equivalent work losses due to real system losses and modeling uncertainty.

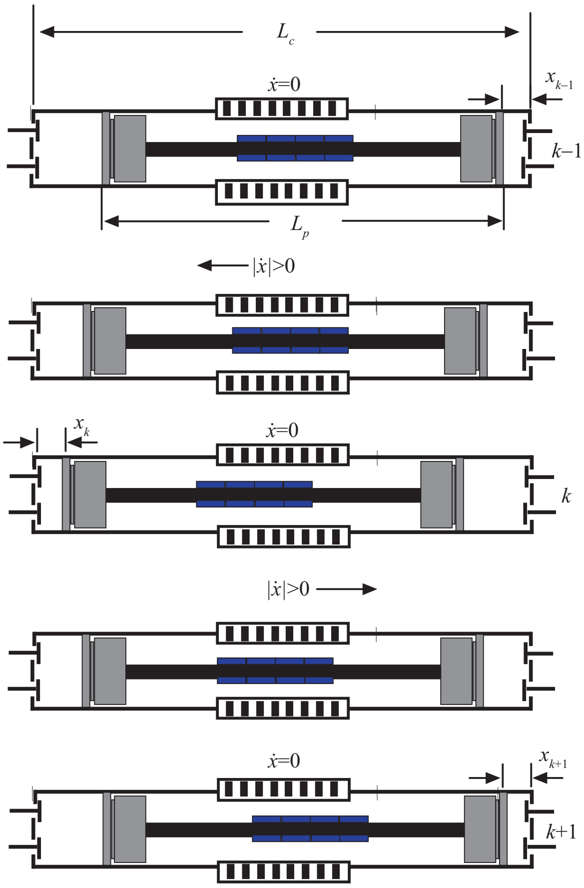

Figure 2 (Color online) Piston position during different phases of the motion including at turn around time instants $k-1$, k, and $k+1$.

As the FPEG is a type of gasoline engines, we assume the compression/expansion processes are considered as isentropic and the volume remains constant during the combustion process. Thus each work term can be expressed as follows:

$$W_{\rm comp} = \frac{P_0 A_p (L - x_{\rm sca} - x_k)}{1-\gamma}\left[\left(\frac{L - x_{\rm sca} - x_k}{x_{k+1}}\right)^{\gamma-1} - 1\right], \tag{2}$$

$$W_{\rm exp} = \left[u_{1,k} H_u + \frac{P_0 A_p}{\gamma - 1}\frac{(L - x_{\rm sca} - x_{k-1})^{\gamma}}{{x_k}^{\gamma-1}}\right]\eta_k, \tag{3}$$

$$W_{\rm mag} = -\int_k^{k+1} u_{2,k} K \dot{x} {\rm d}x = -\int_k^{k+1} u_{2,k} K \dot{x}^2 {\rm d}t, \tag{4}$$

$$W_{\rm int} = -W_{\rm exh}, \tag{5}$$

$$W_d = d_k, \tag{6}$$

where P_0 denotes the ambient pressure, A_p is the cross-sectional area of the piston, γ denotes the ratio of specific heats for air, H_u is the fuel lower heating value, $x_{\rm sca}$ is the scavenging length (piston movement length during scavenging), L is the effective stroke length, and work losses d_k are calibrated based on simulation or experimental data. The variable η_k is similar to the engine cycle efficiency and can be expressed as

$$\eta_k = \left[1 - \left(\frac{x_k}{L - x_{\rm sca} - x_{k+1}}\right)^{\gamma-1}\right]. \tag{7}$$

Detailed derivations of (2) and (3) can be found in Appendix A.2.

Table 1 FPE model parameters

Symbol	Value (Unit)
H_u	4.35×10^7 (J/Kg)
P_0	1.35×10^5 (Pa)
γ	1.37
L	0.0187 (m)
x_{sca}	0.002 (m)
a_m	5821.2
b_m	−29.6
A_p	0.001809 (m^2)

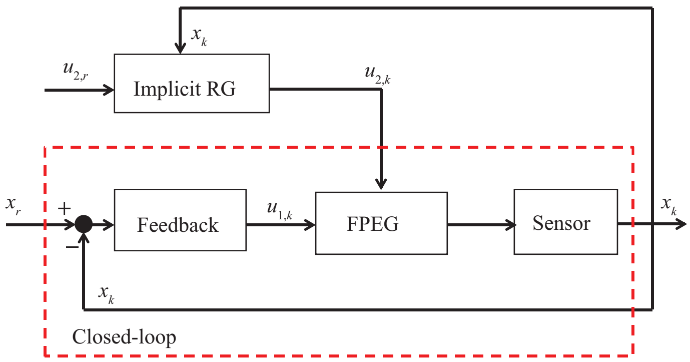

Figure 3 (Color online) Control architecture of FPEG motion control system.

In two stroke engines, the gas exchange process, that includes scavenging, is difficult to model and characterize. As a simplification, we assume that the intake and exhaust processes both occur at the same cylinder pressure and with the same working displacements. As a result, work done during intake and work done during exhaust are equal in magnitude and result in (5). The modeling uncertainty in the scavenging process can be reflected in the term W_d.

The magnetic work (4) is defined in continuous-time. In the discrete-time model, this magnetic work term is considered as a function of the clearance height x_k and power demand $u_{2,k}$, and approximated by the following polynomial:

$$\bar{W}_{\text{mag}} = f_{\text{map}}(x_k, u_{2,k}) \approx u_{2,k}(a_m x_k + b_m), \tag{8}$$

where parameters a_m, b_m are the fitting parameters obtained from experiments or simulations.

The parameters of the control-oriented model of the considered FPEG are provided in Table 1.

3 Overview of FPEG motion control system

This section introduces the control architecture for constrained FPEG motion control. The objectives of FPEG motion control are described as follows:

(1) Design a feedback controller for regulating fuel injection $u_{1,k}$ so that the free piston assembly tracks a specified clearance set-point, i.e., $x_k \to x_r$;

(2) Design a reference governor to manage the load/demand factor $u_{2,k}$, ensuring that $u_{2,k} \to u_{2,r}$, and so that the safety constraint on clearance height is enforced during HEV load transitions.

The proposed control architecture is shown in Figure 3. Note that we treat the state x_k as directly measured or accurately estimated. The state estimation is addressed in [10, 13].

