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A computational framework for Karl Popper's logic of scientific discovery

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Abstract Belief revision is both a philosophical and logical problem. From Popper's logic of scientific discovery, we know that revision is ubiquitous in physics and other sciences. The AGM postulates and R-calculus are approaches from logic, where the R-calculus is a Gentzen-type concrete belief revision operator. Because deduction is undecidable in first-order logic, we apply approximate deduction to derive an R-calculus that is computational and has finite injury. We further develop approximation algorithms for SAT problems to derive a feasible R-calculus based on the relation between deduction and satisfiability. In this manner, we provide a full spectrum of belief revision: from philosophical to feasible revision.

Keywords belief revision, logic of scientific discovery, approximate deduction, approximation algorithms, feasible computation

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1 Introduction

Logic of scientific discovery of Popper [1,2] proposed that a line of scientific research begins with a theory that is falsifiable. When an atomic statement that falsifies the theory is found, the theory should be revised rather than completely rejected. Revision is ubiquitous in physics [3]. Classical Newtonian mechanics described the mechanics of gases and the observed motion of the planets. Young's demonstration of the wave nature of light led to the revision of Newton's particle view. This in turn was challenged by Maxwell's electrodynamics and Einstein's theory of special relativity.

The AGM postulates [4–7] constitute an axiomatized approach to the revision $K \circ \varphi$ of a theory K by a formula φ. The DP postulates [8] apply to iterated revision $(\cdots(K \circ \varphi_1) \circ \cdots) \circ \varphi_n$, where $\varphi, \varphi_1, \ldots, \varphi_n$ are formulas, and K is a theory in propositional logic.

Li [9] proposed a belief revision operator called R-calculus $\boldsymbol{R}$, which is a Gentzen-type deduction system for deducing a consistent theory $\Delta \cup \Gamma'$ from an inconsistent theory $\Delta \cup \Gamma$, where $\Delta \cup \Gamma'$ is proved to be the maximally consistent subtheory of $\Delta \cup \Gamma$ that includes Δ as a subset. Here, $\Delta|\Gamma$ is called an R-configuration, Γ is a consistent set of formulas, and Δ is a consistent set of atomic formulas or the negation of atomic formulas. It can be shown that if $\Delta|\Gamma \Rightarrow \Delta|\Gamma'$ is deducible and $\Delta|\Gamma'$ is an R-termination, i.e., no R-rule exists that allows $\Delta|\Gamma'$ to be reduced to another R-configuration $\Delta|\Gamma''$, then $\Gamma' \cup \Delta$ is a contraction of Γ by Δ.

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Because deduction in first-order logic is undecidable and semidecidable, approximate deduction is employed in R-calculus at the cost of finitely many injuries [10–14]. The finite injury priority method from recursion theory (a branch of mathematical logic, which consists of model theory, set theory, recursion theory and proof theory) is used to derive a computational R-calculus $\boldsymbol{R}^{\mathrm{app}}$ [15].

As the computational complexity of deduction in propositional logic is NP-complete, we can apply feasible computation [16–18] to approximate the deduction in $\boldsymbol{R}^{\mathrm{app}}$ and derive a feasible R-calculus $\boldsymbol{R}^{\mathrm{fea}}$ [19, 20].

In this manner, revision can be addressed from philosophical, scientific, logical, computational, and feasibly computational perspectives.

The remainder of this paper is organized as follows. Section 2 introduces Popper's logic of scientific discovery, and discusses revision from the philosophical perspective. Sections 3–6 discuss revision from the scientific, logical, computational, and feasibility perspectives. Section 7 presents our conclusion.

2 Logic of scientific discovery: revision at the philosophical level

Popper's logic of scientific discovery is a theory of revision at the philosophical level. Popper [1, 2] stated as follows:

(1) If we look for confirmations, it is easy to obtain verifications, or confirmations, for nearly every theory.

(2) If they are the result of risky predictions, confirmations should only count. That is, we should have expected an event, which was incompatible with the theory — an event that would have refuted the theory, if unenlightened by the theory in question.

(3) Every 'good' scientific theory is a prohibition, it forbids certain things from happening. The more a theory forbids, the better it is.

(4) A theory, which is not refutable by any conceivable event, is nonscientific. Irrefutability is not of a theory's a virtue (as people often think), but a vice.

(5) Every genuine test of a theory is an attempt to refute it, or to falsify it. Testability is falsifiability, but there are degrees of testability: some theories are more exposed, or more testable to refutation, than others; they take, as it were, greater risks.

