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Non-fragility of multi-agent controllability

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Abstract Controllability of multi-agent systems is determined by the interconnection topologies. In practice, losing agents can change the topologies of multi-agent systems, which may affect the controllability. In order to preserve controllability, this paper first introduces the concept of non-fragility of controllability. In virtue of the notion of cutsets, necessary and sufficient conditions are established from a graphic perspective, for almost surely strongly/weakly preserving controllability, respectively. Then, the problem of leader selection to preserve controllability is proposed. The tight bounds of the fewest leaders to achieve strongly preserving controllability are estimated in terms of the diameter of the interconnection topology, and the cardinality of the node set. Correspondingly, the tight bounds of the fewest leaders to achieve weakly preserving controllability are estimated in terms of the cutsets of the interconnection topology. Furthermore, two algorithms are established for selecting the fewest leaders to strongly/weakly preserve the controllability. In addition, the algorithm for leaders' locations to maximize non-fragility is also designed. Simulation examples are provided to illuminate the theoretical results and exhibits how the algorithms proceed.

Keywords non-fragility, controllability preserving, cutset, leader selection, almost surely

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1 Introduction

In recent years, distributed coordination control of multi-agent systems (MASs) has become an important topic due to the wide applications of MASs in areas such as flocking in biology groups, unmanned aerial vehicle cooperative formation, flocking of robots, and attitude adjustment of spacecrafts. Some basic and important issues were studied including consensus problem [1–4], formation control [5], flocking [6, 7], controllability and stabilizability [8–12], and equilibrium topologies [13].

Controllability of MASs was proposed by Tanner for the first time in [14], where a necessary and sufficient condition was presented through the Laplacian matrix and the corresponding eigenvalues. Wang et al. [15] studied controllability of MASs with high-order dynamics and generic linear dynamics, and showed that controllability is congruously determined by the interconnection topology. Afterwards, researchers attempted to investigate controllability of MASs from the algebraic point of view and graphic perspective. For example, Zhao et al. [16] designed a leader selection algorithm using the algebraic properties of the Laplacian matrix; Ji et al. [17] proposed a construction procedure for uncontrollable topologies; and Rahmani et al. [18] provided some necessary conditions for controllability utilizing the equitable partition of the interconnection topology. In addition, interesting methods were developed for controllability of

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some special graphs, e.g., tree graphs via analysing the leaders' role with downer branches [19]; paths, cycles and grid graphs via simple rules from number theory [20,21]. A parallel research line in this field is structural controllability of MASs, which was investigated under various models [22–24]. The relationship between controllability and structural controllability was studied in [25], and controllability improvement for structurally controllable MASs was discussed in [16].

In practice, for MASs, losing agents is a common phenomenon. For example, in robot systems and vehicle systems, malfunction of some units may appear during the formation process; in biological systems, individuals of a species might be dead during the migration; and in social groups, members of an organization may quit at any time. Losing agents causes failure of communication links and influences controllability and structural controllability of MASs. Therefore, in [26], an optimal selection of the fewest leaders was shown to improve the reliability of MASs in term of structural controllability. In [27], the robustness of structural controllability was investigated against the failure of agents and communication links simultaneously. However, there is a lack of study on controllable MASs with fixed edge weights, especially how to preserve controllability subject to losing agents. In [16], a leader selection problem is proposed to search for the fewest leaders making MASs controllable. In virtue of this idea, we intend to get the fewest leaders to preserve the controllability of MASs after a part of the agents are removed from the systems. Besides, if the number of external inputs is limited, how to select the corresponding leaders to ensure that, non-fragility of the controllability be as strong as possible, is also worthy of study.

Motivated by the above analysis, this paper studies controllability preserving problems of MASs. The main contributions of this paper are threefold.

(1) As a fundamental concept, non-fragility is proposed for controllability. The difference between structural controllability and non-fragility of controllability is clarified. Utilizing the notion of cutsets and leader-follower cutsets, necessary and sufficient graphic conditions of the interconnection topology to almost surely strongly/weakly preserve controllability are established, respectively. The discussion is clarified into single leader case and multiple leaders case.

(2) Leader selection problems are proposed for achieving almost surely strongly/weakly preserving controllability and maximizing the non-fragility, respectively. By utilizing number theory and some graphic properties of the interconnection topology, the upper and lower bounds of the minimal leaders to achieve strongly or weakly preserving controllability are provided. A difference between the results on strongly/weakly preserving controllability is illustrated.

(3) Three algorithms are designed to select the fewest leaders to preserve the controllability, and to arrange leaders' locations to maximize the non-fragility of controllability. Some representative examples are provided to explain how the algorithms proceed.

This paper is organized as follows: Section 2 introduces the basic concepts and mathematical tools for this paper, and proposes three basic problems. Main results on preserving controllability are shown in three subsections of Section 3. Numerical simulation examples are provided in Section 4 to illuminate the theoretical results. Conclusion is drawn in Section 5.

Notations. $|S|$ represents the cardinality of set S. $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ is the diagonal matrix with principal diagonals $a_1, a_2, \ldots, a_n$. The set of n-dimensional real vectors is denoted by $\mathbb{R}^n$. C_n^p denotes the combination number, selecting p items from n items, $0 \leqslant p \leqslant n$. S/T represents the set of all the elements in S but not in T. $\lfloor r \rfloor$ represents the maximal integer that is not larger than r.

2 Preliminaries

2.1 Graph theory

An undirected graph $G = (V, E)$ consists of a vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and an edge set $E \subseteq V \times V$. In graph G, $e_{ij} \in E$ if and only if $e_{ji} \in E$, and $v_i,\ v_j$ are said to be adjacent with each other. The neighbor set of v_j is denoted by $N_j = \{v_i \in V | (v_i, v_j) \in E\}$. The adjacency matrix of G is $A(G) = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} > 0$ is the weight of edge e_{ji} (as well as e_{ij}), and $a_{ij} = 0$ if $(v_j, v_i) \notin E$. The Laplacian matrix of G is $L(G) = D - A$, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ where $d_k = \sum_{i=1,i\neq k}^{n} a_{ki},\ k = 1, 2, \ldots, n$. Graph

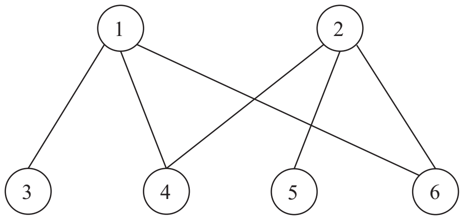

Figure 1 A bipartite graph consists of 6 nodes, where $V_1 = \{1,2\}$,$V_2 = \{3,4,5,6\}$.