3.1 Feedback control for setpoint tracking

We first design a state feedback control for fuel input $u_{1,k}$. The stability of system (1) and the feedback control design have been addressed in [11]. In this paper, we implement a proportional-plus-integral (PI)

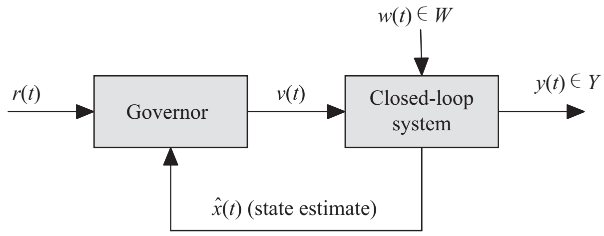

Figure 4 A schematic of reference governor as applied within a closed-loop [15].

controller of the form

$$\begin{aligned} &u_{1,k} = -K_p(x_r - x_k) - K_I\omega_k, \\ &\omega_{k+1} = x_k + \omega_k - x_r, \end{aligned} \tag{9}$$

where ω_k is the integral state and K_p and K_I are tuning parameters. Pole-placement or linear quadratic regulator (LQR) techniques can be applied to determine K_p and K_I and ensure that the closed-loop system is locally asymptotically stable [11, 13, 14]. An additional observer can be augmented for the system disturbance estimation if necessary [10].

3.2 Reference governor for the constraint enforcement

It is important for an FPEG to maintain a consistent clearance height to ensure safe operation and maintain combustion quality. Load changes in HEVs, however, can cause violation of piston position constraints. Plugging feedback control law (9) into (1), the closed-loop system and the constraints assume the form

$$\begin{aligned} &f_c(x_{k+1}, x_k, x_{k-1}, \omega_k, u_{2,k}) = 0, \\ &y_k = x_k \in X, \end{aligned} \tag{10}$$

where X is a specified set that corresponds to safe piston motion during load transitions of $u_{2,k}$, defined as $X = \{x : |x_k - x_r| \leqslant \mathcal{M}\}$ and where $\mathcal{M}$ is specified. The feedback control (9) makes the actual clearance height x_k track the desired value x_r but without the ability to handle constraints. Therefore, we consider augmenting this system with the reference governor for the constraints enforcement. However, the closed loop dynamics (10) are represented by a discrete-time nonlinear implicit model, to which standard reference governors are not directly applicable. Consequently, in this paper, an implicit reference governor is developed which utilizes Newton's method for response prediction and a bisection search algorithm to enforce all constraints over future time instants by adjusting the load command $u_{2,k}$. Sections 4 and 5 discuss the design of the reference governor based on our implicit model and error bound estimation.

4 Reference governor design based on implicit models

The reference governors [15] are a class of predictive control schemes that are appended to well-designed closed-loop systems to enforce pointwise-in-time state and control constraints. As shown in Figure 4, the reference governor modifies the desired reference $r(t)$ to a currently applied reference $v(t)$, based on the measured or estimated states $\hat{x}(t)$ whenever it becomes necessary to avoid the constraint violations for the closed-loop variables. Acting as an add-on control scheme, the reference governor preserves the nominal response of the closed-loop system when there is no danger of constraint violation.

The reference governor was firstly introduced as a continuous-time algorithm [16] and later developed in the discrete-time setting [17]. The formulation of reference governor for linear systems has appeared in [18, 19] and then extended to the treatment of linear systems with uncertainties and bounded disturbances [20]. Reference governor designs for nonlinear systems have been developed in [21, 22]. In the case of nonlinear models, the design can be based on model linearization [23], or by directly applying the reference governor to the nonlinear model [24, 25]. The robust reference governor that deals with uncertainty based on constraint tightening is presented in [26]. In [15], a comprehensive overview of theoretical developments and applications of reference governors is presented. To the best of the authors' knowledge, the existing reference governor design methodologies rely on models in explicit form. Until recently, the reference governor design for implicit systems has remained an open problem which has not been sufficiently addressed.

4.1 Problem generalization

Motivated by the application to FPEG control, we consider the development of the reference governor for a general class of discrete-time closed-loop systems with an implicit model and pointwise-in-time constraints given by

$$f_c(x_{k+1}, x_k, x_{k-1}, \omega_k, v_k) = 0, \tag{11a}$$

$$y_k = Cx_k + Dv_k \in Y, \quad k \in \mathbb{Z}^+, \tag{11b}$$

where k denotes the discrete time-instant, $x_k \in \mathbb{R}$ denotes the state, $\omega_k \in \mathbb{R}$ denotes the integrator state, $y_k \in \mathbb{R}^q$ denotes the system output, and $v_k \in \mathbb{R}^m$ is the command (set-point) modified by the reference governor. The constraints in (11b) are imposed on the output y_k, Y is a prescribed set and $0 \in \text{int } Y$. The constraints can be equivalently expressed as $x_k \in X(v_k)$ where $X(v_k) = \{x_k : H_x x_k + H_v v_k \leqslant s_x\}$, $s_x > 0$, which is a polytope for which the matrices H_x, H_v, s_x are appropriately defined. Note that the state x_{k-1} at the previous time instant is known and the state x_k at the current time instant is assumed to be measured by available sensors or estimated by an observer. While the form of the system (11) is specific, however, it is motivated by our application to the FPEG. The subsequent developments can be extended to the case where the state $x_k \in \mathbb{R}^n$ is a vector of dimension $n > 1$.

In this paper, to simplify the subsequent developments, we consider the case of x_k being a scalar, and assume that the following assumptions hold: (A1) The function $f_c(x_{k+1}, x_k, x_{k-1}, \omega_k, v_k) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable and $\frac{\partial f_c}{\partial x_{k+1}}$ is nonsingular at each point $(x_{k+1}, x_k, x_{k-1}, \omega_k, v_k)$ of an open set $S \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m$. Then there exist neighborhoods $P \subset \mathbb{R}$ of x_{k+1}, $L \subset \mathbb{R}$ of ω_k and $Q \subset \mathbb{R}^m$ of v_k such that for each $x_k \in P$, $x_{k-1} \in P$, $v_k \in Q$, the equation $f(x_{k+1}, x_k, x_{k-1}, \omega_k, v_k) = 0$ has a unique solution $x^*_{k+1} \in P$ [27]. The assumption (A1) is fulfilled in FPEG as $\frac{\partial f_c}{\partial x_{k+1}}$ is continuously differentiable and nonsingular during normal operating conditions.

