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# Several classes of negabent functions over finite fields

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## Appendix A

*Proof of Lemma 2:* Suppose that $A = 0$, then we have $\mathrm{Tr}_1^k(cx^{-1}) = 0$ when $\mathrm{Tr}_1^k(bx) = 0$. We also know $\mathrm{Tr}_1^k(bx)$ is linear, that is, $\sharp Support(\mathrm{Tr}_1^k(bx) + 1) = 2^{k-1}$. Thus, we have $\sharp Support(\mathrm{Tr}_1^k(cx^{-1}) + 1) \geqslant 2^{k-1}$. Since $\mathrm{Tr}_1^k(cx^{-1})$ is a balanced function, we obtain $\mathrm{Tr}_1^k(cx^{-1}) = 1$ if $\mathrm{Tr}_1^k(bx) = 1$. From the above discussion, we have $\mathrm{Tr}_1^k(cx^{-1}) = \mathrm{Tr}_1^k(bx)$, i.e., $\sum_{x\in\mathbb{F}_q^*}(-1)^{\mathrm{Tr}_1^k(bx+cx^{-1})} = q > 2\sqrt{q}$ due to $k > 2$, a contradiction with Lemma 1. □

*Proof of Lemma 4:* According to Lemma 3, $f(x)$ is a permutation polynomial over $\mathbb{F}_{2^n}$ if and only if $\gcd(\frac{x^{3k}-1}{x^k-1}, x^n-1) = 1$. Note that $\gcd(\frac{x^{3k}-1}{x^k-1}, x^k-1) = \gcd(3, x^k-1) = 1$. This implies that $\gcd(x^{3k}-1, x^n-1) = \gcd(\frac{x^{3k}-1}{x^k-1}, x^n-1)\cdot\gcd(x^k-1, x^n-1)$ which leads to $\gcd(\frac{x^{3k}-1}{x^k-1}, x^n-1) = \frac{x^{\gcd(n,3k)}-1}{x^{\gcd(n,k)}-1}$. Thus, $f(x)$ is a permutation polynomial over $\mathbb{F}_{2^n}$ if and only if $\gcd(n, 3k) = \gcd(n, k)$.

If $g(x)$ is the compositional inverse of $f(x)$, then we have $g(x)$ is a 2-polynomial over $\mathbb{F}_2$ due to Lemma 3. Moreover, we have $g(1) = 1$ since $f(1) = 1$, i.e., $g(x)$ has odd number of terms. This leads to $\mathrm{Tr}_1^n(g(x)) = \mathrm{Tr}_1^n(x)$ since $g(x)$ is a 2-polynomial over $\mathbb{F}_2$. □

*Proof of Lemma 5:* Note that $f(x)$ is a $2^k$-polynomial over $\mathbb{F}_{2^k}$. Thus the first assert follows directly from Lemma 3. Further, by Lemma 3 we have that $g(x)$ is also a $2^k$-polynomial over $\mathbb{F}_{2^k}$ if $g(x)$ is the compositional inverse of $f(x)$. Suppose that $g(x) = \sum_{i=0}^{r-1} c_i x^{2^{ki}}$, where $c_i \in \mathbb{F}_{2^k}$. Then, we have $\mathrm{Tr}_1^n(g(x)) = \mathrm{Tr}_1^k(\mathrm{Tr}_k^{rk}(g(x))) = \mathrm{Tr}_1^k(\mathrm{Tr}_k^{rk}(\sum_{i=0}^{r-1} c_i x^{2^{ki}})) = \mathrm{Tr}_1^k(\sum_{i=0}^{r-1} c_i \mathrm{Tr}_k^{rk}(x^{2^{ki}})) = \mathrm{Tr}_1^k(g(1)\mathrm{Tr}_k^{rk}(x))$. Then the result follows from the fact that $g(1) = 1$ since $f(1) = 1$. □

## Appendix B

*Proof of Theorem 2:* According to Theorem 1, to complete this proof, it is sufficient to prove that $f(x)+f(x+a)+\mathrm{Tr}_1^n(ax)$ is balanced for all nonzero $a \in \mathbb{F}_{2^n}$ if and only if $\lambda, u, v$ satisfy one of the conditions given in Theorem 2. A direct calculation gives

$$\begin{aligned} f(x) + f(x+a) + \mathrm{Tr}_1^n(ax) &= \mathrm{Tr}_1^k(\lambda(a^{2^k}x + ax^{2^k})) + \mathrm{Tr}_1^n(ua)\mathrm{Tr}_1^n(vx) + \mathrm{Tr}_1^n(va)\mathrm{Tr}_1^n(ux) + \mathrm{Tr}_1^n(ax) \\ &\quad + \mathrm{Tr}_1^k(\lambda a^{2^k+1}) + \mathrm{Tr}_1^n(ua)\mathrm{Tr}_1^n(va) \\ &= \mathrm{Tr}_1^n((\lambda a^{2^k} + a)x) + \mathrm{Tr}_1^n(v\mathrm{Tr}_1^n(ua)x) + \mathrm{Tr}_1^n(u\mathrm{Tr}_1^n(va)x) \\ &\quad + \mathrm{Tr}_1^k(\lambda a^{2^k+1}) + \mathrm{Tr}_1^n(ua)\mathrm{Tr}_1^n(va). \end{aligned}$$