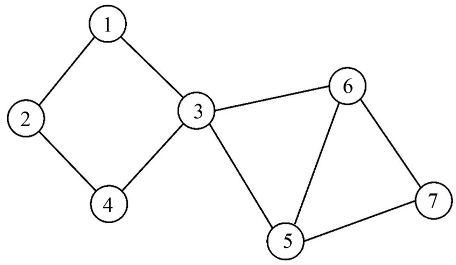

Figure 2 A graph consists of 7 nodes.

$G' = (V', E')$ is called a subgraph of G if $V' \subseteq V$, $E' \subseteq E$. In particular, G' is said to be a spanning subgraph of G if $V' = V$, $E' \subseteq E$. Removing a vertex v from G means deleting v and all the edges connected with v in G, and the remaining subgraph is denoted as $G - v$. Removing a vertex set V' from G means deleting all the vertices in V' and all the edges connected with at least one vertex in V', and denote the remaining subgraph as $G - V'$. $G' = (V', E')$ is said to be the induced subgraph of G by V', if $G' = G - V/V'$.

A path between v_i and v_j is a subgraph of G, whose vertex set is $\{v_i, v_{k_1}, \ldots, v_{k_r}, v_j\}$ and the edge set is $\{(v_i, v_{k_1}), (v_{k_1}, v_{k_2}), \ldots, (v_{k_{r-1}}, v_{k_r}), (v_{k_r}, v_j)\}$, where $1 \leqslant r \leqslant n-2$; $1 \leqslant i, j, k_1, \ldots, k_r \leqslant n$, and $\{i \neq k_1 \neq \cdots \neq k_r \neq j\}$. For two vertices $v_i \neq v_j$, the length of the shortest path connecting them is said to be the distance between v_i and v_i. The longest distance between two nodes in G is call the diameter of the graph. For two different vertices $v_i \neq v_j$, we say they are in the same connected component if there exists a path between them, otherwise, they are in different connected components. The number of connected components of G is denoted as $C(G)$. G is said to be connected if $C(G) = 1$. A bipartite graph consists of two vertex sets $\langle V_1, V_2 \rangle$, where $V_1 \bigcup V_2 = V$ and $V_1 \bigcap V_2 = \emptyset$ such that for each edge in the bipartite graph, one node is from V_1 and the other is from V_2.

Example 1. Figure 1 is a bipartite graph. Let $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, v_5, v_6\}$, then, there is no edge connecting two nodes both in V_1 or in V_2. The distance between v_3 and v_6 is 2, and there exists a path $v_3 - v_1 - v_6$ with length 2, while v_3 and v_6 are not directly adjacent. The diameter of the graph is 4, since the distance between v_3 and v_5 is 4, and the distance between each other pair of nodes is no more than 4.

Definition 1 ([28])**.** For a graph $G = (V, E)$, vertex set $V' \subset V$ is said to be a cutset of G, if when all the nodes in V' are removed, the subgraph $G - V'$ contains more connected components than G, i.e., $C(G - V') > C(G)$, whereas removing any proper subset $V'' \subset V'$ will not increase the connected components of G. The minimal cutset is a cutset of G that contains the fewest vertices. If a cutset contains only one vertex v, we call v a cut vertex. The number of nodes in a minimal cutset of G is called the connectivity of the graph.

Generally speaking, a cutset is a set of nodes (not edges), when removed, will lead to more connected components than in the original graph. For example, in Figure 2, removing a single vertex v_3 will break the connectedness, therefore v_3 is a cut vertex of Figure 2. Apparently, neither of v_5, v_6 is a cut vertex, however, removing vertex set $\{v_5, v_6\}$ makes v_7 separated from the other nodes, which means v_5 and v_6 form a two-node cutset of Figure 2. The connectivity of the Figure 2 is $C(G) = 1$.

2.2 Model formulation

Consider an MAS consisting of n agents with single-integrator dynamics,

$$\dot{x}_i = u_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

where $x_i, u_i \in \mathbb{R}$ represent the state and the control input of agent v_i, respectively. Without loss of generality, the leaders which can be actuated by external inputs are supposed to be $v_1, \ldots, v_m$. Denote the leader set as $V_l = \{v_1, \ldots, v_m\}$. The set of the rest agents, i.e., followers, are denoted as

$V_{\rm f} = \{v_{m+1}, \ldots, v_n\}$. The interconnection topology of system (1) is denoted by G, and $G - V_{\rm l}$ is said to be the follower subgraph. The control inputs are supposed to obey the consensus-based protocol: $u_i = \sum_{j\in N_i} a_{ij}(x_j - x_i) + u_{{\rm o}i}$, where $u_{{\rm o}i} \in \mathbb{R}$ is the external control on the leader agents v_i, $i = 1, 2, \ldots, m$ and $u_{{\rm o}i} = 0$ when $i = m+1, \ldots, n$. The compact form of system (1) under the protocol is summarized as

$$\dot{x} = -Lx + Bu, \tag{2}$$

where $x = (x_1, x_2, \ldots, x_n)^{\rm T} \in \mathbb{R}^n$ and $u = u_{\rm o} = (u_{{\rm o}1}, u_{{\rm o}2}, \ldots, u_{{\rm o}m})^{\rm T} \in \mathbb{R}^m$ is the external control. L is the Laplacian matrix of G and $B = (e_1, e_2, \ldots, e_m) \in \mathbb{R}^{n\times m}$ consists of the first m columns of the n-dimensional identical matrix. System (2) is said to be controllable if for any initial states $x(0) = x_0 \in \mathbb{R}^n$ and any target states $x^* \in \mathbb{R}^n$, there exists a control input $u = u_{\rm o}(t)$ and a finite time instant $T \geqslant 0$, such that $x(T) = x^*$. In the following, we call $[B, -LB, \ldots, (-L)^{n-1}B]$ the controllability matrix of system (2). According to the Rank criterion, system (2) is controllable if and only if the controllability matrix is of full row rank.

