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On the structural controllability of distributed systems with local structure changes

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Abstract This paper analyzes the structural controllability of distributed systems, which are composed of many subsystems and have complicated interconnections. Different from traditional methods in centralized systems where global information is required, the method proposed in this paper is based on local structural properties and simplified interconnections, by which the computational burden is highly decreased and the implementation is tractable. Moreover, a necessary condition for global structural controllability is obtained by combining local information. When the structure in any subsystems is changed, only corresponding local information needs to be re-evaluated instead of whole distributed systems, which makes the analysis easier. Finally, examples are given to illustrate the effectiveness of our proposed method.

Keywords structural controllability, distributed systems, subsystems, structure changes, graph theory

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1 Introduction

According to control theory, a dynamical system is controllable if, with suitable inputs, it can be driven from any initial state to any desired final state within the finite time. In the design phase of an automated system, the system structure tends to be known without some concrete parameters. Considering the structure without concrete parameters, systems can be described in generic representations named structured models. Structural controllability is an important property to be guaranteed by analyzing structured models before the design of system parameters and controllers.

Currently, there are a lot of research methods about structural controllability using algebraic and geometric forms. Ref. [1] firstly introduced the concept of structural controllability, where structured matrices and graphical concept "cactus" are studied to judge structural controllability of single-input linear time-invariant systems. By constructing matrices, the results in [1] were extended to multi-input linear systems [2–4], while the graph method about "cactus" was extended to networks of linear systems in [5]. Based on graphic representations, the structural controllability of a class of switched linear systems was studied in [6]. Ref. [7] proposed a new graph method named "minimum input theorem", which presents a more common case for directed networks. And the results can also be extended to conventional structured automated systems. For example, combined with a graph-theoretic approach named bipartite graphs, impact of the actuators' failures on the structural controllability of linear systems was researched in [8]. Analysis about structural controllability of bilinear systems with a single control and a graphic

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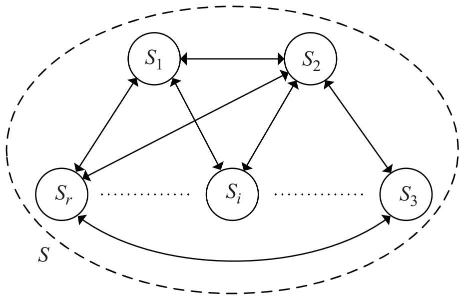

Figure 1 A distributed system.

algorithm to design the location of controllers was presented in [9]. References above mainly focused on structural controllability in centralized form.

As the scale of systems gets larger and larger, distributed systems have received a lot of attention. Based on physical structure and space distribution, the subsystems of distributed systems have been pre-defined. And to the controllability analysis of large-scale systems, the existing results mainly focus on control strategy designs and/or system performance with the assumption that the location of controllers are fixed. Although the fixed controllers can ensure the system structural controllability, the location of them is not necessarily optimal to guarantee the structural controllability [10, 11]. So structural controllability of distributed systems is important to research before study of control strategies. However, few results have been derived so far. Since distributed systems are large scale with complicated interconnections, traditional centralized methods become computationally intractable and need to reanalyze the overall systems when local structure changes [12]. In this paper, a distributed method is proposed to address this requirement and find the optimal location of controllers to guarantee the structural controllability by using the concept of minimum inputs. The proposed method simplifies the interconnections between subsystems. And combining local structural controllability properties of each subsystem and simplified interconnections, global structural controllability properties of a distributed system can be determined. It makes the analysis of global structural controllability for large-scale distributed systems significantly more computationally tractable since systems are split into many smaller subsystems. As the approach developed is constructive, the results allow the analysis of global structural controllability to be updated easily when local structure of the distributed systems changes, through local adjustment of subsystems and interconnections between them.

This paper is organized as follows. Section 2 raises the problem about structural controllability of distributed systems with structure changes, reviews the statements of structural controllability and graph theory (including the minimum input theorem), and provides various notations and necessary definitions. Section 3 develops a graph theory based method to determine a set of minimum control inputs to ensure the global structural controllability of distributed systems and presents solutions to deal with local structure changes. Section 4 gives several simple examples, using a distributed system composed of three subsystems, to show the effectiveness of the proposed method. Finally, conclusion is drawn in Section 5.

2 Preliminaries

A distributed system shown as Figure 1 is composed of many subsystems S_i $(i = 1, 2, \dots,)$ with interconnections. When each time structure changes, existing methods need to reanalyze structural controllability of the whole system, which may be computational intractable due to large scales of distributed systems. Therefore, it is significant to develop a method which can update the analysis using local information when local structure changes instead of re-evaluation of overall.

Before discussing structural controllability, structured systems are chosen as described objects to give out some notations and definitions. These contents will be used in the method to analyze the structural controllability of distributed systems in Section 3.

2.1 System structure

Given a class of systems in the form shown as

$$S : \dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^m$ are the state variables and control inputs, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ represent the dynamics and the control effectiveness, respectively.

The structural controllability normally considers the locations of zero and non-zero entries in the matrices A and B of the systems given by (1).