(6) When it is the result of a genuine test of the theory, confirming evidence should not count except. It means that it can be presented as a serious but unsuccessful attempt to falsify the theory.

(7) When found to be false, some genuinely testable theories are still upheld by their admirers, for example, by re-interpreting the theory ad hoc, or by introducing ad hoc some auxiliary assumption in such a manner that it escapes refutation. Such a procedure is always possible, but only rescues the theory from refutation at the cost of destroying, or at least lowering, its scientific status. One can sum all of this up by saying that the criterion of the scientific status of a theory is its falsifiability, or testability, or refutability.

Popper proposed the following process of scientific discovery to challenge the view that theories are derived inductively from observed facts:

(1) Induction, i.e., inference based on many observations, is a myth. It is neither a fact of ordinary life, nor a psychological fact, nor one of scientific procedure.

(2) The actual procedure of science is to operate with conjectures: to jump to conclusion — often after a single observation (as noticed, for example, by Hume and Born).

(3) Repeated observations and experiments function in science as tests of our conjectures or hypotheses, that is, as attempted refutations.

(4) The mistaken belief in induction is fortified by the need for a criterion of demarcation, which it is traditionally, but wrongly, believed that only the inductive method can provide.

(5) The conception of such an inductive method, like the criterion of verifiability, implies a faulty demarcation.

(6) If we say that induction makes theories only probable rather than certain, none of this is altered in the least.

The process of building a theory involves the following four steps [9]:

• A scientific theory is a consistent set of laws or principles, each of which is a conjecture extracted from and used for specifying a certain domain of knowledge.

• Every prediction of such a theory is subject to verification by empirical evidence (sometimes called facts for short) from experiments or observations.

• A theory is refuted whenever one of its predictions contradicts the facts produced by experiments or observations.

• In such cases, a new scientific theory will be constructed by revision, by discarding the refuted law and proposing new conjectures.

3 Revision at the scientific level

Revision is ubiquitous in physics and other sciences. An example is the revision of Galilean transformations by Lorentz transformations.

Consider two reference frames S and S'. An event has the space-time coordinates (x, y, z, t) for an observer in frame S and (x', y', z', t') for an observer in frame S'. Assuming that time is measured in the same way in all reference frames, we require that $x = x'$ when $t = 0$, the relation between the space-time coordinates of the same event, which is observed from the reference frames S' and S, which are moving at a relative velocity of u in the x direction, is given by

$$\begin{cases} x' = x - u \cdot t, \\ y' = y, \\ z' = z, \\ t' = t. \end{cases}$$

Galilean transformations have the following consequence:

$$\begin{cases} v_x = v'_x + u, \\ v_y = v'_y, \\ v_z = v'_z, \end{cases} \qquad \begin{cases} a'_x = a_x, \\ a'_y = a_y, \\ a'_z = a_z. \end{cases}$$

That is,

$$\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{u},$$
$$\boldsymbol{a}' = \boldsymbol{a},$$
$$\boldsymbol{F}' = \boldsymbol{F},$$

which means that

• The velocity $\boldsymbol{v}$ of a particle seen from the perspective of S is higher by $\boldsymbol{u}$ than its velocity $\boldsymbol{v}'$ seen from the perspective of S'.

• The acceleration of a particle is the same in any inertial reference frame.

• The force on a particle is the same in any inertial reference frame.

• In classical mechanics, the speed of light is not a constant, and the special status it possesses in relativistic mechanics has no counterpart in classical mechanics.

Einstein's special relativity is based on two postulates:

⋄ The laws of physics are invariant (i.e., identical) in all inertial systems (non-accelerating frames of reference).

⋄ Regardless of the motion of the light source, the speed of light in a vacuum is the same for all observers.

Correspondingly, Galilean transformations are revised to Lorentz transformations:

$$\begin{cases} x = x' + vt' \Big/ \sqrt{1 - \dfrac{v^2}{c^2}}, \\ y = y', \\ z = z', \\ t = t' + \dfrac{v}{c^2}x' \Big/ \sqrt{1 - \dfrac{v^2}{c^2}}. \end{cases}$$

4 Revision at the logical level

Let K be a theory to be revised, and φ a revising formula, in propositional logic. The AGM postulates are as follows.