Remark 1. System (2) in the above is the simplest model of MASs. The controllability of MASs with high-order dynamic agents or generic linear dynamic agents is also an important topic. The MAS consists of h-order dynamic agents is modeled by $\dot{x}_i^{(1)} = x_i^{(2)}, \ldots, \dot{x}_i^{(h-1)} = x_i^{(h)}$, $\dot{x}_i^{(h)} = u_i$, $i = 1, 2, \ldots, n$, and is with the protocols $u_i = \sum_{j\in N_i}\sum_{k=0}^{h} a_{ij}(\dot{x}_j^{(k)} - \dot{x}_i^{(k)}) + u_{{\rm o}i}$ or $u_i = \sum_{j\in N_i}\sum_{k=0}^{h-1} a_{ij}(\dot{x}_j^{(k)} - \dot{x}_i^{(k)}) + \sum_{j\in N_i}\theta\dot{x}_i^{(h)} + u_{{\rm o}i}$, where θ is the feedback gain. The MAS with generic linear dynamic agents is modeled by $\dot{x}_i = Ax_i + bu_i$, $i = 1, 2, \ldots, n$, $A \in \mathbb{R}^{s\times s}$, $b \in \mathbb{R}^s$, and (A, b) is completely controllable. The system is with the protocol $u_i = \alpha^{\rm T}T^{-1}x_i + \sum_{j\in N_i}\sum_{k=0}^{s} a_{ij}\beta^{\rm T}T^{-1}(x_j - x_i) + u_{{\rm o}i}$, where α, β, T^{-1} are proper vectors and matrix to be designed. However, by [15], the controllability of the two kinds of systems is equivalent to that of system (2). The following discussion is under single-integrator model for convenience. Obviously, the results obtained in this paper can be applied to MASs with high-order and generic linear dynamic agents. Considering that controllability is congruously determined by the interconnection topology, "controllability of the system" and "controllability of the topology" are used alternatively without causing misunderstanding.

For the interconnection topologies of system (2), leader-follower cutset is introduced.

Definition 2. For the interconnection topology $G = (V, E)$ of system (2), vertex set $V' \subset V$ is said to be a leader-follower cutset of G, if V' contains no leader and when all the nodes in V' are removed, there must be at least one follower that no path exists in the subgraph which connecting the follower and any leader, whereas removing any proper subset $V'' \subset V'$ will not satisfy this condition. The minimal leader-follower cutset is a cutset of G that contains the fewest vertices. If a leader-follower cutset contains only one vertex v, we call v a leader-follower cut-vertex.

A detailed difference between cutsets and leader-follower cutsets is explained in the following proposition.

Proposition 1. For system (2) with some agents being leaders, node set $\tilde{V}$ is a leader-follower cutset of the interconnection topology G only if $\tilde{V}$ is a cutset of the graph and $\tilde{V}$ contains no leader.

Proof. If $\tilde{V}$ is a leader-follower cutset of G, we know that $\tilde{V}$ contains no leader. Since deleting $\tilde{V}$ makes the leaders and some node be divided into different connected components, $\tilde{V}$ is also a cutset of G.

Although $\{v_5, v_6\}$ is a cutset of Figure 2, if v_3 and v_7 are selected as the leaders, $\{v_5, v_6\}$ do not form a leader-follower cutset. When they are removed, all the three followers are able to get information from the leader v_3. This explains why the condition provided in Proposition 1 is only necessary, but not sufficient. However, the set $\{v_1, v_4\}$ is both a cutset and a leader-follower cutset of the topology. In this paper, we assume that the leaders can not be removed.

Definition 3. If MAS (2) is controllable, for any removal of p followers ($1 \leqslant p \leqslant n - m$), let r_p be the minimal rank of the controllability matrices of all the remaining subsystems. If $r_p = n - p$, the controllability of the original system is said to be p-nodes non-fragile (p-nodes NF). If for all $1 \leqslant p \leqslant n - m$, the controllability of system (2) is p-nodes NF, we then say the controllability is strongly non-fragile (SNF). Otherwise, if the controllability is p-nodes NF for all $1 \leqslant p \leqslant k$, but is not $(k+1)$-NF,

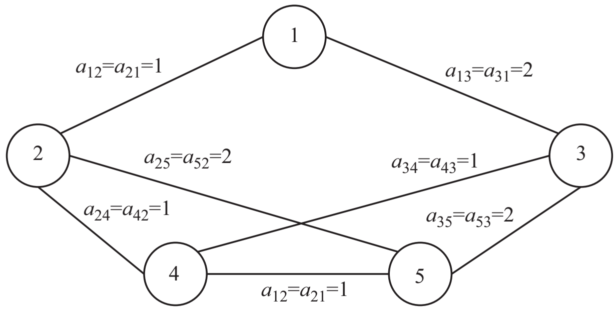

Figure 3 A weighted graph with 5 nodes.

$k \leqslant n-m-1$, we say the controllability is k-weakly non-fragile (k-WNF). Especially, if it is not 1-node NF, the controllability is said to be fragile.

Example 2. For system (2) with the interconnection topology depicted in Figure 3, where the weights are labeled on the edges, the system is obviously controllable. No matter which agent in $\{v_2, v_3, v_4, v_5\}$ is removed, rank of the controllability matrix of the remaining subsystem is always 4, thus the remaining subsystems are all controllable, i.e., the controllability of system (2) is 1-node NF. Although removing v_4 and v_5 makes the remaining subsystem also controllable, if v_2 and v_3 are both removed, rank of the controllability matrix of the remaining subsystem is 1, hence the controllability of the system is not 2-nodes NF, which means the controllability of system (2) is 1-WNF.

Remark 2. Equivalently, SNF is another expression of $(n-m)$-WNF, and fragile can be also treated as 0-WNF. Be worth mentioning, non-fragility is discussed only for controllable MASs, which should have fixed edge weights. Besides, if controllability of an MAS is k-WNF, it does not mean that the controllability is not p-nodes NF for all $p \geqslant k+1$, see Example 3.

Remark 3. Non-fragility of controllability is different from structural controllability. The concept of non-fragility is proposed for controllable systems when losing agents, which means the edge weights of the systems are fixed. Structural controllability is proposed for structural systems, which means the edges are not weighted. In other words, to study structural controllability, the edge weights are adjustable and the structure never changes, whereas when studying non-fragility of controllability, the edge weights are fixed and the structure changes. The relationship between non-fragility and structural controllability will be discussed later.

Example 3. Under the interconnection topology depicted in Figure 4, system (2) is controllable. Since the system will become uncontrollable if v_4 is removed, the controllability is fragile. However, removing any two nodes in $\{v_2, v_3, v_4\}$ will make the remaining subsystem controllable. Therefore, the system is 2-nodes NF, but is not even 1-WNF.

Definition 4. The interconnection topology of system (2) is said to almost surely strongly preserve the controllability (ASSPC) if for almost all sets of edge weights [1]), the system is controllable, and the controllability is SNF. Similarly, if for almost all sets of edge weights, the system is controllable and the controllability is k-WNF, we then say topology almost surely k-weakly preserves the controllability (ASkWPC). If the topology of system (2) is neither ASSPC nor ASkWPC for any $k \geqslant 1$, we say the topology is fragile for controllability.