Their corresponding structured systems are denoted as

$$\bar{S} : \dot{\boldsymbol{x}} = \bar{A}\boldsymbol{x} + \bar{B}\boldsymbol{u}, \tag{2}$$

where structured matrices $\bar{A}, \bar{B}$ are obtained from the matrices A, B of (1) as

$$\bar{a}_{ij} = \begin{cases} \times, & a_{ij} \neq 0, \\ 0, & a_{ij} = 0, \end{cases} \quad \bar{b}_{ij} = \begin{cases} \times, & b_{ij} \neq 0, \\ 0, & b_{ij} = 0. \end{cases}$$

"$\times$" means a free parameter, so $\bar{a}_{ij} = \times$ and $\bar{b}_{ij} = \times$ mean state x_j and input u_j have effect on state x_i, respectively, while "0" means no effect.

System (A, B) has the same structure as the structured system $(\bar{A}, \bar{B})$ if locations of the fixed zero and non-zero entries in A and B are corresponding to those in $\bar{A}$ and $\bar{B}$ [1].

Definition 1 (Structural Controllability [1])**.** A structured system $(\bar{A}, \bar{B})$ is structurally controllable if there exists a concrete system (A, B) of the same structure to be controllable in the usual sense.

Definition 2 (General rank [2])**.** The general rank of a structured matrix M is defined to be the maximum rank that M can attain as a function of the free parameters in M, and denoted by g-rank(M).

g-rank$(\bar{B})$ is equal to the number of control inputs on the structured system $(\bar{A}, \bar{B})$.

Generally, to a given system, structured matrix $\bar{A}$ is fixed, study about structural controllability mainly lies in obtaining a structured matrix $\bar{B}$ with minimum general rank to ensure the structural controllability of the system, i.e., obtaining a set of minimum control inputs.

2.2 Graph theory

Graph theory is intuitive to help to understand the components of structured systems. Each structured system can easily be mapped into a graph, as a collection of two basic elements named vertices and edges.

Definition 3 (Vertices)**.** The vertices represent the state and the control input components of (2). It can also be named as nodes.

Definition 4 (Edges)**.** The directed edges model the static or dynamic relationships between the vertices defined above.

Given two vertices v_i, v_j, $v_i v_j$ denotes the directed edge from v_i to v_j.

Digraph $G_{(\bar{A},\bar{B})}$ is called the graph of the pair $(\bar{A}, \bar{B})$ given by (2), and can be denoted by $G(V, E)$ since constituted by a vertex set V and a directed edge set E. More precisely, $V = X \bigcup U$, where $X = \{x_1, x_2, \ldots, x_n\}$ is the set of state vertices and $U = \{u_1, u_2, \ldots, u_m\}$ is the set of known control input vertices, and $E = \bar{A}$-edges $\bigcup \bar{B}$-edges, with $\bar{A}$-edges $= \{x_j x_i | \forall \bar{a}_{ij} \neq 0, i \neq j\}$ and $\bar{B}$-edges$=$ $\{u_j x_i | \forall \bar{b}_{ij} \neq 0\}$.

Ref. [7] introduced an attempt in investigating the controllability of directed networks, where the proposed "minimum input theorem" is an effective method to determine the needed minimum number of control inputs to ensure the directed networks to be controllable, and can also be applied to structured systems [13, 14].

Definition 5 (Matching set)**.** A set of directed edges between state vertices, denoted as $E*$, is called a matching set, if every pair of edges in $E*$ does not have common starting vertices nor common ending vertices.

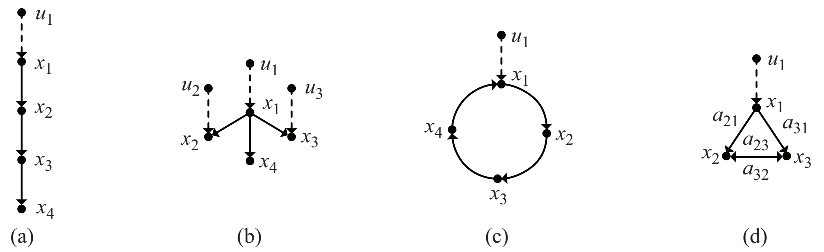

Figure 2 Examples for the minimum input theorem. (a) A chain; (b) a tree; (c) a perfect match; (d) a bidirected edge.

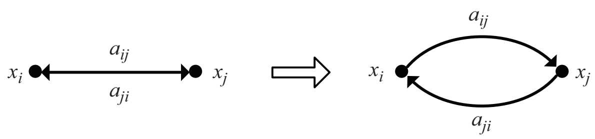

Figure 3 A bidirected edge to two unidirected edges.

Definition 6 (Matched vertex)**.** A vertex is called a matched vertex if it is the end point of an edge in $E*$; otherwise, it is an unmatched vertex.

To a digraph $G(V, E)$, control inputs on all unmatched vertices can ensure it structurally controllable. A matching set with the maximum number of matched vertices is called a maximum matching set, corresponding to minimum control inputs. Of course, a digraph may have more than one maximum matching set, but their numbers of matched vertices are same. Moreover, a matching set is said to be perfect if all vertices of the digraph are matched vertices; thus, the largest possible perfect matching set is the digraph itself.

As shown in Figure 2(a), $\{x_1x_2, x_2x_3, x_3x_4\}$ is the maximum matching set of the system, then state vertex x_1 is the only unmatched vertex, so control input u_1 can ensure the system structurally controllable; in Figure 2(b), x_1x_2, x_1x_3, x_1x_4 have the common starting vertex x_1, so any of them composes the maximum matching set, there need three control inputs to ensure structural controllability [7, 13].