- Closure: $K \circ \varphi = \mathrm{Cn}(K \circ \varphi)$.
- Success: $\varphi \in \mathrm{Cn}(K \circ \varphi)$.
- Inclusion: $K \circ \varphi \subseteq K + \varphi$.
- Vacuity: if $\neg\varphi \notin K$, then $K \circ \varphi = K + \varphi$.
- Extensionality: $K \circ \varphi$ is consistent if φ is consistent.
- Extensionality: if $(\varphi \leftrightarrow \psi) \in \mathrm{Cn}(\emptyset)$, then $K \circ \varphi = K \circ \psi$.
- Superexpansion: $K \circ (\varphi \wedge \psi) \subseteq (K \circ \varphi) + \psi$.
- Subexpansion: if $\neg\psi \notin \mathrm{Cn}(K \circ \varphi)$ then $(K \circ \varphi) + \psi \subseteq K \circ (\varphi \wedge \psi)$.

When an existing theory is found to contradict certain observed facts, a four-step revision procedure begins:

- The laws that produce the contradiction are discarded.
- A maximal subset of the existing theory that is consistent with the observed facts is formed.
- Those facts that are supported by experiments or observations are extracted and used to derive new laws or principles.
- These new laws or principles are merged with the maximal subset and used to establish a new scientific theory.

As this example shows, scientists confront two issues in their work: the need for a theory to be revised, and the recording of facts from experiments or observations. As we have seen, the scientific theory to be revised can be described by a formal theory (say Γ) of a first-order language L. A fact relevant to the scientific theory and supported by experiments or observations can be specified by a formal theory Δ, comprising the atomic formulas and negations of atomic formulas of L only. The logic is as follows:

- Because of the rapid development of modern sensor technology, almost all the information from experiments or observations may be available in a digitized form.
- This data (perhaps big data) represents some constant, function, or set.
- The facts, which are extracted from the data, can be represented by equations, inequalities, or predicates describing sets of data that share certain common attributes.
- The set of facts, which is denoted by Δ, is therefore a formal theory comprising atomic formulas or negations of atomic formulas.

The R-calculus is a belief-revision operator that has been shown to satisfy all the AGM postulates. Let $\Delta|\Gamma$ be a configuration, where Δ is a set of formulas and Γ is a consistent theory that is to be revised.

The R-calculus $\boldsymbol{R}$ [9] is a Gentzen-type deduction system comprising the following deduction rules:

$$(S^{\rm con})\ \frac{\Delta \nvdash \neg A}{\Delta|A,\Gamma \Rightarrow \Delta, A|\Gamma} \qquad (S^{\neg})\ \frac{\Delta \vdash \neg l}{\Delta|l,\Gamma \Rightarrow \Delta|\Gamma}$$
$$(S_1^{\wedge})\ \frac{\Delta|A_1,\Gamma \Rightarrow \Delta|\Gamma}{\Delta|A_1 \wedge A_2,\Gamma \Rightarrow \Delta|\Gamma} \qquad (S^{\vee})\ \frac{\Delta|A_1,\Gamma \Rightarrow \Delta|\Gamma \quad \Delta|A_2,\Gamma \Rightarrow \Delta|\Gamma}{\Delta|A_1 \vee A_2,\Gamma \Rightarrow \Delta|\Gamma}$$
$$(S_2^{\wedge})\ \frac{\Delta,[A_1]|A_2,\Gamma \Rightarrow \Delta,[A_1]|\Gamma}{\Delta|A_1 \wedge A_2,\Gamma \Rightarrow \Delta|\Gamma}$$
$$(S^{\forall})\ \frac{\Delta|A_1(t),\Gamma \Rightarrow \Delta|\Gamma}{\Delta|\forall x A_1(x),\Gamma \Rightarrow \Delta|\Gamma} \qquad (S^{\exists})\ \frac{\Delta|A_1(x),\Gamma \Rightarrow \Delta|\Gamma}{\Delta|\exists x A_1(x),\Gamma \Rightarrow \Delta|\Gamma}.$$

The semantics is the $\subseteq$-minimal change.

Definition 1. Given any theories Δ and Γ, a theory Θ is a $\subseteq$-minimal change of Γ by Δ, which is denoted by $\models^{\rm FOL}_{\boldsymbol{S}} \Delta|\Gamma \Rightarrow \Delta,\Theta$, if $\Theta \subseteq \Gamma; \Theta$ is consistent with Δ, and any theory Ξ in which $\Theta \subset \Xi \subseteq \Gamma$ is inconsistent with Δ.