Remark 4. Followed from Example 2, we check whether the topology of system (2) is ASSPC, or ASkWPC for some k, or fragile for controllability. This interconnection topology contains $m=7$ edges. Since the topology is connected, the set of edge weights, denoted as $\mathcal{C}_0$, making the system not controllable has Lebesgue measure 0 in $\mathbb{R}^7$. If v_2 is removed, the interconnection topology turns to be Figure 5, which is also connected. This means the set of edge weights making the subsystem not controllable has Lebesgue measure 0 in $\mathbb{R}^4$. Denoting the set of edge weights for the original system satisfying the subsystem not being controllable as $\mathcal{C}_1$, we get $\mathcal{C}_1$ has Lebesgue measure 0 in $\mathbb{R}^7$. If v_5 is removed, the

1) The combinations of edge weights have Lebesgue measure 1 in $\mathbb{R}^{|E|}$.

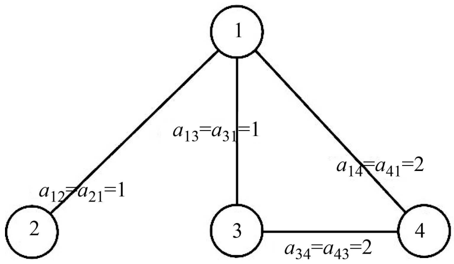

Figure 4 A weighted graph with 4 nodes.

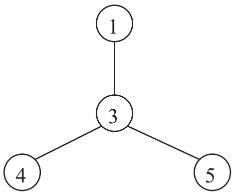

Figure 5 A graph with 4 nodes.

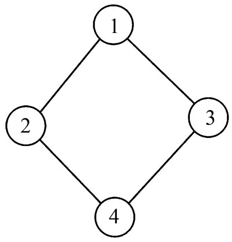

Figure 6 A graph with 4 nodes.

interconnection topology turns to be Figure 6, and the set of edge weights making the subsystem not controllable also has Lebesgue measure 0 in $\mathbb{R}^4$. Denoting the set of edge weights of system (2) making the subsystem not controllable as $\mathcal{C}_4$, we know $\mathcal{C}_4$ also has Lebesgue measure 0 in $\mathbb{R}^7$. Similarly, If v_3 or v_4 is removed, the interconnection topology turns to be Figures 5 or 6 as well. Denoting the corresponding edge weight sets as $\mathcal{C}_2$ and $\mathcal{C}_4$, we know $\mathcal{C}_2$ and $\mathcal{C}_4$ also have Lebesgue measure 0 in $\mathbb{R}^7$. Let $\mathcal{C} = \bigcup_{i=0}^{4} \mathcal{C}_i$, apparently $\mathcal{C}$ has Lebesgue measure 0 in $\mathbb{R}^7$. Consequently, if the edge weights of G are selected from $\mathbb{R}^7/\mathcal{C}$, the system is always controllable, and the controllability is 1-node NF. However, for any edge weights, if v_2 and v_3 are both removed, the remaining subsystem is not controllable, which means the system is not 2-nodes NF for any weight combination. According to Definition 4, the topology of system (2) is AS1WPC.

To thoroughly investigate non-fragility of controllability for system (2), we investigate three problems to get the conditions and select proper leaders to preserve controllability.

Problem 1. Controllability preserving conditions: find the necessary and/or sufficient graphic conditions for the interconnection topologies of system (2) to ensure the topologies be ASSPC or ASkWPC.

Problem 2. Leader selection for preserving controllability: find the fewest leaders for the interconnection topology of system (2) to achieve ASSPC and ASkWPC, and show the corresponding locations of the fewest leaders.

Problem 3. Leaders' locations to maximize non-fragility: assume that the maximal number of leaders in system (2) are given, find the locations of the leaders to maximize k, such that the controllability is k-WNF.

3 Main results on preserving controllability

3.1 Controllability preserving conditions

3.1.1 *Single leader case*

As defined in Definition 3, non-fragility is a measure of controllability that how difficult it would be broken. Intuitively, an SNF controllable system is the MAS which can best preserve controllability, i.e., no matter how many followers are removed, the remaining subsystems are still controllable. To investigate controllability preserving, we first propose the condition from a graphic perspective to show what topologies of single leader MASs are ASSPC. A basic lemma is needed.

Lemma 1. Denote the set of all possible weight combinations of G as $\mathfrak{S}$. The set of weight combinations making the system be controllable but not SNF has Lebesgue measure 0 in $\mathfrak{S}$ if and only if all the followers are directly adjacent with the leader.

Proof. Necessity: Suppose that there exists a follower, denoted as v_s, which is not directly adjacent with the leader. When all the other followers are removed, v_s and the leader are not adjacent with each other, therefore, for all edge weights, the system is not SNF, i.e., the set of weight combinations making the controllability not SNF has Lebesgue measure 1 in $\mathfrak{S}$. This makes a contradiction.

Sufficiency: Suppose there are $s \geqslant 0$ edges in G, with the weights $w_1, w_2, \ldots, w_s$. The determinant of the controllability matrix is mathematically a function of $w = (w_1, \ldots, w_s)^{\rm T}$, denoted as $f(w)$. Apparently, the set of $\tilde{w}$ satisfying $f(\tilde{w}) = 0$, denoted as W_0, has Lebesgue measure 0 in $\mathbb{R}^s$. Let $G_1, G_2, \ldots, G_{2^{n-1}}$ be the interconnection topologies of the subsystems of (2) generated by removing the followers. For each G_i, there is an $f_i(w)$ representing the determinant of the controllability matrix, $i = 1, 2, \ldots, 2^{n-1}$ and the sets of $\tilde{w}_i$ satisfying $f_i(\tilde{w}) = 0$ have Lebesgue measure 0 in $\mathbb{R}^s$, denoted as $W_1, W_2, \ldots, W_{2^{n-1}}$ respectively. Since $W = \bigcup_{i=0}^{2^{n-1}} W_i$ also has Lebesgue measure 0 in $\mathbb{R}^s$, we declare that the set of weight combinations making the controlability not SNF has Lebesgue measure 0 in $\mathfrak{S}$.

Theorem 1. The interconnection topology of system (2) is ASSPC if and only if all the followers are directly adjacent with the leader.

Proof. In the following, the set of weight combinations will be called the weight set for short. According to Lemma 1, the weight sets making the controllability not SNF have Lebesgue measure 0 in $\mathfrak{S}$ if and only if all the followers are directly adjacent with the leader. In fact, "the weight sets making the controllability not SNF have Lebesgue measure 0 in $\mathfrak{S}$" means "the weight sets making the controllability SNF have Lebesgue measure 1 in $\mathfrak{S}$". By Definition 4, the latter statement means the topology is ASSPC.