The objective of the reference governor is to manage the applied reference v_k, which should be as close as possible to the desired reference $v_{r,k}$, and to guarantee that the system constraint $y_k \in Y$ ($x_k \in X(v_k)$) is enforced. To do so, we consider the scalar reference governor [28] that uses the following update:

$$v_k = v_{k-1} + \beta_k(v_{r,k} - v_{k-1}), \tag{12}$$

where $\beta_k \in [0, 1]$ is a scalar adjustable parameter. If no danger of constraint violation exists, $\beta_k = 1$, and $v_k = v_{r,k}$ in which case the reference governor does not interfere with the operation of the system. If the potential for constraint violation exists, the value of β_k is decreased by the reference governor. In the extreme case, $\beta_k = 0$, $v_k = v_{k-1}$ so that the reference governor isolates the system from further application of reference command in order to enforce constraints.

The adjustment of the reference is based on solving for the value of β_k:

$$\beta_k = \max\{\beta_k \in [0, 1] : x_{k+t|k} \in X(v_k); t \in \mathbb{Z}_{[0,p]}\}, \tag{13}$$

where $x_{k+t|k}$ denotes the predicted response t steps ahead from the time instant k with the constant reference input v_k applied, i.e., $v_{k+1|k} = v_k$, $k = 0, 1, \ldots, p$. Here p is the prediction horizon. The bisection search method [26] can be applied to search for the maximum β_k at every time step. See [22, 28] for necessary assumptions and theoretical results which apply even if the system model is given in implicit form, assuming that the accurate response prediction $x_{k+1|k}$ can be made.

4.2 Iterative reference governor based on Newton's method

Since the model is given in an implicit form, the response prediction requires solving (11a) online. Specifically, the following system of equations needs to be solved:

$$\begin{aligned} &f_c(x_{k+1|k}, x_k, x_{k-1}, \omega_k, v_k) = 0, \\ &f_c(x_{k+2|k}, x_{k+1|k}, x_k, \omega_{k+1|k}, v_k) = 0, \\ &\qquad\vdots \\ &f_c(x_{k+p|k}, x_{k+p-1|k}, x_{k+p-2|k}, \omega_{k+p-1|k}, v_k) = 0. \end{aligned} \tag{14}$$

The set of equations (14) includes p equations with future states as unknowns that can be stacked up into a vector

$$\mathcal{X}_k = [x_{k+1|k}, x_{k+2|k}, \ldots, x_{k+p-1|k}, x_{k+p|k}]^{\mathrm{T}} \in \mathbb{R}^p. \tag{15}$$

The set of equations (14) can be rewritten as

$$F(\mathcal{X}_k) = 0 \tag{16}$$

for an appropriately defined F.

By now, the state prediction for the future p steps has been translated into the root finding problem for $\mathcal{X}_k = \mathcal{X}_k^*$ of (16). In this paper, Newton's method is implemented to approximately solve it. Newton's method uses updates of the form

$$\hat{\mathcal{X}}_{k,i+1} = \hat{\mathcal{X}}_{k,i} - \nabla_J F(\hat{\mathcal{X}}_{k,i})^{-1} \cdot F(\hat{\mathcal{X}}_{k,i}), \tag{17}$$

where the subscript i denotes the i-th Newton iteration and $\nabla_J F(\hat{\mathcal{X}}_{k,i}) \in \mathbb{R}^{p\times p}$ denotes the Jacobian matrix of (16) evaluated at $\hat{\mathcal{X}}_{k,i}$. Several iterations are performed until the convergence criterion

$$\|\hat{\mathcal{X}}_{k,i+1} - \hat{\mathcal{X}}_{k,i}\| \leqslant \epsilon \tag{18}$$

is satisfied where ϵ denotes the convergence threshold at which point $\hat{\mathcal{X}}_{k,i} \approx \mathcal{X}_k^*$.

Remark 1. The particular form of (14) assumes that the integral control is used and the integrator state $\omega_{k+p-1|k}$ depends on the state sequence of $x_{k+p-1|k}, \ldots, x_{k+1|k}$. Thus the corresponding Jacobian matrix $\nabla_J F(\mathcal{X}_k)$ is a lower trianglular matrix of the form

$$\nabla_J F(\mathcal{X}_k) = \begin{bmatrix} \nabla F_{11} & 0 & 0 & \cdots & 0 \\ \nabla F_{21} & \nabla F_{22} & 0 & \cdots & 0 \\ \vdots & \nabla F_{31} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \nabla F_{p1} & \cdots & \cdots & \nabla F_{p(p-1)} & \nabla F_{pp} \end{bmatrix}, \tag{19}$$

which implies that it is nonsingular if diagonal elements which correspond to $\frac{\partial f_c}{\partial x_{k+1|k}}$ are non-zero, where the latter is our assumption.

Note that if the integral control is not used, $\nabla_J F(\mathcal{X}_k)$ would be a band-like matrix of the form

$$\nabla_J F(\mathcal{X}_k) = \begin{bmatrix} \nabla F_{11} & 0 & 0 & \cdots & 0 \\ \nabla F_{21} & \nabla F_{22} & 0 & \cdots & 0 \\ 0 & \nabla F_{31} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \nabla F_{p(p-1)} & \nabla F_{pp} \end{bmatrix}, \tag{20}$$

and its nonsingularity still holds [29].

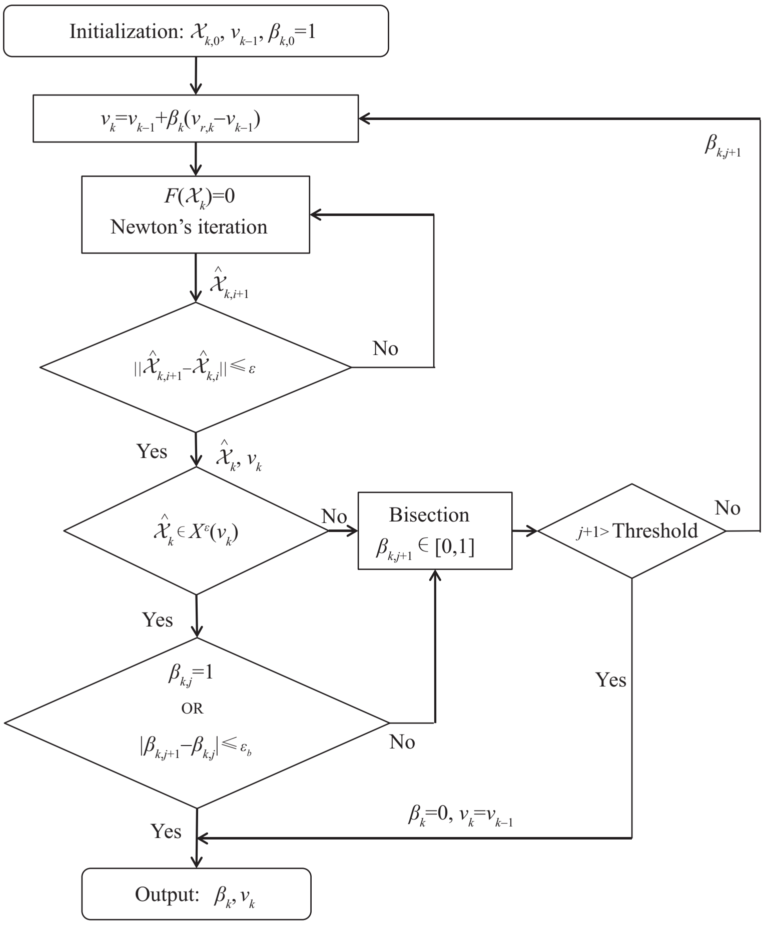

Figure 5 Computing flow diagram of the iterative reference governor.