Definition 2. $\Delta|\Gamma \Rightarrow \Delta,\Theta$ is provable in $\boldsymbol{S}^{\rm FOL}$, denoted by $\vdash^{\rm FOL}_{\boldsymbol{S}} \Delta|\Gamma \Rightarrow \Delta,\Theta$, if a sequence $\{S_1,\ldots,S_m\}$ of statements exists such that

$$S_1 = \Delta|\Gamma \Rightarrow \Delta_1|\Gamma_1,$$
$$\cdots$$
$$S_m = \Delta_{m-1}|\Gamma_{m-1} \Rightarrow \Delta_m|\Gamma_m = \Delta,\Theta,$$

and for each $i < m, S_{i+1}$ is an axiom or is deduced from the previous statements by a deduction rule in $\boldsymbol{S}^{\rm FOL}$.

Theorem 1 (Soundness theorem)**.** For any consistent formula sets Θ,Δ and any finite consistent formula set Γ, if $\Delta|\Gamma \Rightarrow \Delta,\Theta$ is provable in $\boldsymbol{S}^{\rm FOL}$ and Θ is a $\subseteq$-minimal change of Γ by Δ. That is,

$$\vdash^{\rm FOL}_{\boldsymbol{S}} \Delta|\Gamma \Rightarrow \Delta,\Theta \text{ implies } \models^{\rm FOL}_{\boldsymbol{S}} \Delta|\Gamma \Rightarrow \Delta,\Theta.$$

Theorem 2 (Completeness theorem)**.** For any consistent formula sets Θ,Δ and any finite consistent formula set Γ, if Θ is a $\subseteq$-minimal change of Γ by Δ then $\Delta|\Gamma \Rightarrow \Delta,\Theta$ is provable in $\boldsymbol{S}^{\rm FOL}$. That is,

$$\models^{\rm FOL}_{\boldsymbol{S}} \Delta|\Gamma \Rightarrow \Delta,\Theta \text{ implies } \vdash^{\rm FOL}_{\boldsymbol{S}} \Delta|\Gamma \Rightarrow \Delta,\Theta.$$

5 Revision at the computational level

Because the deduction $\Delta \nvdash \neg A$ is undecidable in first-order logic, we cannot decompose the rule

$$(S^{\rm con})\ \frac{\Delta \nvdash \neg A}{\Delta|A,\Gamma \Rightarrow \Delta, A|\Gamma}$$

into the atomic form

$$(S^{\rm con})'\ \frac{\Delta \nvdash \neg l}{\Delta|l,\Gamma \Rightarrow \Delta, l|\Gamma},$$

as is the case for $\Delta \vdash \neg l$:

$$(S^{\neg})\ \frac{\Delta \vdash \neg l}{\Delta|l,\Gamma \Rightarrow \Delta|\Gamma},$$

because $\Delta \vdash \neg l$ is semi-decidable.

The semi-decidable $\Gamma \vdash A$ can be approximated by the computational deduction $\Gamma \vdash_s A$. Correspondingly, $\Delta|A \Rightarrow \Delta$ is approximated by $\Delta|A \Rightarrow_s \Delta$, implying that we have the following:

	First-order logic	R-calculus
Monotonic in s	$\Delta \vdash_s A$	$\Delta\|A \Rightarrow_s \Delta$
Nonmonotonic in s	$\Delta \nvdash_s A$	$\Delta\|A \Rightarrow_s \Delta, A$

Here, the following hold

- $\Delta \vdash_s A$ is monotonic in s, i.e., for any stage $s, \Delta \vdash_s A$ implies that for any $t \geqslant s, \Delta \vdash_t A$.
- $\Delta \nvdash_s A$ is nonmonotonic in s, i.e., for any stage $s, \Delta \nvdash_s A$ does not imply that for any $t \geqslant s, \Delta \nvdash_t A$.
- $\Delta|A \Rightarrow_s \Delta$ is monotonic in s, i.e., for any stage $s, \Delta|A \Rightarrow_s \Delta$ implies that for any $t \geqslant s, \Delta|A \Rightarrow_t \Delta$.
- $\Delta|A \Rightarrow_s \Delta, A$ is nonmonotonic in s, i.e., for any stage $s, \Delta|A \Rightarrow_s \Delta, A$ does not imply that for any $t \geqslant s, \Delta|A \Rightarrow_t \Delta, A$.

5.1 Approximate deduction $G^{\rm app}$ for first-order logic

The Gentzen-type approximate deduction system $\boldsymbol{G}^{\rm app}$ comprises the following axioms and deduction rules.