If the topology of system (2) is ASSPC, it does not mean that controllability of the system is always SNF for all the edge weights making system (2) controllable. For example, if the weighted interconnection topology of system (2) is Figure 4, each follower is directly adjacent with the leader agent v_1. When v_4 is removed, the subsystem is not controllable. Refer to the proof of Lemma 1, this situation appears only for a 0 Lebesgue measure set of edge weight combinations of system (2).

Theorem 1 provides a necessary and sufficient graphic condition to check if the controllability of system (2) can be almost surely preserved after any loss of followers. For arbitrary graphs, we generalize the result to weakly preserving controllability, and a property of minimal cutsets is shown in advance. Especially, the concept of (minimal) cutset(s) in the following should not contain the leader.

Lemma 2. For the interconnection topology G of system (2), if the minimal cutset(s) of G contains $k+1$ followers, then, the controllability is k-WNF for almost all sets of edge weights.

Proof. If the minimal cutset(s) of G contains $k+1$ followers, for any $0 \leqslant p \leqslant k$, any choice of p followers removed from G, the remaining subgraph is connected. According to [25], for the remaining subgraph, the sets of edge weights those make the subsystem uncontrollable have Lebesgue measure zero in $\mathbb{R}^{|E|-p}$. Therefore, for the original graph, the sets of edge weights making the subsystem uncontrollable, denoted as W_i, also have Lebesgue measure zero in $\mathbb{R}^{|E|}$, $i = 1, 2, \ldots, C_n^p$. Since $\sum_{p=1}^{k} C_n^p < 2^{|E|} < +\infty$, we get the set $W_u = \bigcup_{p=1}^{k} \bigcup_{i=1}^{C_n^p} W_i$ has Lebesgue measure zero in $\mathbb{R}^{|E|}$, where W_u is the set of edge weights making the system uncontrollable by removing any choice of no more than k followers. This implies that the controllability of system (2) is k-WNF.

Theorem 2. The interconnection topology of system (2) is ASkWPC if and only if the connectivity of G is $k+1$.

Proof. Necessity: If the minimal cutset(s) of G contains $k+2$ (or more) followers, by Lemma 2, the controllability is (at least) $(k+1)$-WNF [2], which contradicts the k-WNF controllability. If the minimal cutset of G only contains k (or less) followers, by Lemma 2, the controllability is (at most) $(k-1)$-WNF,

2) The controllability is $(k+1)$-WNF and may be t-WNF for $t > k+1$.

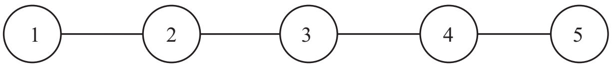

Figure 7 A path with 5 nodes.

which also contradicts the k-WNF controllability. This means the minimal cutset(s) of the interconnection topology contains exactly $k+1$ followers if it is ASkWPC.

Sufficiency: This can be directly derived from Lemma 2.

Corollary 1. The following assertions hold:

(1) The interconnection topology of system (2) is fragile for controllability if and only if the topology contains at least one follower as a cut vertex.

(2) If the controllability of system (2) is k-WNF, there exist at least $k+1$ followers directly adjacent with the leader.

Proof. (1) The first assertion follows directly from Definition 3 and Theorem 2.

(2) According to Theorem 2, the minimal cutset(s) of G contains $k+1$ followers. If there exist only k (or less) followers adjacent with the leader, when these nodes are removed, the rest followers will not be able to recieve information from the leader, which means the remaining subsystem is not controllable. This contradicts the assumption that the system is k-WNF.

Remark 5. Theorems 1 and 2 provide a method to judge non-fragility of controllability of system (2). This method is from a graphic manner, and the probability to successfully judge the non-fragility is 1. For Figure 3, if v_1 is selected as the leader, since $\{v_2, v_4\}$ is the minimal cutset of the graph, by Theorem 2, the topology is AS1WPC; if v_2 is selected as the leader, since every follower is directly adjacent with v_2, by Theorem 1, the topology is ASSPC. For Figure 4, if v_1 is selected as the leader, since every follower is directly adjacent with v_1, the topology is ASSPC; if v_2 is selected as the leader, considering than v_1 is a cut vertex of the graph, we declare that the topology is fragile for controllability.

Apparently, to discuss the controllability preserving problem, system (2) is required to be structurally controllable, i.e., the interconnection topology is connected. However, even if the system is strongly structurally controllable (which means the system is controllable for all sets of edge weights [29]), the topology may also be fragile for controllability. For example, if the interconnection topology of system (2) is a path topology depicted in Figure 7 and v_1 or v_5 is selected as the leader, the system is strongly structurally controllable. However, removing any node of v_2, v_3, v_4 breaks the connectedness of the graph, which means the topology is fragile for controllability.

3.1.2 *Multiple leaders case*

If the interconnection topology G is not connected, the system is impossible to be controllable for any single leader case, let alone to discuss the non-fragility or preserving controllability. When the connected components are more than one in G, we investigate the situation for $m > 1$ leaders in system (2). Be worth mentioning, the results obtained in the subsection also hold for single leader case.

Theorem 3. The interconnection topology of system (2) is ASSPC if and only if all the followers are directly adjacent with at least one leader.

Proof. Necessity: If there exists a follower, denoted as v_s, that is not directly adjacent with any leader. When all the other followers are removed, v_s is isolated from every leader, therefore, for all edge weights, controllability of the system is not SNF. This makes a contradiction.

Sufficiency: Comparing the Laplacian matrix L with A_c, B_c in [25], we get $-L = \left(\begin{smallmatrix} D_c - A_c & -B_c \\ -B_c^{\mathrm{T}} & D_c \end{smallmatrix}\right)$, where $D_c = \mathrm{diag}\{B_c \mathbf{1}_m\}$, and it follows that the controllability of system (2) in this paper is equivalent to that of system (3) in [25]. Since every follower is adjacent with at least one leader, the three assumptions in [25] are satisfied, and system (2) is weight controllable according to Theorem 3 in [25]. This means for almost all sets of edge weights of G, controllability of system (2) is SNF, i.e., the interconnection topology is ASSNF.

Accordingly, the results on weakly preserving controllability are obtained as follows.

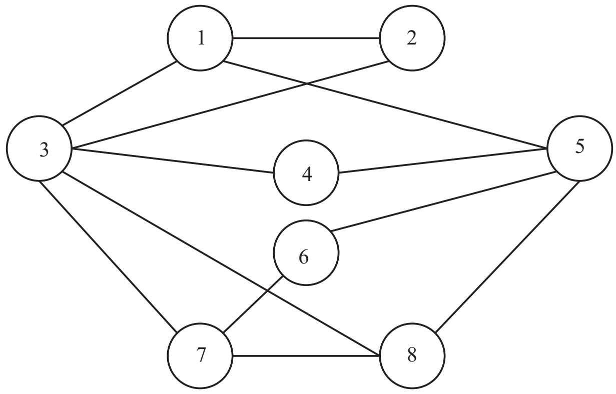

Figure 8 A graph with 8 nodes.