Remark 2. For the system of the form (14), if the prediction horizon p is fixed, the matrix $\nabla_J F(\mathcal{X}_k)$ and the corresponding inverse matrix which depend on $\mathcal{X}_{k,i}$ and v_k, can be calculated off-line in symbolic form.

With the approximated prediction sequence $\hat{\mathcal{X}}_k \approx \mathcal{X}_k^*$, combining the condition (11b) and optimization problem (13), a feasible β_k that satisfies the constraints can be obtained by bisection search at every time instant k. Considering the small discrepancy between $\hat{\mathcal{X}}_k$ and $\mathcal{X}_k^*$, the constraint set X should be tightened by a small margin $\varepsilon > 0$ for safety. Thus, for system (11), applying the iterative procedure of (17), the design of iterative reference governor is described by the following optimization problem:

$$\max \quad \beta_k \in [0,1], \quad v_k = v_{k-1} + \beta_k(v_{r,k} - v_{k-1}), \quad \hat{\mathcal{X}}_k \in X^\varepsilon(v_k), \tag{21}$$

where $X^\varepsilon(v_k)$ is a tightened constraint set.

The procedure to determine β_k can be summarized as follows. At every time instant k, with the initial $\mathcal{X}_{k,0} = [x_0, \ldots, x_0] \in \mathbb{R}^p$, $\beta_{k,0} = 1$ and previous v_{k-1}, v_k is updated by (12). According to (14), the prediction sequence $\hat{\mathcal{X}}_k$ over the interval $[k, k+p]$ is calculated by using Newton's iteration of (17). If $\hat{\mathcal{X}}_k \in X^\varepsilon(v_k)$, the current v_k is the feasible output. Otherwise, a smaller value for $\beta_k = \beta_{k,j} \in [0,1]$ is searched by using the bisection algorithm to update v_k until the updated state prediction sequence is in the constraint set. At the bisection searching stage, in order to approximately maximize β_k, a searching threshold ϵ_b is set as $|\beta_{k,j+1} - \beta_{k,j}| \leqslant \epsilon_b$. The number of bisection iterations can also be constrained to an upper bound j_n to avoid excessive computational overhead. Since only the scalar parameter β_k is optimized on-line, the computational complexity of this approach is relatively low. The flow diagram of the iterative reference governor computations is shown in Figure 5.

5 Error bound estimation and one-step iterative reference governor

As shown in Figure 5, the proposed iterative reference governor algorithm includes two iterative loops: one is for the Newton method to determine the state prediction sequence, the other is for the bisection iterative search for the feasible β_k. Being the inner iterative loop, the Newton iteration is executed for the updated state prediction sequence every time $\beta_{k,j}$ is updated by the bisection loop. This implementation can lead to a large number of iterations and high computational burden while the computational resources are limited in real-time applications. In the following, with the aim to reduce the computational effort, the proposed reference governor is modified to a one-step iterative reference governor.

The main idea is briefly summarized as follows: as its name suggests, only one Newton iteration is implemented to predict the future state sequence with each updated $\beta_{k,j}$. As a result, the computing effort is decreased but it becomes necessary to handle errors in the state sequence prediction given by $\hat{\mathcal{X}}_{k,1}$. To ensure constraint enforcement, the constraints must be tightened to account for approximation errors.

5.1 Error bound estimation

The error bound on $\|\mathcal{X}_k^* - \hat{\mathcal{X}}_{k,1}\|$ can be computed based on the following proposition.

Proposition 1. If the mapping $F(\mathcal{X}_k)$ is continuously differentiable and globally invertible over an open set $U \subset \mathbb{R}^p$ and there exits an upper bound $L_{\rm inv} \geqslant \sup_{\theta\in[0,1]} \|[\nabla_J F(\theta\ \mathcal{X}_k + (1-\theta)\mathcal{X}_k^*)^{-1}]^{-1}\|$, where $F(\mathcal{X}_k^*) = 0$, $\mathcal{X}_k^* \in U$, $\mathcal{X}_k \in U$,

$$\|\mathcal{X}_k - \mathcal{X}_k^*\| \leqslant \mathcal{L}_{\rm inv}\|\mathcal{E}_k\|, \tag{22}$$

where $\mathcal{E}_k \in \mathbb{R}^p$ is the residual, i.e., $F(\mathcal{X}_k)$.

The proof of Proposition 1 can be found in Appendix A.1.

Eq. (22) can be used to compute the error bound between the actual solution $\mathcal{X}_k^*$ and the approximate solution at the i-th iteration of Newton's algorithm, $\mathcal{X}_{k,i}$. In order to reduce the online computational complexity for the state sequence prediction, a single Newton iteration can be used yielding $\hat{\mathcal{X}}_{k,1}$. Then the error bound $\|\mathcal{X}_k^* - \hat{\mathcal{X}}_{k,1}\|$ can be computed and used to tighten the constraint set.

Remark 3. The bound $\mathcal{L}_{\rm inv}$ may be determined or estimated based on the off-line study of a system model for different operating scenarios and across the operating range. Meanwhile, the residual $\|\mathcal{E}_k\|$ needs to be calculated on-line.

5.2 One-step iterative reference governor

The one-step iterative reference governor refers to the governor that only implements a single Newton iteration for the state sequence prediction and accounts for the error bound in enforcing constraints.