Then, the needed minimum number of control inputs is $N_D = \max\{N - |E*|, 1\}$, where N is the number of all state vertices, $E*$ is a maximum matching set and $|E*|$ is the number of elements in $E*$. If mapping the results to the state space expression, the minimum control inputs mean the structured matrix $\bar{B}$ is with the minimum general rank to ensure structural controllability of the system. Particularly, if a digraph has a perfect matching set, then $N_D = 1$, which means any vertex can be chosen to put a control input, as shown in Figure 2(c); otherwise $N_D = N - |E*|$ is the number of unmatched vertices corresponding to any maximum matching set of the digraph, and the unmatched vertices can be chosen to put control inputs similar to pinning control described in [13, 15, 16].

A path is composed of several edges end to end in sequence, such as $x_1x_2 \to x_2x_3 \to x_3x_4$ in Figure 2(a). In a maximum matching set, each vertex x_i can uniquely trace to an unmatched vertex with a control input (assume to contain the selected vertex with the control input in the case of perfect match) along a path.

When it comes to the condition that there is a couple of a_{ij} and a_{ji} to be non-zero, which means there is a bidirected edge between x_i and x_j. Take an example from the supplementary information, Section III.A of [7] as shown in Figure 2(d). Although this system is uncontrollable in some pathological cases, e.g., those weights satisfy the constraint $a_{32}a_{21}^2 = a_{23}a_{31}^2$ exactly, it is completely controllable for almost all weights combinations. So the system is structurally controllable.

In another way, a bidirected edge can be divided into two unidirected edges with opposite directions shown as Figure 3, where a bidirected edge between x_i and x_j is divided into an edge from x_i to x_j and an edge from x_j to x_i. $\{x_1x_2, x_2x_3\}$ and $\{x_1x_3, x_3x_2\}$ are both the maximum matching sets of the system in Figure 2(d), x_1 is the only unmatched vertex needed to be put a control input to ensure the system structurally controllable. So if there is a bidirected edge, it can be divided into two unidirected edges of opposite directions, then the minimum input theorem still applies.

3 Structural controllability of distributed systems

This paper studies the structural controllability of distributed systems with possible local structure changes. Considering distributed systems are large scale, analysis of the whole system using existing methods is computationally intractable. Besides, if local structure changes, it needs to reanalyze the whole system. In this section, a method using the minimum input theorem is developed to analyze the structural controllability by analyzing each subsystem with local information and then combining the interconnections between subsystems to correct the global property of structural controllability [5, 17].

3.1 Structured distributed systems

Structured distributed systems are of the form shown as

$$S_i : \dot{\boldsymbol{x}}_i = \bar{A}_i \boldsymbol{x}_i + \bar{B}_i \boldsymbol{u}_i + \sum_{j \neq i} \bar{A}_{ij} \boldsymbol{x}_j, \ i = 1, \ldots, r, \tag{3}$$

where $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ and $\boldsymbol{u}_i \in \mathbb{R}^{m_i}$ are the local state variables and local control inputs, $\bar{A}_i \in \mathbb{R}^{n_i \times n_i}$ and $\bar{B}_i \in \mathbb{R}^{n_i \times m_i}$ represent the local structure and the local control effectiveness of subsystem S_i respectively, and $\bar{A}_{ij} \in \mathbb{R}^{n_i \times n_j}$ represents the interconnections from subsystem S_j to S_i.

3.2 Graphical representation

Given a structured distributed system S, graphical form of each subsystem $S_i, \ i = 1, \ldots, r$, can be denoted as $G_i(V_i, E_i, E_i^*)$, where the vertex set $V_i = X_i \bigcup U_i$, with $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$ is the set of state vertices and $U_i = \{u_{i1}, u_{i2}, \ldots, u_{im_i}\}$ is a set of minimum control inputs to be determined, E_i is the set of edges in subsystem S_i and E_i^* is the set of edges from S_i to its neighbor subsystems. X_i and $E_i, \ i \in \{1, \ldots, r\}$ are local information of subsystem S_i, while $E_i^*, \ i \in \{1, \ldots, r\}$ denotes the interconnections from S_i to its neighbor subsystems.

In distributed systems of practical engineering application, considering that subsystems are pre-defined based on physical structure and space distribution, perfect match in the minimum input theorem is incapable to involve several subsystems with edges between them, since cycles are needed to contain all related vertices.

Assumption 1. To a distributed system, a possible perfect matching set may exist in a single subsystem or relate only to two interrelated subsystems.

To a distributed system, if a perfect match originally existed and involved more than two subsystems, assume that involved subsystems are taken as a whole subsystem and the newly constructed distributed system is obtained before analysis of structural controllability.

Definition 7 (Local structural controllability)**.** For a subsystem S_i, according to the given local information X_i and E_i without considering its interconnections with its neighbor subsystems, if it can be structurally controllable with a set of minimum control inputs U_i, then subsystem S_i is said to be of local structural controllability with U_i.

To each subsystem S_i, local structural controllability can be analyzed on the basis of the minimum input theorem. Then its maximum matching set E_i*, corresponding set of matched vertices V_i* and corresponding set of minimum control inputs U_i are obtained.

According to the minimum input theorem, $E* = \{E_i*, \ i = 1, \ldots, r\}$ is a matching set of the structured distributed system S, and $U = \{U_i, \ i = 1, \ldots, r\}$ can ensure the structural controllability of S. However, without taking the interconnections between subsystems into consideration, $E* = \{E_i*, \ i = 1, \ldots, r\}$ is not a maximum matching set, therefore $U = \{U_i, \ i = 1, \ldots, r\}$ is not a set of minimum control inputs to ensure structural controllability of S. It is necessary to analyze the structural controllability of S with minimum control inputs U by combining the interconnections.