Theorem 4. The interconnection topology of system (2) is ASkWPC if and only if the minimal leader-follower cutset(s) of G contains $k+1$ followers.

Proof. Sufficiency: If the minimal leader-follower cutset(s) of G contains $k+1$ followers, then, for any removal of k followers, the subgraph is always leader-follower connected. This implies that for any removal of k followers, the sets of edge weights making the subsystem uncontrollable have Lebesgue measure 0 in $\mathfrak{S}$. Since the removal of followers has finite choices, we get that the sets of edge weights making any of the subsystems uncontrollable also have Lebesgue measure 0 in $\mathfrak{S}$, i.e., almost all sets of edge weight combinations of system (2) make the controllability k-WNF.

Necessity: If the minimal cutset(s) of G contains $k+2$ (or more) followers, according to the sufficiency proof, the topology is (at least) AS$(k+1)$WPC, which contradicts the assumption. If the minimal leader-follower cutset of G only contains k (or less) followers, then, there exists k followers such that when they are removed, the subsystem is not leader-follower connected. Thus the system is not controllable for any set of edge weights of G, i.e., the topology of system (2) is not ASkWPC.

Corollary 2. The following assertions hold:

(1) The interconnection topology of system (2) is fragile for controllability if and only if the topology contains at least one follower as a leader-follower cut-vertex.

(2) If the controllability of system (2) is k-WNF, there must be at least $k+1$ followers directly adjacent with at least one leader.

Proof. (1) The first assertion follows directly from Definition 3 and Theorem 4.

(2) According to Theorem 4, if system (2) is k-WNF, the minimal cutset(s) of G contains $k+1$ followers. If there exist only k (or less) followers adjacent with the leader, when these nodes are removed, the remaining subgraph is not leader-follower connected, which means the remaining subsystem is not controllable for any set of edge weights. This contradicts the assumption that the system is k-WNF.

Example 4. For system (2) with the interconnection topology depicted in Figure 8, if v_1 and v_7 are selected as leaders, removing any single follower does not break the connectedness, whereas removing v_3 and v_5 makes v_4 be isolated. This means the minimal leader-follower cutset of the topology is $\{v_3, v_5\}$, and the topology is AS1WPC. If we add v_4 into the leader set, each follower is adjacent with at least one leader, i.e., the topology is ASSPC. However, if v_3 and v_5 are selected as leaders, the topology also satisfies that each follower is adjacent with at least one leader, which also makes the topology ASSPC. In the condition that the diameter of Figure 8 is 3, any single leader will make the system have a follower not directly adjacent with the leader. We hence declare that at least two leaders are needed to make the topology of system (2) ASSPC.

3.2 Leader selection for achieving non-fragility

From Subsection 3.1 we can see, for a controllable MAS, if the minimal cutset(s) of G contains only a few agents, the preservation of controllability is weak for the topology. We intend to search for the

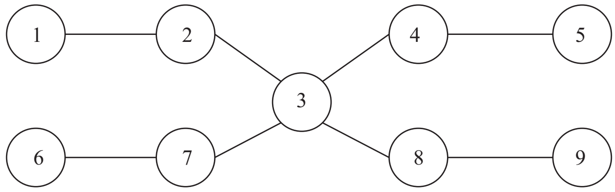

Figure 9 A graph with 9 nodes.

fewst leaders to preserve the controllability of system (2) against losing a number of agents, therefore, the problem of leader selection for preserving controllability is proposed.

In this and the next section, we assume that the interconnection topology of system (2) is connected. Meanwhile, the edge weights of G are supposed to satisfy that, for any selection of leaders, the system is always controllable. We introduce the algorithm to search for the fewest leaders making the topology of system (2) ASSPC. First, the upper and lower bounds of the number of the fewest leaders are estimated.

Proposition 2. Finding the fewest leaders making the interconnection topology of system (2) ASSPC is equal to finding the bipartite spanning subgraph of G with the maximal difference between the cardinality of the two node sets, and selecting the node set with fewer agents as the leader set.

Proof. It is equal to proving that if the node set of the bipartite induced subgraph with the fewer nodes is selected as the leader set, the interconnection topology of system (2) is ASSPC and any fewer leaders can not make the controllability ASSPC.

Feasibility: Since the leader set is the node set of a bipartite induced subgraph of G, we know that each follower is adjacent with at least one leader, which means the topology is ASSPC.

Minimality: Since the difference between the cardinality of the two sets is maximal, it means any fewer nodes cannot constitute a node set of a bipartite induced subgraph of G. Therefore, the node set with the fewer leaders is the leader set with the fewest leaders making the topology of system (2) ASSPC.

Theorem 5. For MAS (2), denote the minimal number of leaders making the interconnection topology ASSPC as l_{m}, then, it holds that

$$\left\lfloor \frac{l}{3} \right\rfloor + 1 \leqslant l_{\mathrm{m}} \leqslant \left\lfloor \frac{n}{2} \right\rfloor, \tag{3}$$

where l is the diameter of G.

Proof. Upper bound: If n is even, the conclusion is obvious; if n is odd, since $l_{\mathrm{m}} \leqslant \frac{n}{2}$ and l_{m} must be an integer, it follows that $l_{\mathrm{m}} \leqslant \lfloor \frac{n}{2} \rfloor$.

Lower bound: Since the diameter of G is l, there exist two nodes v_i, v_j such that the length of the shortest path connecting them is l, and no path of any two nodes in G is longer than l. On this path, each agent is either a leader, or directly adjacent with a leader. Therefore, length of the path between any two followers is no less than 2, and for any two followers on the path, either they are neighbors, or there is a leader adjacent with both of them, which leads to $l \leqslant 3l_{\mathrm{m}}$. This means $\lfloor \frac{l}{3} \rfloor + 1 \leqslant l_{\mathrm{m}}$.