We use the infinity norm ($\|\cdot\|_\infty$) in (22) and define the error bound as $h_{\mathcal{E}}(\mathcal{L}_{\rm inv}, \mathcal{E}_{k,1}) = \mathcal{L}_{\rm inv}\|\mathcal{E}_{k,1}\|_\infty$, where $\mathcal{E}_{k,1}$ denotes the residual after the first Newton iteration. The prediction sequence based on a single Newton iteration for a given v_k is denoted by $\hat{\mathcal{X}}_{k,1}(v_k)$. We note that

$$\mathcal{X}_k^* \in \{\hat{\mathcal{X}}_{k,1}(v_k)\} \oplus h_{\mathcal{E}}(\mathcal{L}_{\rm inv}, \mathcal{E}_{k,1})\mathcal{B}_\infty, \tag{23}$$

where $\mathcal{B}_\infty = \{x : \|x\|_\infty \leqslant 1\}$. Hence to enforce our constraints we can enforce the condition

$$\hat{\mathcal{X}}_{k,1} \in X_{\rm tight}(v_k), \tag{24}$$

where

$$X_{\rm tight}(v_k) = \{\hat{\mathcal{X}}_{k,1} : H_x \hat{x}_{k+t|k,1} + H_v v_k \leqslant s_{\rm tight},\ t = 1, 2, \ldots, p\}, \tag{25}$$

$$[s_{\rm tight}]_i = [s_x]_i - \max[H_x]_i g, \quad g \in h_{\mathcal{E}}(\mathcal{L}_{\rm inv}, \mathcal{E}_k)\mathcal{B}_\infty, \tag{26}$$

and $[\cdot]_i$ denotes the i-th row.

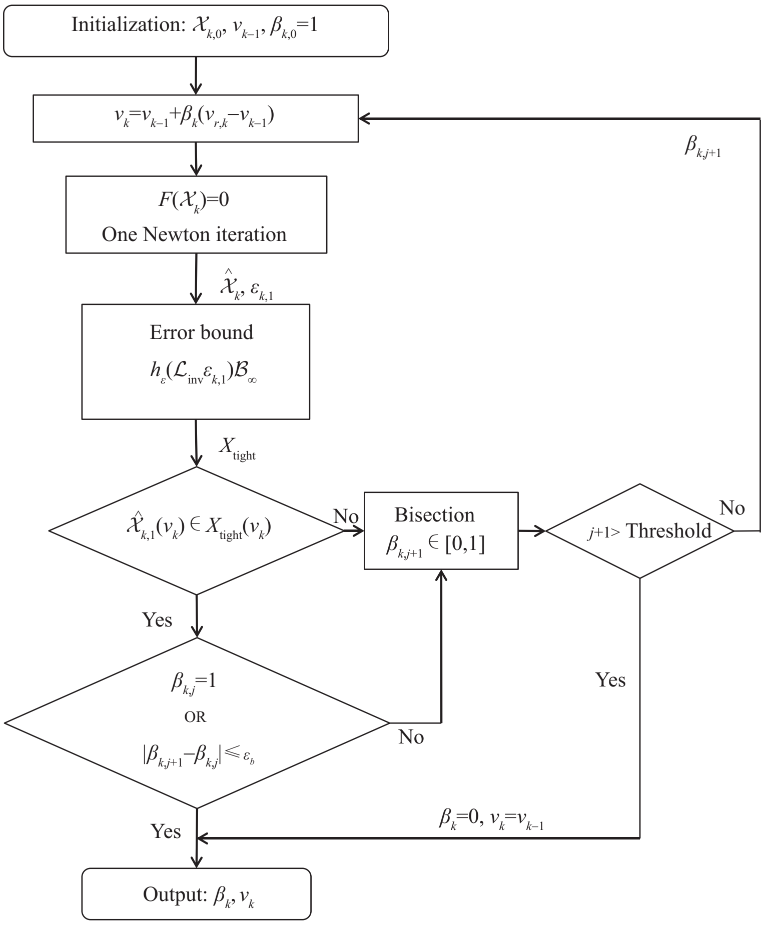

Figure 6 Computing flow diagram for the one-step iterative reference governor.

Remark 4. If the system constraints are fixed, H_x, H_v, s_x can be computed off-line and stored for online use. The constraints set $X_{\rm tight}(v_k)$ needs to be updated online by computing $s_{\rm tight}$ based on our error bound.

Thus, considering the system (11), the one-step iterative reference governor for our constrained nonlinear implicit system is defined by the solution of the following optimization problem:

$$\max \quad \beta_k \in [0,1], \quad v_k = v_{k-1} + \beta_k(v_{r,k} - v_{k-1}), \quad \hat{\mathcal{X}}_{k,1}(v_k) \in X_{\rm tight}(v_k). \tag{27}$$

To solve for β_k, the bisection algorithm is still utilized. If no feasible solution exists, β_k is set to zero and v_k is maintained at previous value, v_{k-1}. The computing flow diagram of one-step iterative reference governor is shown in Figure 6.

6 Simulation results

This section reports the simulation results of applying the proposed iterative and one-step iterative reference governors for FPEG piston motion control. The simulations are set up in Matlab/Simulink (R2013b/8.2) on a Windows 7 64 bit PC (CPU Core(TM) i7-4790 3.6 GHz, Memory 8 GB). Parameters relating to the iterative reference governor are selected as follows: convergence threshold of Newton's method is set to $\epsilon = 10^{-7}$, the convergence criterion of the bisection algorithm is chosen as $|\beta_{k,j+1} - \beta_{k,j}| \leqslant 10^{-3}$ with the maximum number of iterations limited to $j_n = 30$. Parameters of the feedback control for set-point tracking are given as $K_p = 4.2 \times 10^{-4}$, $K_i = 1.4 \times 10^{-2}$.