Definition 8 (Virtual control input)**.** In each set of minimum control inputs U_i to subsystem S_i, virtual control inputs are those to be redundant when interconnections between subsystems are taken into consideration.

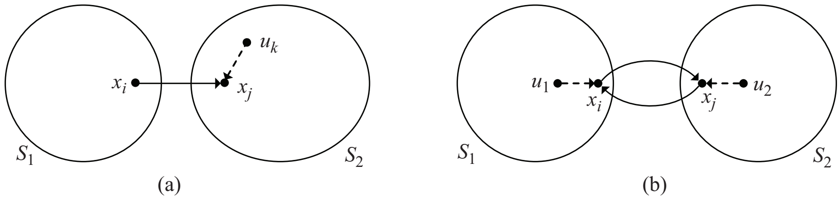

Figure 4 Virtual control inputs. (a) Definition of a virtual control input; (b) a simplest case of perfect match.

Take a distributed system S composed of two subsystems S_1, S_2 shown in Figure 4(a) for instance, there is only one edge $x_i x_j$ between S_1 and S_2. E_1*, E_2* are maximum matching sets of S_1, S_2, and U_1, U_2 are sets of minimum control inputs which can separately ensure the local structural controllability of S_1 and S_2. x_j is an unmatched vertex in S_2, so there is a control input $u_k \in U_2$ acting on x_j. If x_i is not a starting vertex of any edge in E_1*, $E_1* \bigcup E_2* \bigcup \{x_i x_j\}$ is a maximum matching set of S. x_j is a matched vertex of $S_1 \bigcup S_2$ and $U_1 \bigcup \{U_2 \setminus u_k\}$ is a set of minimum control inputs ensuring the structural controllability of distributed system S. Then the control input u_k is redundant and can be removed as a virtual control input.

Remark 1. Note that, edges in maximum matching sets of several subsystems and edges between these subsystems form a cycle like Figure 2(c), it needs to keep one control input since it corresponds to a perfect matching set. Considering Assumption 1, the case may occur in a single subsystem or two neighbor subsystems. To an edge $x_k x_l$ between two subsystems S_i, S_j, assume x_l is an unmatched vertex with a control input. If edges in maximum matching sets E_i*, E_j* and an edge from S_j to S_i, which has been analyzed to remove a control input on its ending point, can form a path from x_l to x_k, the path and the edge $x_k x_l$ form a perfect match. Then the control input on x_l cannot be removed as a virtual control input any more.

Figure 4(b) shows an example about bidirected edges between subsystems, and it is also a simplest example of the above case that a perfect match involves two subsystems. There are only two subsystems S_1 and S_2 in the distributed system, and each subsystem only has one state vertex, so S_1 needs a control input u_1 to confirm structural controllability, while S_2 needs u_2. $x_1 x_2, x_2 x_1$ mean that there is a bidirected edge between S_1 and S_2. If consider $x_1 x_2$ first, u_2 is a virtual control input to be removed. Then $x_2 x_1$ cannot be chosen to remove u_1 as a virtual control input since a path from x_1 to x_2 exists to form a perfect match with $x_2 x_1$.

Based on the above description, after analyzing local structural controllability of subsystems, relevant information needs to be extracted from interconnections between subsystems to find out and remove virtual control inputs. Several definitions of subgraphs are needed before the analysis of global structural controllability in Subsection 3.3.

Definition 9 (Border nodes)**.** Border nodes of subsystem S_i are those state vertices which have edges from/to other state vertices of its neighbor subsystems.

It means that any endpoint of each edge between two subsystems is a border node of its corresponding subsystem.

Definition 10 (Expansion set)**.** The edge $x_k x_l \in E_i^*$, which comes from border node x_k of S_i to border node x_l of S_j neighbor to S_i, can be chosen as an element of the expansion set of S_i, denoted as D_i, if it meets the following conditions.

(1) There is no edge from x_k belonging to the maximum matching set E_i* of S_i, unless the maximum match of S_i is perfect and there is an edge from x_k to the selected vertex with a control input.

(2) Every pair of edges in the expansion set does not have common starting vertices nor common ending vertices.

(3) There is no path from x_l to x_k satisfying that it is formed by edges in E_i*, E_j* and an edge $x_q x_p \in D_j$ with x_p being an unmatch vertex of S_i.

According to the definition, to a subsystem S_i with graphical form $G_i(V_i, E_i, E_i^*)$, its expansion set $D_i \subseteq E_i^*$.

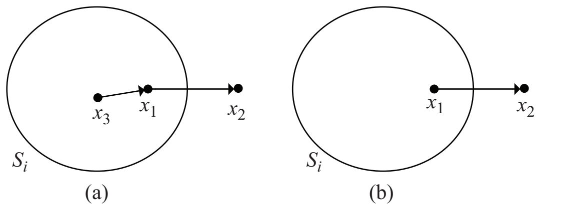

Figure 5 Condition (1) in Definition 10. (a) x_1 is a matched vertex of S_i; (b) x_1 is an unmatched vertex of S_i.

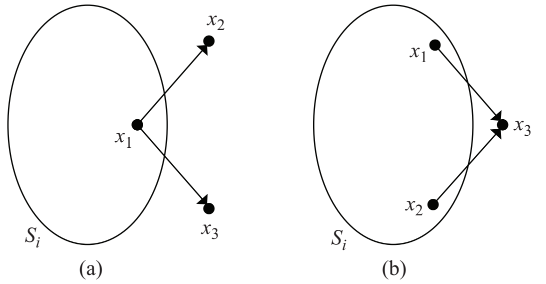

Figure 6 Condition (2) in Definition 10. (a) A common starting vertex; (b) a common ending vertex.