Remark 6. The bounds in Theorem 5 are tight. Several graphs ensure that Eq. (3) holds with equality, such as Figure 8, the diameter of which is 3, needs at least two leaders to make the topology ASSPC. In this circumstance, $l_{\mathrm{m}} = 2 = \lfloor \frac{3}{3} \rfloor + 1$ holds. For Figure 9, at least four leaders are needed to make the topology ASSPC, and $l_{\mathrm{m}} = 4 = \lfloor \frac{9}{2} \rfloor$ holds. For some graphs, both equalities may be satisfied, such as Figure 7, whose length is $l = 4$. Apparently it needs at least 2 leaders, e.g., v_2 and v_4, to make the topology ASSPC. One can check that $\lfloor \frac{5}{3} \rfloor + 1 = 2 = \lfloor \frac{5}{2} \rfloor$, and meanwhile, $\langle \{v_2, v_4\}, \{v_1, v_3, v_5\} \rangle$ constitute a bipartite partition of G.

Next we show Algorithm 1 to search for the fewest leaders making the topology of system (2) ASSPC.

If we do not have enough leaders to make the topology ASSPC, we can consider ASkWPC situations. Before we show the algorithm to search for the fewest leaders, the upper and lower bounds of the fewest number are also estimated.

Theorem 6. The minimal number l_{m} of leaders making the topology of system (2) at least ASkWPC satisfies $l_{\min} \leqslant l_{\mathrm{m}} \leqslant l_{\max}$, where

Algorithm 1 Searching for the fewest leaders making the topology ASSPC

Get the diameter of G, and denote it as l;
for $i = \lfloor \frac{l-1}{3} \rfloor + 1 : \lfloor \frac{n}{2} \rfloor$ **do**
  Get all combinations of i nodes from V, and denote them as $V_1, \ldots, V_{C_n^i}$;
  for $j = 1 : C_n^i$ **do**
    if $\langle V_j, V/V_j \rangle$ constitutes a bipartite graph **then**
      Output "The topology is ASSPC with the minimal leader set V_j" and exit.
    end if
  end for
end for

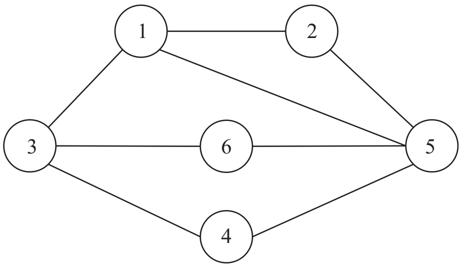

Figure 10 A graph with 6 nodes.

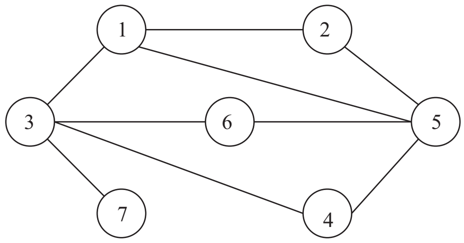

Figure 11 A graph with 7 nodes.

(1) $l_{\min} = 1$ when there only exist k'-node cutsets of G, and $k' \leqslant k$, meanwhile, a leader must be in one of these cutsets; $l_{\min} = 2$ when there exist k'-node cutsets and k''-node cutsets of G, and $k' < k'' \leqslant k$, meanwhile, at least one leader is from one of these cutsets; and

(2) $l_{\max}$ is the cardinality of the set generated by uniting all the k'-node cutsets of G, $1 \leqslant k' \leqslant k$.

Proof. Lower bound: When there only exist k'-node cutsets of G, $k' \leqslant k$, and no leader is from the cutsets, we can see that removing any k'-node cutset makes the system not leader-follower connected. Suppose that there exist k'-node cutsets and k''-node cutsets of G satisfying $k' < k'' \leqslant k$. If the leader of system (2) is single, it must be in one of these cutsets, otherwise, removing any cutset makes the system not leader-follower connected. However, no matter which cutset the leader comes from, there exists another cutset $\tilde{V}$, such that when $\tilde{V}$ is removed, the system becomes not leader-follower connected. Therefore, at least two leaders are needed.

Upper bound: If for all $1 \leqslant k' \leqslant k$, uniting all the k'-node cutsets gets $\tilde{V}$, by Proposition 1, selecting all nodes in $\tilde{V}$ makes G contains no k-node leader-follower cutset, which ensures the topology be ASkWPC.

Example 5. Similarly, the upper and lower bounds in Theorem 6 are also tight. For system (2) with the topology depicted in Figure 10, there are two minimal cutsets of the graph, which are $\{v_1, v_5\}$ and $\{v_3, v_5\}$. If we intend to make the topology AS2WPC, by Theorem 6, $k' = k = 2$, therefore $l_{\min} = 1$. Actually, when v_5 is selected as the single leader, the topology contains no leader-follower cut-vertex and no 2-node leader-follower cutset, i.e., the topology is AS2WPC with only one leader. If the topology of the system is depicted in Figure 11 and we still intend to make the topology AS2WPC, single leader is not enough in the condition that the graph contains both cut vertex v_7 and 2-node cutset $\{v_1, v_5\}$. By Theorem 6, $l_{\min} = 2$ and at least one leader should be v_2 or v_5 or v_7. Proper selections of two leaders making the topology AS2WPC are not unique, however, if none of the two leaders is selected from $\{v_2, v_5, v_7\}$, removing v_7 or $\{v_2, v_5\}$ also breaks the leader-follower connectedness. This example implies that the lower bound in Theorem 6 is tight.

The upper bound is also tight. For system (2) with the topology depicted in Figure 11, there are two cut vertices v_2 and v_3. If we intend to ensure the topology be at least AS1WPC, any single leader is not enough. For example, if v_2 is the single leader, when v_3 ie removed, the topology is not leader-follower connected; if v_3 is the single leader, when v_2 ie removed, v_1 becomes isolated. However, if both v_2 and v_3 are selected as leaders, the topology is ASSPC, which also satisfies the demand to be at least AS1WPC.

Algorithm 2 is to search for the fewest leaders making the topology of system (2) at least ASkWPC.

Algorithm 2 Searching for the fewest leaders making the topology at least ASkWPC

Let $k' = k$;
if no cutset of G contains k or less nodes **then**
    Output "The topology is at least ASk'WPC for any selection of leaders" and exit;
end if
while $k \neq 1$ **do**
    if there exists a k-node cutset of G **then**
        Find all the k-node cutsets, and denote them as $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{s_k}$;
        Break;
    else
        $k = k - 1$;
    end if
end while
for $l_m = 2 : n - 1$ **do**
    Denote the selections of l_m nodes from G as $V_1, \ldots, V_{C_n^{l_m}}$;
    for $j = 1 : C_n^{l_m}$ **do**
        if no leader-follower cutset of G contains k or less nodes **then**
            Output "The topology is ASk'WPC with the leader set V_j" and exit.
        end if
    end for
end for

3.3 Leaders' locations to maximize non-fragility

When the number of leaders is fixed, we intend to make non-fragility of the controllability of system (2) as strong as possible. In addition, if we ensure that the topology is at most ASkWPC with M leaders 3), a question arises that whether the controllability can be preserved with fewer leaders against losing k agents. The next lemma gives the answer.