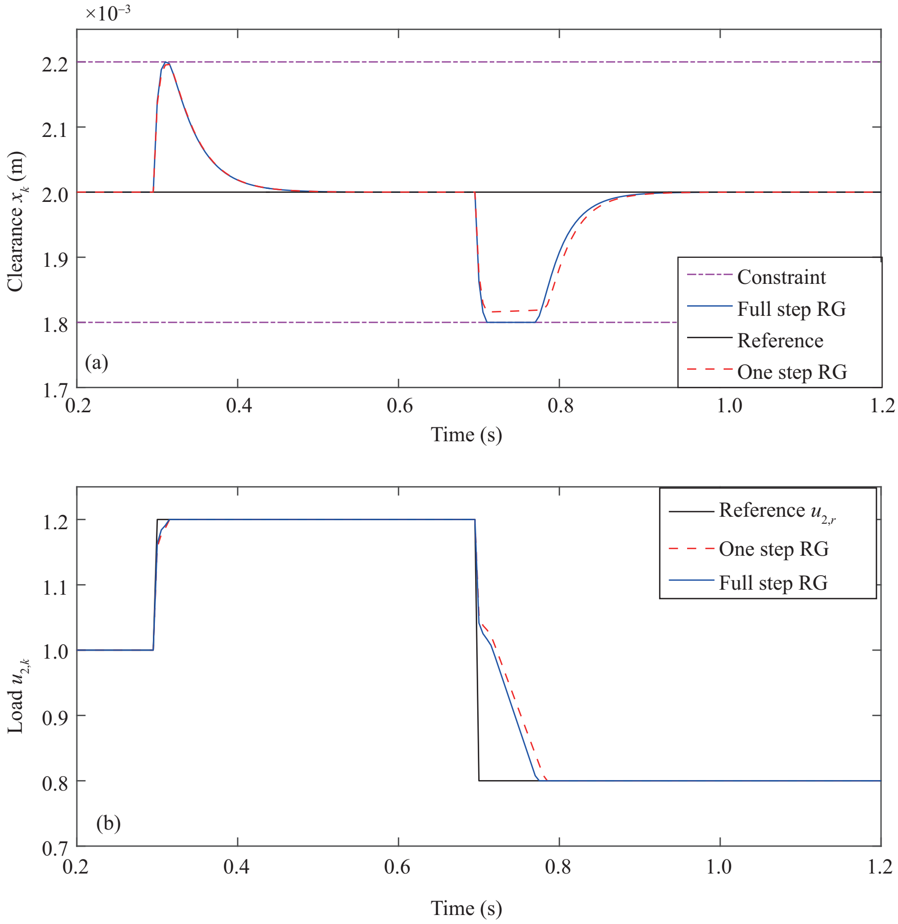

Figure 7 (Color online) Comparison between full iterative RG and one-step iterative RG. (a) The time history of clearance height, set-point and the constraints; (b) the time history of reference load and actual load.

Table 2 Computation performance comparison: initial value $x_0 = x_r = 2$ mm, simulation time 1.5 s

	Iterative RG	One-step iterative RG	MPC [10]
Total iterations	1213	449	–
Total computation time (ms)	37.35	29.92	401.2

6.1 Simulation test using the control oriented model

The comparison of responses with the full iterative reference governor described in Section 4 and that with the one step iterative reference governor described in Section 5 is shown in Figure 7 in which the initial clearance height trajectory for the Newton iterations is set to a vector of constant values corresponding to $x_0 = x_r = 2$ mm. In the simulation, the desired load $u_{2,r}$ follows a step profile. From the simulation results we can observe that both reference governors are able to enforce the constraints. The one-step iterative reference governor is more conservative than the iterative reference governor which manifests itself in slower transient load response. With both reference governors and with the help of feedback controller to regulate the fuel, the piston position can quickly track the reference clearance height after sudden changes of the load. The computational performance is compared in Table 2 for 1.5 s simulation. The number of iterations is reduced significantly when one-step Newton's method is implemented, where the iteration count reported combines inner loop (Newton method) and outer loop (bisection algorithm) iterations. As expected, the computation time is also reduced as the number of iterations decreases. The comparison with MPC controller is also considered. The computation times for the reference governors are smaller than that of the MPC controller in [10] which relies for the numerical solution on 'quadprog'

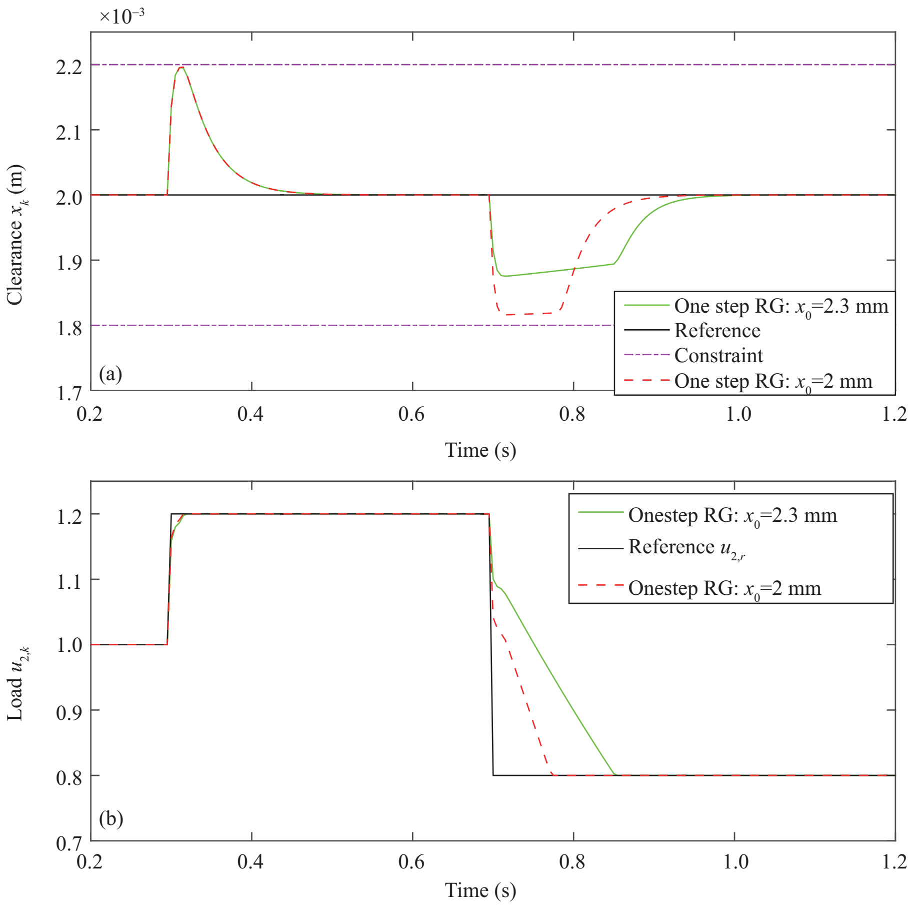

Figure 8 (Color online) Comparison of one-step iterative RG with different initial values. (a) The time history of clearance height, set-point and the constraints; (b) the time history of reference load and actual load.

Table 3 Computation performance comparison: initial value $x_0 = 2.3$ mm, simulation 1.5 s

	Iterative RG	One-step iterative RG
Total iterations	1548	506
Total computation time (ms)	45.01	33.69

function of Matlab.

Figure 8 compares the performances of the proposed one-step iterative reference governor for different initializations of the Newton's iteration. The response becomes more conservative and slower when the initial guess for the solution is further away from the actual solution. Table 3 compares the computational performance of the two governors with the initial clearance height trajectory for Newton iterations set to a vector of constant values of $x_0 = 2.3$ mm. It shows that, for the same set-point ($x_r = 2$ mm), the one-step iterative reference governor requires less computation time. Note that since the normal operating frequency of the free piston engine is 20–30 Hz, the proposed one step reference governor is computationally feasible in each update interval and is promising for real time application.