Condition (1) in Definition 10 can be divided into two cases as shown in Figure 5.

(i) x_k is a matched vertex of S_i, i.e., there is an edge from other vertex to x_k belonging to E_i*. As shown in Figure 5(a), there is an edge $x_3x_1 \in E_i*$ ending in x_1. Besides, there is no edge from x_1 to any other vertex in E_i*, so x_1x_2 belongs to the expansion set D_i of S_i. Specially, if S_i has a perfect match and there is an edge in E_i* from x_1 to the selected vertex with a control input, x_1x_2 also belongs to the expansion set D_i.

(ii) x_k is an unmatched vertex of S_i, i.e., there is no edge from other vertices to x_k belonging to E_i*. As shown in Figure 5(b), there is no edge from/to x_1 in E_i*, so x_1x_2 belongs to the expansion set D_i of S_i.

Condition (2) in Definition 10 is as shown in Figure 6. If x_1 in Figure 6(a) meets Conditions (1) and (3), edges x_1x_2 and x_1x_3 have a common starting vertex x_1, so only one of the two edges can be chosen as an element of the expansion set D_i; similarly, if x_1 and x_2 in Figure 6(b) both meet Conditions (1) and (3), edges x_1x_3 and x_2x_3 have a common ending vertex x_3, so only one of them can be chosen as an element of the expansion set D_i.

Note that, since expansion sets are defined to find and remove virtual control inputs as many as possible, there also needs a supplement for Condition (2) of Definition 10: when it comes to the condition in Figure 6(a), the edge, whose ending vertex is an unmatched vertex with a control input (it can be removed as a virtual control input), is preferred to be chosen to the expansion set.

Condition (3) is used to solve the case that there is a perfect match related to two subsystems and edges between them. Similarly to the circumstance in Figure 4(b), if there exist edges in E_i*, E_j* and an edge $x_qx_p \in D_j$ that can form a path from x_l to x_k with x_p being an unmatch vertex of S_i, then x_kx_l cannot be chosen as an element of D_i. It is the case of bidirected edges between subsystems, when x_p is same to x_k and x_q same to x_l.

Given a structured system S shown in Figure 7, denote one maximum matching set of it as E_S* and the corresponding set of minimum control inputs U_S. So it can be structurally controllable with U_S. Then analyze its expansion set D_S. Assume that x_1x_4, x_1x_5, x_2x_6 all meet Conditions (1) and (2) of Definition 10, it is easy to know that $x_2x_6 \in D_S$. The case of x_1 is similar to the condition in Figure 6(a), if x_4 and x_5 are both matched vertices or unmatched vertices, then any of them can be chosen to D_S; if one of them is unmatched vertex, just choose the unmatched one to D_S. Here assumes to choose x_1x_5 to D_S. So its expansion set $D_S = \{x_1x_5, x_2x_6\}$. Definite a new system S^* composed of S and D_S, then $E_S* \bigcup \{x_1x_5, x_2x_6\}$ is a maximum matching set of S^*. So S^* meets the minimum input theorem with U_S, i.e., S^* is still structurally controllable without adding any new control input. This is the meaning of the expansion set.

Since there is the above situation that x_4 and x_5 are both matched vertices or unmatched vertices, the expansion set of a subsystem is not unique. However, expansion sets contain all the useful information of interconnections between subsystems with respect to the structural controllability of distributed systems.

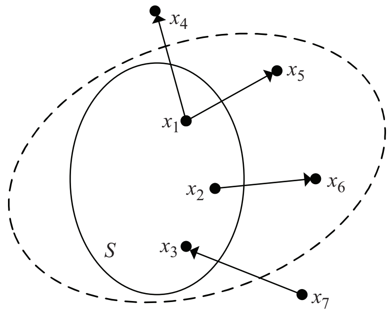

Figure 7 Meaning of the expansion set.

3.3 Global structural controllability

As Assumption 1 is imposed to reflect some physical structure characteristics of distributed systems, structural controllability of distributed systems can be analyzed based on the local structural controllability of subsystems combining with their expansion sets.

The structured distributed system of (3) is of structural controllability equivalent with the condition that every state variable is controllable. On the premise of the condition, the locations of all control inputs acting on the digraph corresponding to (3) need to meet the minimum input theorem.

According to Subsection 3.2, by analyzing the local information of each subsystem S_i using the minimum input theorem, a lot of useful information can be obtained, containing the maximum matching set E_i*, the set of matched vertices V_i* and the set of unmatched vertices $\bar{V}_i*$. Then the set of minimum control inputs U_i acting on $\bar{V}_i*$ separately can be determined to ensure the local structural controllability, where those virtual control inputs need to be determined with the help of the expansion sets.

Combining with the definitions of virtual control input and expansion set, global structural controllability is analyzed as follows.

Theorem 1 (Global structural controllability)**.** The structured distributed system of (3) is of global structural controllability with a set of control inputs as few as possible if
(C1) Each subsystem is local structural controllability with respect to a set of control inputs obtained by the minimum input theorem;
(C2) To each element x_kx_l in the expansion set D_i of every subsystem S_i, if x_l is a vertex with a control input of S_i's neighbor subsystem, then the control input is a virtual control input redundant to be removed.