- Axioms:

$$\Gamma, A \Rightarrow_0 A, \Delta \ (I).$$

- Deduction rules:

$$
\begin{array}{llll}
(\wedge_1^L) & \dfrac{\Gamma, A \Rightarrow_i \Delta}{\Gamma, A \wedge B \Rightarrow_{i+1} \Delta} & (\wedge^R) & \dfrac{\Gamma \Rightarrow_i A, \Delta \quad \Gamma \Rightarrow_i B, \Delta}{\Gamma \Rightarrow_{i+1} A \wedge B, \Delta} \\
(\wedge_2^L) & \dfrac{\Gamma, B \Rightarrow_i \Delta}{\Gamma, A \wedge B \Rightarrow_{i+1} \Delta} & & \\
(\vee^L) & \dfrac{\Gamma, A \Rightarrow_i \Delta \quad \Gamma, B \Rightarrow_i \Delta}{\Gamma, A \vee B \Rightarrow_{i+1} \Delta} & (\vee_1^R) & \dfrac{\Gamma \Rightarrow_i A, \Delta}{\Gamma \Rightarrow_{i+1} A \vee B, \Delta} \\
 & & (\vee_1^R) & \dfrac{\Gamma \Rightarrow_i B, \Delta}{\Gamma \Rightarrow_{i+1} A \vee B, \Delta} \\
(\neg^L) & \dfrac{\Gamma \Rightarrow_i A, \Delta}{\Gamma, \neg A \Rightarrow_{i+1} \Delta} & (\neg^R) & \dfrac{\Gamma, A \Rightarrow_i \Delta}{\Gamma \Rightarrow_{i+1} \neg A, \Delta} \\
(\forall^L) & \dfrac{\Gamma, A(t) \Rightarrow_i \Delta}{\Gamma, \forall x A(x) \Rightarrow_{i+1} \Delta} & (\forall^R) & \dfrac{\Gamma \Rightarrow_i A(x), \Delta}{\Gamma \Rightarrow_{i+1} \forall x A(x), \Delta} \\
(\exists^L) & \dfrac{\Gamma, A(x) \Rightarrow_i \Delta}{\Gamma, \exists x A(x) \Rightarrow_{i+1} \Delta} & (\exists^R) & \dfrac{\Gamma \Rightarrow_i A(t), \Delta}{\Gamma \Rightarrow_{i+1} \exists x A(x), \Delta}.
\end{array}
$$

where in the $\forall$-rules, t is an arbitrary term, and x does not occur in the lower sequent; and in the $\exists$-rules, x does not occur in the lower sequent, and t is an arbitrary term.

Definition 3. A sequent $\Gamma \Rightarrow \Delta$ is i-deducible, denoted by $\vdash_i \Gamma \Rightarrow \Delta$, if there exists a sequence $\Gamma \Rightarrow_0 \Delta, \ldots, \Gamma_{i+1} \Rightarrow_{i+1} \Delta_{i+1}$ that is a proof in $\boldsymbol{G}^{\rm app}$ and $\Gamma_{i+1} = \Gamma, \Delta_{i+1} = \Delta$.

Proposition 1. (i) For any sequent $\Gamma \Rightarrow \Delta$, if $\vdash_i \Gamma \Rightarrow \Delta$ then $\vdash \Gamma \Rightarrow \Delta$.

(ii) For any sequent $\Gamma \Rightarrow \Delta$, if $\vdash \Gamma \Rightarrow \Delta$, then there exists $i \in \omega$ such that $\vdash_i \Gamma \Rightarrow \Delta$.

5.2 Approximate R-calculus $R^{\rm app}$

Let $\Gamma_{i,s}$ be the formulas in $\Gamma_i = \{A_0, \ldots, A_{i-1}\}$ that are not deleted at the end of stage s, and for each $j < i, A_j \in \Gamma_{i,s}$ iff $\Delta, \Gamma_{i,s} \nvdash \neg A_j$.

The approximate R-calculus $\boldsymbol{R}^{\rm app}$ comprises two sets of rules. One set is designed to eliminate formulas from Γ, and the other to prevent their elimination.