Proposition 3. If the interconnection topology of system (2) is at most ASkWPC and $k < n - m$, then, for any choice of fewer than m leaders, the topology is at most ASkWPC.

Proof. Assume that there exist $m' < m$ followers to make the topology of system (2) AS$(k+1)$WPC. Since adding leaders will never break the controllability of an MAS, adding any other $m - m'$ followers into the leader set will also make the topology at least AS$(k+1)$WPC. This contradicts the fact that the topology is at most ASkWPC with m leaders.

Remark 7. The condition $k < n - m$ in Lemma 3 is essential. If $k = n - m$, the topology is actually ASSPC, whereas fewer leaders may also make the topology ASSPC, see Example 8.

Based on Proposition 3, Algorithm 3 for leaders' locations to maximize non-fragility is proposed.

Algorithm 3 Leaders' distribution to maximize non-fragility

Get the maximum number of leaders M, find all the M-node subsets of V, and denote them as $V_1, \ldots, V_{C_n^M}$, $\tilde{V} = \emptyset$, $\tilde{k} = 0$;
for $i = 1 : C_n^M$ **do**
    Select V_i as the leader set;
    if the topology is ASkWPC and $k > \tilde{k}$ **then**
        $\tilde{V} = V_i, \tilde{k} = k$;
    end if
end for
if $k == n - M$ **then**
    Run algorithm 1 to search for the fewest leaders making the topology ASSPC, denote the number of the fewest leaders as m, $k = n - m$, and update the leader set $\tilde{V}$;
end if
Output "The topology is at most ASkWPC with no more than M leaders, and achieves ASkWPC with the fewest leaders $\tilde{V}$".

3) The topology is not AStWPC with M leaders for all $k + 1 \leqslant t$.

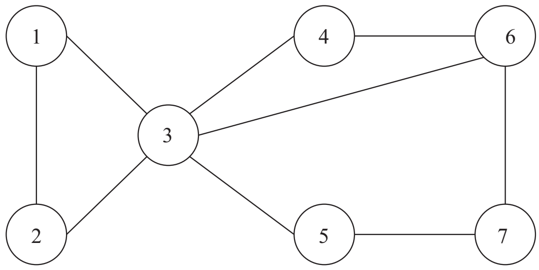

Figure 12 A graph consists of 7 nodes.

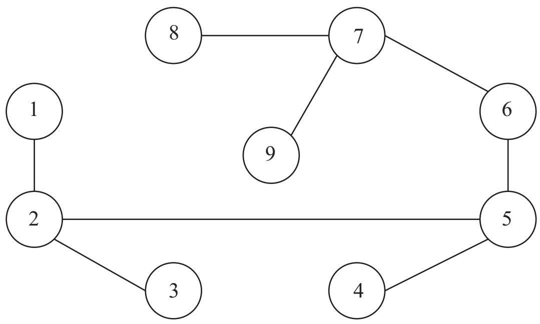

Figure 13 A graph consists of 8 nodes.

4 Examples

In this section, we provide three examples to illuminate how Algorithms 1–3 proceed, respectively.

Example 6. Assume that the interconnection topology of system (2) is depicted as Figure 12. The diameter of the graph is $l = 3$ and $n = 7$, therefore the number of the fewest leaders satisfies $2 \leqslant l_{\mathrm{m}} \leqslant 3$. Select v_3 and v_6 as leaders, and we can see that each follower is either adjacent with v_3 or adjacent with v_6, which means the topology is ASSPC.

Example 7. Assume that the interconnection topology of system (2) is also depicted as Figure 12. If we intend to ensure the topology be AS2WPC, we first search for the cut vertices and 2-node cutsets of G. The 1-node cutset is $\{v_3\}$ and the 2-node cutsets are $\{v_4, v_7\}, \{v_4, v_5\}$. Since both 1-node cutset and 2-node cutsets exist, we need at least two leaders. If we select v_3 and v_4 as leaders, or select v_3 and v_7 as leaders, or select v_1 and v_7 as leaders, or select v_2 and v_7 as leaders, the system will remain leader-follower connected after removing any two followers, i.e., the topology is AS2WPC. On the one hand, not all choices of two nodes from the cutsets are feasible selections. If we select v_3 and v_5 as leaders, the topology is not AS2WPC, owing to that when v_4 and v_7 are removed, the system is not leader-follower connected anymore. On the other hand, feasible selections do not require that both leaders are from the 1-node and 2-node cutsets. However, if we relax the condition to ensuring the topology be AS1WPC, we only need to find the cut vertex of G, which is only v_3. Therefore, selecting v_3 as the single leader makes the topology AS1WPC. In comparison with Example 6, we see that for the topology which is ASkWPC, the smaller k is, the fewer leaders we need, and this also supports the tightness of the lower bound in Theorem 6.

Example 8. Assume that the interconnection topology of system (2) is depicted as Figure 13, and the maximal number of leaders is 3. According to Algorithm 3, step 1, get the $C_9^3 = 84$ selections of three leaders. Step 2, among all these selections, selecting v_2, v_5, v_7 as leaders, which makes the topology ASSPC, i.e., AS6WPC. Step 3, check if there is a selection of two leaders making the topology at least AS6WPC. Obviously, selecting v_2 and v_7 makes the topology also be ASSPC, i.e., AS7WPC, while any single leader can not guarantee this. Step 4, the algorithm outputs: "The topology is at least AS6WPC with no more than 3 leaders, and achieves AS7WPC with the fewest leaders $\{v_2, v_7\}$".

5 Conclusion

Controllability preserving conditions were investigated for MASs under undirected interconnection topologies, as well as algorithms for leader selection to preserve controllability. It was proved that the topology is ASSPC if and only if each follower is directly adjacent with at least one leader, and the necessary and sufficient condition of the topology to be ASkWPC is that the minimal leader-follower cutset(s) of the interconnection graph contains $k+1$ followers. The leader selection problem to achieve ASSPC was equivalently transformed into finding a kind of bipartite spanning subgraph of the interconnection topology. The upper and lower bounds of the fewest leaders ensuring the topology to be ASSPC are limited by $[\frac{l}{3}]$ and $\frac{n}{2}$, respectively, where l is the diameter of G and $n = |V|$. The upper and lower bounds of the fewest leaders ensuring the topology to be ASkWPC are limited by the k'-node cutsets of G, $k' \leqslant k$. Three algorithms for leader selections were proposed to preserve controllability of the system. In the future, we shall discuss non-fragility and controllability preserving for MASs with data sample-based protocols and discrete-time systems.

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