Proof. To each subsystem S_i, $i = 1, \ldots, r$, analyze its local structural controllability based on the minimum input theorem to get its maximum matching set E_i*, the set of matched vertices V_i*, the set of unmatched vertices $\bar{V}_i*$ and the set of minimum control inputs U_i corresponding to $\bar{V}_i*$, so $V_i* \bigcap \bar{V}_i* = \emptyset, V_i* \bigcup \bar{V}_i* = X_i$, where X_i is the set of all state vertices of S_i.

For i from 1 to r, E_i* belongs to the maximum matching set $E*$ of the distributed system S and V_i* belongs to the set of matched vertices $V*$ of S, then obtain and analyze the expansion set D_i of S_i one after another based on Definition 10. To $\forall x_kx_l \in D_i$ with $x_l \in X_j$, if there is a control input acting on x_l (an unmatched vertex or specially the selected vertex with a control input of a perfect match), this control input is redundant to be removed as a virtual control input, and at the same time, x_kx_l belongs to $E*$, x_l belongs to $V*$. Specially, if there is an edge from x_k to the selected vertex with a control input of a perfect match in S_i, the edge needs to be removed from $E*$, and if x_l is the selected vertex with a control input of a perfect match in S_i's neighbor subsystem, the edge from other vertex to x_l in the perfect match needs be removed from $E*$.

After analyzing the expansion sets of all subsystems, virtual control inputs can be removed as many as possible according to Definition 10, the number of the elements in the matching set $E*$ approaches to be maximum and correspondingly the number of control inputs needed approaches to be minimum.

Although all the control inputs containing virtual control inputs can ensure the distributed system structurally controllable since $\{E_i*,\ i = 1, \ldots, r\}$ is a matching set of the distributed system, Theorem 1 is aimed to remove virtual control inputs as many as possible and get a set of minimum control inputs. During the analysis, $V*$ is consistent with $E*$ all the time, so Theorem 1 satisfies the minimum input theorem. If the determination of E_i* and D_i is proper, the maximum matching set $E*$ of S obtained in Proof at last can meet the minimum input theorem.

However, to any subsystem $S_i,\ i=1,\dots,r$, its maximum matching set E_i* and expansion set D_i may not be unique, although there are several screening conditions in the minimum input theorem and the definition of expansion sets. So the method above is a necessary but not sufficient condition for the global structural controllability.

By this way, structural controllability analysis of a distributed system can be resolved into a series of local structural controllability analysis of subsystems. The basic operation process is given in Algorithm 1 as follows.

Algorithm 1 Distributed algorithm of structural controllability analysis

1: Given a structured distributed system S as in (3) composed of subsystems $S_i,\ i=1,\dots,r$ with graphical form denoted as $G_i(V_i,E_i,E_i^*)$;
2: **for** i from 1 to r **do**
3:   To subsystem S_i, analyze its local structural controllability using the minimum input theorem;
4:   Obtain its maximum matching set E_i*, the set of matched vertices V_i*, the set of unmatched vertices $\bar{V}_i*$, and the corresponding set of control inputs U_i;
5: **end for**
6: **for** i from 1 to r **do**
7:   Determine the expansion set D_i of S_i according to E_i^* containing the interconnection information with its neighbor subsystems;
8:   **for** each $x_kx_l\in D_i$ **do**
9:     If x_l is a vertex with a control input in S_j, the control input on x_l is redundant to be removed as a virtual control input;
10:   **end for**
11: **end for**

In the end, a matching set $E*$ of the structured distributed system S is obtained as large as possible, and correspondingly the set of minimum control inputs U is obtained as small as possible.

Given a structured distributed system S with N vertices and L edges totally, denote that each subsystem S_i $(i=1,\dots,r)$ has N_i vertices, L_{i1} edges in it and L_{i2} edges from it to its neighbor subsystems. So $N=\sum_{i=1}^{r}N_i$ and $L=\sum_{i=1}^{r}(L_{i1}+L_{i2})$. According to [7], line 3 in Algorithm 1 is executed with computation complexity $O(N_i^{1/2}L_{i1})$. Based on Definition 10, line 7 incurs in linear time $O(L_{i2})$, and the for loop (lines 8–10) also runs in at most $O(L_{i2})$ steps. Then the overall complexity of lines 2–11 is $O(\sum_{i=1}^{r}(N_i^{1/2}L_{i1}+L_{i2}))$, no more than $O((\max_{i=1,\dots,r}N_i)^{1/2}\sum_{i=1}^{r}(L_{i1}+L_{i2}))$, i.e., $O((\max_{i=1,\dots,r}N_i)^{1/2}L)$. In conclusion, the complexity of Algorithm 1 is $O((\max_{i=1,\dots,r}N_i)^{1/2}L)$.

Remark 2. Note that, if a distributed system is contrary to Assumption 1, i.e., a perfect match is known to involve several subsystems, take these subsystems and edges between them as a whole. Then a new equivalent distributed system is obtained, where the possible perfect match is local information of the newly constructed subsystem, and Theorem 1 still applies as well as Algorithm 1.

3.4 Solutions to structure changes

This method decomposes a large-scale distributed system into a series of subsystems when analyzing the structural controllability, which guarantees it can deal with local structure changes, such as adding/removing subsystems or some other structure changes [6], without recalculating the controllability properties of the whole system.

To a structured distributed system S of (3), its structure changes can be classified into 4 situations as follows with corresponding solutions extended from the proposed method:

(1) A subsystem S_{r+1} is added to S.