$$(R^{\mathrm{con,app}})\ \frac{\Delta,\Gamma_{i,s}\nvdash_s \neg p}{\Delta,\Gamma_{i,s}|p,\Gamma_{i+2,s}\Rightarrow_i \Delta,\Gamma_{i,s},p|\Gamma_{i+2,s+1}};$$

$$(R_{\wedge,\mathrm{app}})\ \frac{\begin{array}{c}\Delta,\Gamma_{i,s}|A_{i1},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},A_{i1}|\Gamma_{i+2,s}\\ \Delta,\Gamma_{i,s}|A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},A_{i2}|\Gamma_{i+2,s}\end{array}}{\Delta,\Gamma_{i,s}|A_{i1}\wedge A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma_{i+2,s+1}};$$

$$(R^1_{\vee,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}|A_{i1},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},A_{i1}|\Gamma_{i+2,s}}{\Delta,\Gamma_{i,s}|A_{i1}\vee A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma_{i+2,s+1}};$$

$$(R^2_{\vee,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}|A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},A_{i2}|\Gamma_{i+2,s}}{\Delta,\Gamma_{i,s}|A_{i1}\vee A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma_{i+2,s+1}};$$

$$(R^1_{\rightarrow,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}|\neg A_{i1},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},\neg A_{i1}|\Gamma_{i+2,s}}{\Delta,\Gamma_{i,s}|A_{i1}\to A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma_{i+2,s+1}};$$

$$(R^2_{\rightarrow,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}|\neg A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},\neg A_{i2}|\Gamma_{i+2,s}}{\Delta,\Gamma_{i,s}|A_{i1}\to A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma_{i+2,s+1}};$$

$$(R_{\forall,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}|A_i(t),\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},A_i(t)|\Gamma_{i+2,s}}{\Delta,\Gamma_{i,s}|\forall xA_i(x),\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma_{i+2,s+1}};$$

$$(R_{\exists,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}|A_i(x),\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},A_i(x)|\Gamma_{i+2,s}}{\Delta,\Gamma_{i,s}|\exists xA_i(x),\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma_{i+2,s+1}};$$

where $\Gamma_{i,s+1}=\Gamma_{i,s}\cup\{A_i\}$, $A_i=p|\neg p|A_{i1}\wedge A_{i2}|A_{i1}\vee A_{i2}|A_{i1}\to A_{i2}|\forall xA_i(x)|\exists xA_i(x)$, and

$$(R^{\neg,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}\vdash_s \neg p}{\Delta,\Gamma_{i,s}|p,\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma'_{i+2,s+1}};$$

$$(R_1^{\wedge,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}|A_{i1},\Gamma_{i+1,s}\Rightarrow \Delta,\Gamma_{i,s}|\Gamma'_{i+2,s}}{\Delta,\Gamma_{i,s}|A_{i1}\wedge A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma'_{i+2,s+1}};$$

$$(R_2^{\wedge,\mathrm{app}})\ \frac{\begin{array}{c}\Delta,\Gamma_{i,s}|A_{i1},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},A_{i1}|\Gamma_{i+2,s}\\ \Delta,\Gamma_{i,s},A_{i1}|A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s},A_{i1}|\Gamma'_{i+2,s}\end{array}}{\Delta,\Gamma_{i,s}|A_{i1}\wedge A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma'_{i+2,s}};$$

$$(R^{\vee,\mathrm{app}})\ \frac{\begin{array}{c}\Delta,\Gamma_{i,s}|A_{i1},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s}|\Gamma'_{i+2,s}\\ \Delta,\Gamma_{i,s}|A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s}|\Gamma'_{i+2,s}\end{array}}{\Delta,\Gamma_{i,s}|A_{i1}\vee A_{i2},\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma'_{i+2,s+1}};$$

$$(R^{\forall,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}|A_i(t),\Gamma_{i+1,s}\Rightarrow \Delta,\Gamma_{i,s}|\Gamma'_{i+2,s}}{\Delta,\Gamma_{i,s}|\forall xA_i(x),\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma'_{i+2,s+1}};$$

$$(R^{\exists,\mathrm{app}})\ \frac{\Delta,\Gamma_{i,s}|A_i(x),\Gamma_{i+1,s}\Rightarrow \Delta,\Gamma_{i,s}|\Gamma'_{i+2,s}}{\Delta,\Gamma_{i,s}|\exists xA_i(x),\Gamma_{i+2,s}\Rightarrow \Delta,\Gamma_{i,s+1}|\Gamma'_{i+2,s+1}};$$

where $\Delta,\Gamma_{i,s}|A,\Gamma_{i+2,s}$ means that $\Delta,\Gamma_{i,s}$ is s-consistent, and

$$\Gamma_{i,s+1}=\begin{cases}\Gamma_{i,s}\cup\{A_i\}, & \text{if } \Gamma_{i,s}\nvdash_s \neg A_i,\\ \Gamma_{i,s}, & \text{otherwise.}\end{cases}$$