6.2 Simulation test in AMESim model

The effectiveness of the proposed reference governor is next tested using a high fidelity model of FPEG which is implemented in commercial software package AMESim [12]. The internal combustion engine model in the AMESim is selected as CFM 1D model from IFP-Engine library which includes the detailed mechanical parts of the engine and is able to capture most of significant dynamics in cylinder such as pressure dynamics, the wall heat exchange, and the combustion process.

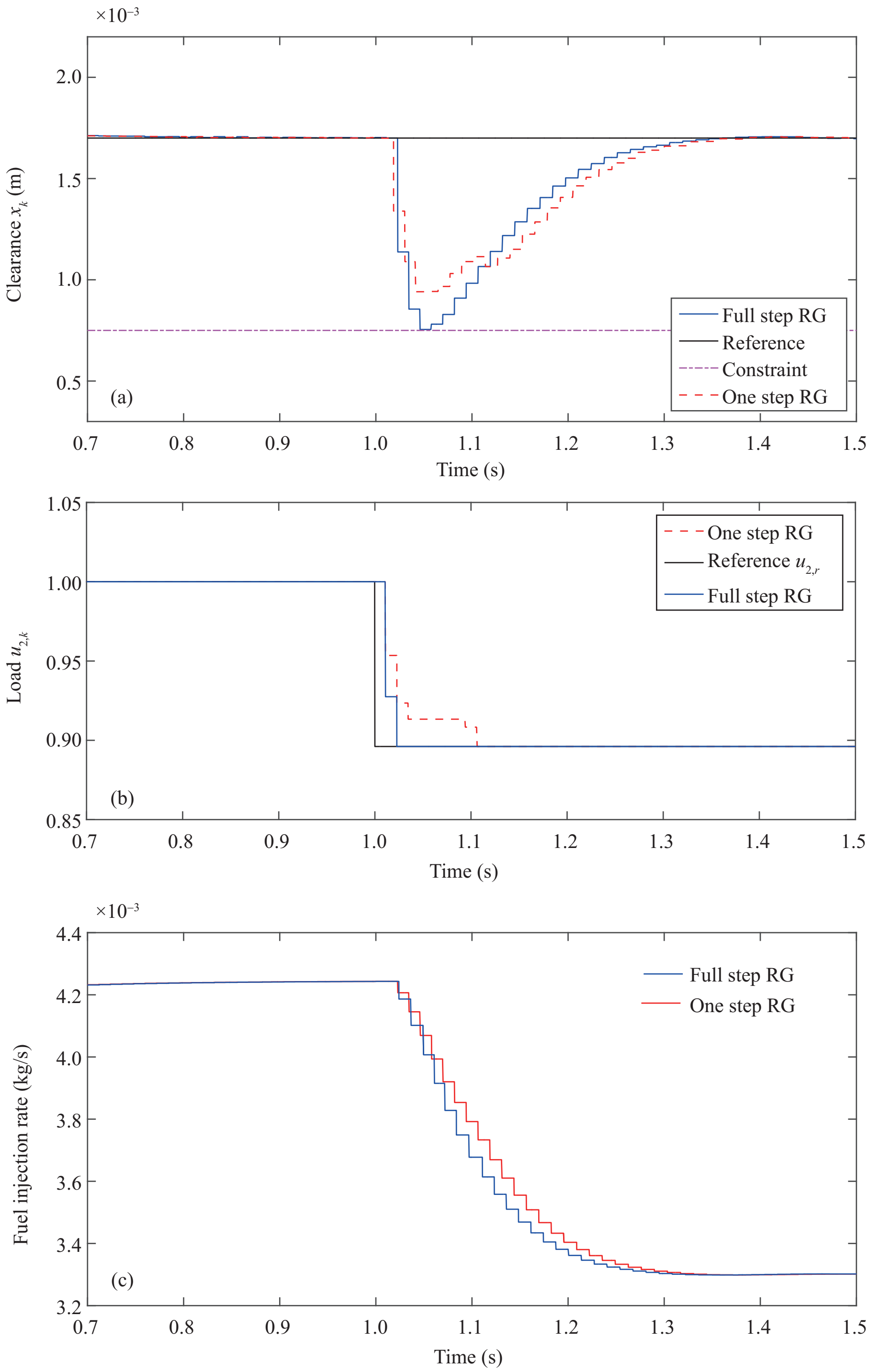

Figure 9 (Color online) Comparison between full iterative RG and one-step iterative RG in AMESim model. (a) The time history of clearance height, set-point and the constraints; (b) the time history of load regulation; (c) the time history of fuel regulation.

Figure 9 shows the closed-loop response of the AMESim model to changes in load demand (from $u_{2,r} = 1$ to $u_{2,r} = 0.9$). Both the iterative reference governor and the one step reference governor can successfully confine the piston motion within 1 mm of the desired piston turnaround position. Compared with the iterative reference governor, the one step iterative reference governor results in a slower load following and a more conservative constraints enforcement. The clearance height converges to the target value within 0.3 s as the load converges to the target load within 0.1 s. As we do not have access to an experimental FPEG, in this paper, we only focus on the controller testing in simulation on a high fidelity model. We leave the development of an experimental platform and the actual controller experiments to future work.

7 Conclusion

This paper considered FPEG piston motion control while enforcing constraints during load transients using implicit reference governors. A nonlinear, discrete-time implicit, control oriented model of FPEG based on energy balance was considered. The proposed controller consists of two main components: a feedback control loop applied to regulate the fuel to track the piston position set-point and a reference governor to manage the load changes by only applying loads that do not lead to constraint violation over the future prediction horizon. To accommodate the implicit form of the FPEG system model, an implicit reference governor, that we refer to as the iterative reference governor, has been developed which utilizes the Newton's method for response prediction and a bisection search algorithm to enforce constraints by adjusting the reference load command. In order to reduce the computational complexity, a one-step iterative reference governor has been developed that accounts for the error bound in approximating the predicted trajectory using one iteration of Newton's method. Simulation results show that the proposed control scheme can effectively track the piston position set-point and handle the constraints during load transients.

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Appendix A

Appendix A.1 Proof of Proposition 1

The proof of Proposition 1 follows standard arguments and is included for completeness.