Firstly, analyze the local structural controllability of subsystem S_{r+1}.

Secondly, according to the expansion set D_{r+1} of it, find out and remove those virtual control inputs acting on its neighbor subsystems.

Thirdly, update the expansion sets of its neighbor subsystems and remove the virtual control inputs acting on subsystem S_{r+1} itself.

By reference to this situation, the specific implementation of Algorithm 1 can be in an iterative manner as a process of adding subsystems in sequence to the distributed system, shown as Algorithm 2.

Algorithm 2 Iterative algorithm

1: Given a structured distributed system S as in (3) composed of subsystems S_i $(i = 1, \dots, r)$ with graphical form denoted as $G_i(V_i, E_i, E_i^*)$;
2: To subsystem S_1, analyze its local structural controllability using the minimum input theorem;
3: **for** i from 2 to r **do**
4:     To subsystem S_i, analyze its local structural controllability using the minimum input theorem;
5:     Obtain its maximum matching set E_i*, the set of matched vertices V_i*, the set of unmatched vertices $\bar{V}_i*$ and the corresponding set of control inputs U_i;
6:     Determine its expansion set D_i based on its neighbor subsystems in subsystems $S_j,\ j = 1, \dots, i-1$, then find out and remove virtual control inputs in its neighbor subsystems;
7:     **for** each $S_j (j \in \{1, \dots, i-1\}$ neighbor to S_i **do**
8:         Update the expansion sets of S_j, then find out and remove virtual control inputs in S_i;
9:     **end for**
10: **end for**

Here Algorithm 2 is another executing process of Algorithm 1. They have the same computation complexity, so omitted here for simplicity. And due to difference sequences of subsystems, maybe this algorithm gets different analysis results.

(2) The subsystem $S_j,\ j \in \{1, \dots, r\}$ is removed from S.

According to the expansion set D_j of subsystem S_j, find out those state vertices used to be unmatched vertices in its neighbor subsystems and add control inputs which have been removed as virtual control inputs. Then, correct the expansion sets of S_j's neighbor subsystems. In the end, the remaining distributed system of S is still structural controllability.

Take the distributed system S composed of two subsystems S_1, S_2 shown in Figure 4(a) for instance, when the subsystem S_1 is removed from S, because the state variable x_j used to be an unmatched vertex in S_2, the control input u_k, which used to be a virtual control input, needs to be added to ensure structural controllability of the remaining system.

(3) The structure of subsystem $S_j,\ j \in \{1, \dots, r\}$ changes.

Reanalyze the local structural controllability of subsystem S_j and obtain the new expansion set of it with its corresponding virtual control inputs in its neighbor subsystems. Then correct the expansion sets of its neighbor subsystems and remove virtual control inputs in the new S_j.

This situation can also be understood as a process of removing an original subsystem S_j and adding a new S_j, i.e., a combination of Situations (2) and (1).

(4) The interconnection between two subsystems changes.

Update the expansion sets of the two subsystems and re-determine virtual control inputs in these two subsystems and their neighbor subsystems to be removed.

4 Examples

This section presents some applications of above results about global structural controllability of distributed systems in the following simple examples. Given a distributed system S composed of three subsystems S_1, S_2 and S_3 [5], the system model is

$$S_i : \dot{\boldsymbol{x}}_i = \bar{A}_i \boldsymbol{x}_i + \bar{B}_i \boldsymbol{u}_i + \sum_{j \neq i} \bar{A}_{ij} \boldsymbol{x}_j,\ i = 1, 2, 3,$$

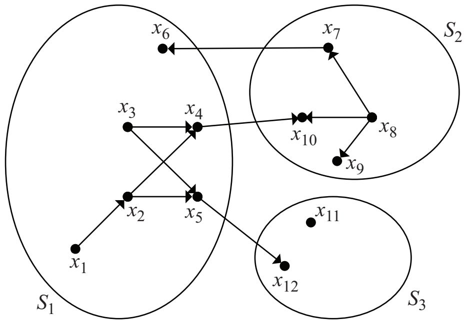

Figure 8 A distributed system composed of three subsystems.

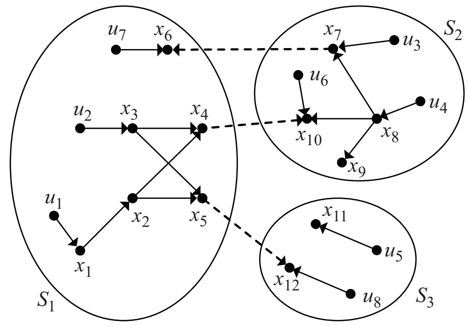

Figure 9 Local structural controllability.

where

$$\bar{A}_1=\begin{bmatrix}0&0&0&0&0&0\\ \times&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&\times&\times&0&0&0\\ 0&\times&\times&0&0&0\\ 0&0&0&0&0&0\end{bmatrix},\quad \bar{A}_{12}=\begin{bmatrix}0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ \times&0&0&0\end{bmatrix},\quad \bar{A}_{13}=\begin{bmatrix}0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\\ 0&0\end{bmatrix},$$

$$\bar{A}_{21}=\begin{bmatrix}0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&\times&0&0\end{bmatrix},\quad \bar{A}_2=\begin{bmatrix}0&\times&0&0\\ 0&0&0&0\\ 0&\times&0&0\\ 0&\times&0&0\end{bmatrix},\quad \bar{A}_{23}=\begin{bmatrix}0&0\\ 0&0\\ 0&0\\ 0&0\end{bmatrix},$$

$$\bar{A}_{31}=\begin{bmatrix}0&0&0&0&0&0\\ 0&0&0&0&\times&0\end{bmatrix},\quad \bar{A}_{32}=\begin{bmatrix}0&0&0&0\\ 0&0&0&0\end{bmatrix},\quad \bar{A}_3=\begin{bmatrix}0&0\\ 0&0\end{bmatrix}.$$

The graphical representation of S is shown as Figure 8. Analysis of structural controllability is to obtain a structured matrix $\bar{B}=\mathrm{diag}(\bar{B}_1,\bar{B}_2,\bar{B}_3)$ with minimum general rank to ensure the structural controllability of S, i.e., to obtain a set of minimum control inputs.