By the soundness and completeness theorems [9] of the R-calculus $\boldsymbol{R}$, for any $i\in\omega$, there is a stage s_i such that

$$\Gamma_{i,s_i}=\lim_{s\to\infty}\Gamma_{i,s},$$

and $\Delta|\Gamma\Rightarrow\Sigma$ is provable in $\boldsymbol{R}^{\mathrm{app}}$, where

$$\Sigma=\Delta\cup\bigcup_i\Gamma_{i,s_i}.$$

Definition 4. $\Delta|\Gamma \Rightarrow \Theta$ is provable in $\boldsymbol{R}^{\mathrm{app}}$, denoted by $\vdash_{\boldsymbol{R}^{\mathrm{app}}} \Delta|\Gamma \Rightarrow \Theta$, if there exists a sequence $\{\Delta_1|\Gamma_1,\ldots,\Delta_n|\Gamma_n,\ldots\}$ such that $\Delta_1|\Gamma_1 = \Delta|\Gamma, \Theta = \lim_{n\to\infty}\Delta_n$, and for each $j < n, \Delta_j|\Gamma_j$ is an axiom or is deduced from the previous statements by the deduction rules in $\boldsymbol{R}^{\mathrm{app}}$.

Therefore, we have the following theorem.

Theorem 3. Given two theories Γ and Δ, $\vdash_{\boldsymbol{R}^{\mathrm{app}}} \Delta|\Gamma \Rightarrow \Sigma$. Conversely, if $\vdash_{\boldsymbol{R}^{\mathrm{app}}} \Delta|\Gamma \Rightarrow \Theta$, then $\Theta = \Sigma$.

6 Revision at the feasible level

In approximation revision, we revise a theory of first-order logic in which quantifiers are less important than they are at the logical level. The theory can therefore be taken as one of propositional logic.

The approximate R-calculus is NP-complete. Using approximate algorithms, we can define a feasible R-calculus, denoted by $\boldsymbol{R}^{\mathrm{fea}}$, which is related to an approximate algorithm for addressing the SAT problem [16].

Definition 5. An optimization problem $P = (I, sol, m, opt)$ is in the class $\mathcal{NPO}$ if

• The set of instances I is recognizable in polynomial time.

• There exists a polynomial q such that given an instance $x \in I$ and for any $y \in sol(x), |y| < q(|x|)$; and for any y with $|y| < q(|x|)$, it can be decided in polynomial time whether $y \in sol(x)$.

• $m(x, y)$ is computable in (deterministic) polynomial time, and

• $opt \in \{\max, \min\}$ specifies whether one has a maximization or minimization problem.

Here, given an instance $x \in I, sol(x)$ is the set of feasible solutions of x, and $m(x, y)$ is the objective function for a feasible solution y of x.

For the SAT problem, the following conditions must be satisfied:

• I represents the sets U of Boolean variables and a collection $C = c_1, \ldots, c_m$ of clauses over U. A set $W = w_1, \ldots, w_m$ of integers (weights) is associated with the clauses.

• Given an instance $x \in I, sol(x)$ is the set of assignments v. Moreover, $|U| < (|x|)$, and it is possible to decide whether a string is a truth-assignment for the formula in polynomial time.

• Given an instance $x \in I$ and a feasible solution y of $x, m(x, y)$ is the sum of the weights associated with the satisfied clauses, and m is trivially computable in polynomial time.

• $opt = \max$.

Given an $\mathcal{NPO}$-problem $P = (I, sol, m, opt)$, if for any given instance $x \in I$, an algorithm A returns an approximation solution, it is an approximation algorithm; that is, a feasible solution $A(x) \in sol(x)$.

Given an $\mathcal{NPO}$-problem P, an instance x, and a feasible solution y, the performance ratio of y can be defined to be

$$R(x, y) = \frac{m(x, y)}{m^\star(x)},$$

where $m^\star(x)$ is the optimal value of the instance x.

In Johnson's approximate algorithm (Algorithm 1), the literal that occurs in the maximum number of clauses is chosen at each step. If the literal is positive, then the corresponding variable is set to true; otherwise, it is set to false. The clauses, which are satisfied by the literal, are deleted from the formula, and when the formula is satisfied or when values have been assigned to all variables, the algorithm stops.

Proposition 2 ([16])**.** The Greedy Johnson algorithm is a polynomial time 1/2-approximate algorithm for MAX-SAT.