Proof. Let $h \in U$, and consider

$$\mathcal{G}_{i_p}(\sigma) := F_{i_p}(\mathcal{X}_k + \sigma h), \tag{A1}$$

where $0 \leqslant \sigma \leqslant 1$, $i_p = 1, 2, \ldots, p$.

Define $\mathcal{X}_{k,j_p}$ as the j_p-th component of $\mathcal{X}_k$ and h_{j_p} as the j_p-th component of h. Then the component $F_{i_p}(\cdot)$ of $F(\cdot)$ satisfies

$$F_{i_p}(\mathcal{X}_k + h) - F_{i_p}(\mathcal{X}_k) = \mathcal{G}_{i_p}(1) - \mathcal{G}_{i_p}(0) = \int_0^1 \mathcal{G}'_{i_p}(\sigma)\mathrm{d}\sigma = \sum_{j_p=1}^{p} \left(\int_0^1 \frac{\partial F_{i_p}}{\partial \mathcal{X}_{k,j_p}}(\mathcal{X}_k + \sigma h)\mathrm{d}\sigma \right) h_{j_p}. \tag{A2}$$

Then, combining the components, (A2) leads to

$$F(\mathcal{X}_k + h) - F(\mathcal{X}_k) = \left(\int_0^1 \triangledown_J F(\mathcal{X}_k + \sigma h)\mathrm{d}\sigma \right) h, \tag{A3}$$

and

$$\|F(\mathcal{X}_k + h) - F(\mathcal{X}_k)\| = \left\| \left(\int_0^1 \triangledown_J F(\mathcal{X}_k + \sigma h)\mathrm{d}\sigma \right) h \right\| \leqslant \left\| \left(\int_0^1 \triangledown_J F(\mathcal{X}_k + \sigma h) \right) \mathrm{d}\sigma \right\| \|h\|. \tag{A4}$$

Let $\mathcal{L}_{\mathrm{or}} \geqslant \sup_{\phi \in U} \|\triangledown_J F(\phi)\|$, then

$$\|F(\mathcal{X}_k + h) - F(\mathcal{X}_k)\| \leqslant \sup_{\phi \in U} \|\triangledown_J F(\phi)\| \cdot \left\| \int_0^1 \mathrm{d}\sigma \right\| \cdot \|h\| \leqslant \mathcal{L}_{\mathrm{or}} \|h\|. \tag{A5}$$

Define $\mathcal{X}_{k1} = \mathcal{X}_k + h$, $\mathcal{X}_{k2} = \mathcal{X}_k$, for $\mathcal{X}_{k1}$, $\mathcal{X}_{k2} \in U$. Then, (A5) implies

$$\|F(\mathcal{X}_{k1}) - F(\mathcal{X}_{k2})\| \leqslant \mathcal{L}_{\mathrm{or}} \|\mathcal{X}_{k1} - \mathcal{X}_{k2}\|. \tag{A6}$$

Define $\Psi_k = F(\mathcal{X}_k)$, and the inverse function $\mathcal{X}_k = F^{-1}(\Psi_k)$. Further, based on the inverse function theorem [29], it follows that

$$\triangledown_J F^{-1}(\Psi_k) = [\triangledown_J F(\mathcal{X}_k)]^{-1}, \tag{A7}$$

where F^{-1} denotes the inverse of F, thus

$$\|\mathcal{X}_k - \mathcal{X}^*\| = \|F^{-1}(\mathcal{E}_k) - F^{-1}(0)\| \leqslant \sup_{\theta \in [0,1]} \left\| [\triangledown_J F(\mathcal{X}_k\ \theta + (1-\theta)\mathcal{X}^*]^{-1} \right\| \cdot \|\mathcal{E}_k\| \leqslant \mathcal{L}_{\mathrm{inv}} \|\mathcal{E}_k\|. \tag{A8}$$

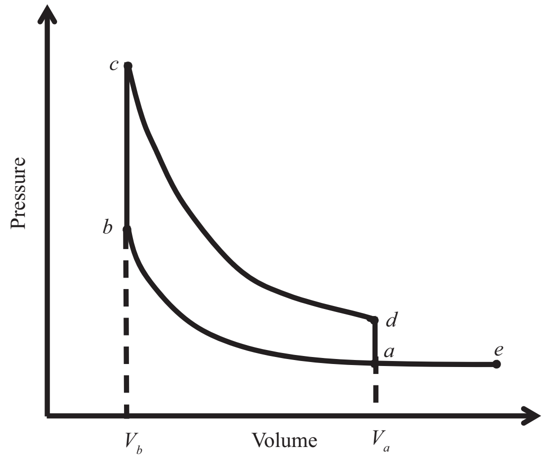

Figure A1 Two-stroke FPEG pressure-volume diagram, where V_a is the maximum volume when valves are closed and V_b is the clearance volume. Process $a \to b$ is isentropic compression, process $b \to c$ is constant volume combustion, process $c \to d$ is isentropic expansion, and process $a \to e \to a$ is ideal scavenging.

Appendix A.2 Control-oriented model derivation

We model the considered 2-stroke FPEG based on pressure-volume (P-V) diagram as shown in Figure A1. The detailed derivation of the compression work and the expansion work are given as follows.

The compression work W_{comp} corresponding to $a \to b$ is derived as

$$W_{\text{comp}} = \int_{V_b}^{V_a} P\mathrm{d}V = \frac{1}{\gamma - 1}(P_a V_a - P_b V_b). \tag{A9}$$

Using the defined clearance height x_k, cylinder volume yields $V_a = [L - (x_{\text{sca}} + x_k)]A_p$ and $V_b = x_{k+1}A_p$. Considering the characteristic of isentropic process, the corresponding boundary work of compression can be expressed as function of the measured state x_k as (2).

Similarly, the expansion work W_{exp} is derived as

$$W_{\text{exp}} = \frac{1}{\gamma - 1}(P_c V_c - P_d V_d). \tag{A10}$$

The cylinder pressure at point c is related to the heat addition and can be calculated as

$$P_c = \frac{u_{1,k} H_u(\gamma - 1)}{x_k A_p} + P_a \left(\frac{L - x_{\text{sca}} - x_{k-1}}{x_k}\right)^{\gamma}. \tag{A11}$$

Combining (A10) and (A11), the expansion work is characterized as

$$W_{\text{exp}} = \left[u_{1,k} H_u + \frac{P_0 A_p}{\gamma - 1}\frac{(L - x_{\text{sca}} - x_{k-1})^{\gamma}}{{x_k}^{\gamma-1}}\right]\left[1 - \frac{x_k^{\gamma-1}}{(L - x_{\text{sca}} - x_{k+1})^{\gamma-1}}\right]. \tag{A12}$$