4.1 Global structural controllability

Firstly, analyze local structural controllability of three subsystems without considering the interconnections between them, i.e., the edges x_4x_{10}, x_5x_{12} and x_7x_6 shown as dotted arrows in Figure 9.

Just the same as the condition in Figure 2(b), the maximum matching set of subsystem S_2 only has one edge of $\{x_8x_7, x_8x_{10}, x_8x_9\}$ and corresponding set of matched vertices picks one of $\{x_7, x_{10}, x_9\}$. Then the rest of state vertices need to have control inputs acting on them. Here, to S_2 in Figure 9, the maximum matching set $E_2* = \{x_8x_9\}$, the set of matched vertices $V_2* = \{x_9\}$ and the set of minimum control inputs $U_2 = \{u_3, u_4, u_6\}$ acting on unmatched vertices $\bar{V}_2* = \{x_7, x_8, x_{10}\}$ correspondingly. So subsystem S_2 can be of structural controllability with U_2.

Similarly, subsystems S_1 and S_3 can be of structural controllability separately with the sets of minimum control inputs $\{u_1, u_2, u_7\}$ and $\{u_5, u_8\}$.

Because $\{x_6\}$, $\{x_{10}\}$ and $\{x_{12}\}$ are separately the unmatched vertices of subsystem S_1, S_2 and S_3, the expansion sets of three subsystems respectively are $D_1 = \{x_4x_{10}, x_5x_{12}\}$, $D_2 = \{x_7x_6\}$ and $D_3 = \emptyset$. Then control inputs u_6, u_7, u_8 need to be removed as virtual control inputs when considering global structural controllability of S as shown in Figure 10, where dotted arrows represent the virtual control inputs needed to be removed. In the end, the distributed system S is globally structurally controllable with control inputs $\{u_1, u_2, u_3, u_4, u_5\}$.

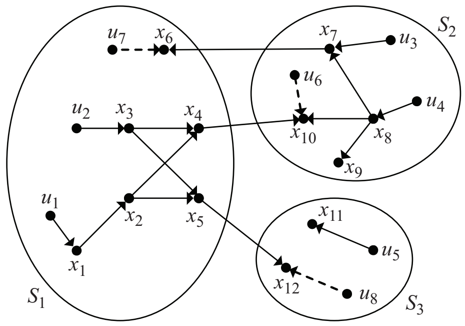

Figure 10 Global structural controllability.

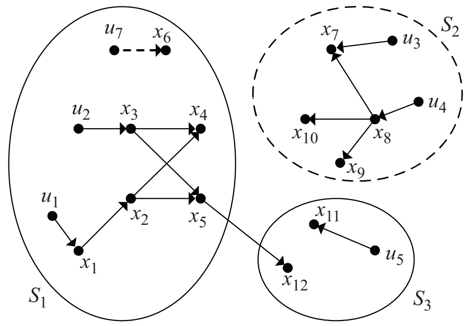

Figure 11 Remove a subsystem.

So the minimum number of control inputs needed is 5. If analyze the structural controllability of S using the minimum input theorem directly, the minimum number of control inputs is also 5. That is to say, the proposed method is effective in this example at least.

4.2 Situations about structure changes

Here just discusses Situation (2) about removing a subsystem from a distributed system described in Subsection 3.4. Situations (1), (3) and (4) can be discussed similarly.

As shown in Figure 11, subsystem S_2 is removed from the system S. The expansion set of S_2 is $D_2 = \{x_7x_6\}$, where x_6 is an unmatched vertex of subsystem S_1, so the control input u_7, which used to be a virtual control input, needs to be added. The expansion set D_1 of subsystem S_1 neighbor to S_2 is corrected from $\{x_4x_{10}, x_5x_{12}\}$ to $\{x_5x_{12}\}$. Then the remaining system of S is still globally structurally controllable with control inputs $\{u_1, u_2, u_5, u_7\}$.

The above examples use only a distributed system with three subsystems and twelve state vertices, which may be not enough to represent the advantage of the proposed method. When it comes to a large-scale distributed system, the proposed method is more computationally tractable and effective than directly analyzing the structural controllability of the whole system.

5 Conclusion

This paper proposes a distributed method to analyze the structural controllability of large-scale distributed systems based on local information. By introducing the definition of virtual control inputs, subsystems' expansion sets and global structural controllability, interconnections between subsystems are simplified and a necessary condition is obtained to analyze global structural controllability. These results provide a computationally tractable method to determine the structural controllability since splitting the whole distributed system into many subsystems. Additionally, the method can be realized in an iterative manner, so it has the capability to analyze the structural controllability of the distributed system with structure changes, such as adding/removing subsystems, without requiring to reanalyze the whole distributed system.

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Conflict of interest The authors declare that they have no conflict of interest.

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