By the correspondence between satisfiability and deduction,

$$\Gamma \cup \{A\} \text{ is satisfiable iff } \Gamma \nvdash \neg A,$$
$$\Gamma \cup \{A\} \text{ is app-satisfiable iff } \Gamma \nvdash^{\mathrm{app}} \neg A,$$

providing feasible deduction and feasible revision for theories in propositional logic. Here, for any feasible SAT algorithm $*$, there exists a feasible deduction $\vdash^*$ such that for any theory Γ and formula A,

$$\Gamma \cup \{A\} \text{ is } *\text{-satisfiable iff } \Gamma \nvdash^{*} \neg A,$$

Algorithm 1 Greedy Johnson

Input: Boolean CNF formula $C = \{c_1, \ldots, c_m\}$;
Output: An assignment v;

```
S ← ∅ (S : The set of satisfied clauses);
Left ← C;
V ← {u : u occurs in C};
repeat
Find l with u(l) ∈ V, which is in the maximal number of clauses in Left;
Solve the ties arbitrarily;
Let {c_{n_1}, ..., c_{n_k}} be the clauses in which l occurs;
S ← S ∪ {c_{n_1}, ..., c_{n_k}};
Left ← Left − {c_{n_1}, ..., c_{n_k}};
if l is positive then
    v(l) = 1;
else
    v(¬l) = 0;
end if
V ← V − {u(l)};
until no literal l with u(l) ∈ V is contained in any clause of Left;
if V ≠ ∅ then
    for all u ∈ V do
        v(u) = 1;
    end for
end if
return U;
```

The R-calculus $\boldsymbol{R}^*$ is a Gentzen-type deduction system comprising the following axioms and deduction rules:

$$(*^{\rm con})\ \frac{\Delta \not\vdash^* \neg A}{\Delta|A,\Gamma \Rightarrow \Delta, A|\Gamma} \qquad (*^{\neg})\ \frac{\Delta \vdash^* \neg l}{\Delta|l,\Gamma \Rightarrow \Delta|\Gamma}$$

$$(*^{\wedge}_1)\ \frac{\Delta|A_1,\Gamma \Rightarrow \Delta|\Gamma}{\Delta|A_1 \wedge A_2,\Gamma \Rightarrow \Delta|\Gamma} \qquad (*^{\vee})\ \frac{\Delta|A_1,\Gamma \Rightarrow \Delta|\Gamma \quad \Delta|A_2,\Gamma \Rightarrow \Delta|\Gamma}{\Delta|A_1 \vee A_2,\Gamma \Rightarrow \Delta|\Gamma}$$

$$(*^{\wedge}_2)\ \frac{\Delta,[A_1]|A_2,\Gamma \Rightarrow \Delta,[A_1]|\Gamma}{\Delta|A_1 \wedge A_2,\Gamma \Rightarrow \Delta|\Gamma}$$

$$(*^{\forall})\ \frac{\Delta|A_1(t),\Gamma \Rightarrow \Delta|\Gamma}{\Delta|\forall x A_1(x),\Gamma \Rightarrow \Delta|\Gamma} \qquad (*^{\exists})\ \frac{\Delta|A_1(x),\Gamma \Rightarrow \Delta|\Gamma}{\Delta|\exists x A_1(x),\Gamma \Rightarrow \Delta|\Gamma}.$$

7 Conclusion

The revision of Popper's logic of scientific discovery to a feasible R-calculus $\boldsymbol{R}^{\rm fea}$ is a transition,

- from philosophy to feasible computation;
- from informal theories to formal theories;
- from non-computability to computability.

Hence, we have the following:

$\boldsymbol{R}^{\rm phy}$	$\boldsymbol{R}^{\rm sci}$	$\boldsymbol{R}^{\rm log}$	$\boldsymbol{R}^{\rm app}$	$\boldsymbol{R}^{\rm fea}$
philosophical	||scientific	||logical	||comp	|| feasibly-comp
	informal	||formal		
		non-computational	||comp	||practically-comp

.

Here, $\boldsymbol{R}^{\rm phy}$ and $\boldsymbol{R}^{\rm sci}$ denote revisions at the philosophical and physical levels, respectively.

By applying the finite injury priority method from recursion theory, we provide a recursive construction of Θ such that $\Delta|\Gamma \Rightarrow \Theta$ is provable in R-calculus. This makes the construction oracle recursive, with the cost of finite injuries (some formulas in Γ may be enumerated in Γ or moved out of Γ finitely often).

Therefore, the full spectrum of belief revision comprises $\boldsymbol{R}^{\rm phy}, \boldsymbol{R}^{\rm sci}, \boldsymbol{R}^{\rm log}, \boldsymbol{R}^{\rm app}$, and $\boldsymbol{R}^{\rm fea}$.

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