This implies that $f(x) + f(x+a) + \mathrm{Tr}_1^n(ax)$ is balanced if and only if $\lambda a^{2^k} + a + v\mathrm{Tr}_1^n(ua) + u\mathrm{Tr}_1^n(va) \neq 0$. Notice that $\lambda a^{2^k} + a$ is a $2^k$-polynomial and $\gcd(\lambda a^k + 1, a^{2k} + 1) = \gcd(\lambda a^k + 1, (a^k+1)^2) = \gcd(\lambda + 1, a^k + 1) = 1$ only if $\lambda \neq 1$. This together with Lemma 3 shows that $\lambda a^{2^k} + a$ is permutation polynomial over $\mathbb{F}_{2^n}$ if $\lambda \neq 1$. Moreover, for any $\lambda \neq 1$ and $b \in \mathbb{F}_{2^n}$, if $\lambda a^{2^k} + a = b$, then one gets $\lambda a + a^{2^k} = b^{2^k}$ since $n = 2k$ and $\lambda \in \mathbb{F}_{2^k}$. These two identities lead to

$$a = \frac{b + \lambda b^{2^k}}{\lambda^2 + 1}, \tag{B1}$$

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which is the unique solution to $\lambda a^{2^k}+a=b$.

For simplicity, define $h(a)=\lambda a^{2^k}+a+v\mathrm{Tr}_1^n(ua)+u\mathrm{Tr}_1^n(va)$. Then by (B1), for $\lambda\neq 1$ we have

1) $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(0,0)$: For this case, $h(a)=0$ has the only solution $a=0$.

2) $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(0,1)$: By (B1), $a=\frac{u+\lambda u^{2^k}}{\lambda^2+1}$ is the unique solution to $\lambda a^{2^k}+a+u=0$. Note that $\mathrm{Tr}_1^n(ua)=\mathrm{Tr}_1^n(u\cdot\frac{\lambda u^{2^k}+u}{1+\lambda^2})=\mathrm{Tr}_1^n(\frac{\lambda u^{2^k+1}}{1+\lambda^2})+\mathrm{Tr}_1^n(\frac{u^2}{1+\lambda^2})=\mathrm{Tr}_1^n(\frac{u}{1+\lambda})$ since $n=2k$ and $\frac{\lambda u^{2^k+1}}{1+\lambda^2}\in\mathbb{F}_{2^k}$. Thus, in this case $h(a)=0$ has the only solution $a=\frac{u+\lambda u^{2^k}}{\lambda^2+1}$ if and only if $\mathrm{Tr}_1^n(\frac{u}{1+\lambda})=0$ and $\mathrm{Tr}_1^n(va)=\mathrm{Tr}_1^n(v\cdot\frac{\lambda u^{2^k}+u}{1+\lambda^2})=1$.

3) $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(1,0)$: Similar as above, for this case $h(a)=0$ has the only solution $a=\frac{v+\lambda v^{2^k}}{\lambda^2+1}$ if and only if $\mathrm{Tr}_1^n(\frac{v}{1+\lambda})=0$ and $\mathrm{Tr}_1^n(ua)=\mathrm{Tr}_1^n(u\cdot\frac{\lambda v^{2^k}+v}{1+\lambda^2})=1$.

4) $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(1,1)$: In this case, $a=\frac{u+v+\lambda(u+v)^{2^k}}{\lambda^2+1}$ is the unique solution to $\lambda a^{2^k}+a+u+v=0$ due to (B1). By the same techniques used in Cases 2) and 3) one can conclude that $h(a)=0$ has the only solution if and only if $\mathrm{Tr}_1^n(\frac{u}{1+\lambda}+\frac{(\lambda v^{2^k}+v)u}{1+\lambda^2})=1$ and $\mathrm{Tr}_1^n(\frac{v}{1+\lambda}+\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2})=1$.

Notice that $\mathrm{Tr}_1^n(\frac{(\lambda v^{2^k}+v)u}{1+\lambda^2})=\mathrm{Tr}_1^n(\frac{(\lambda vu^{2^k})^{2^k}}{(1+\lambda^2)^{2^k}})+\mathrm{Tr}_1^n(\frac{vu}{1+\lambda^2})=\mathrm{Tr}_1^n(\frac{\lambda vu^{2^k}}{1+\lambda^2})+\mathrm{Tr}_1^n(\frac{vu}{1+\lambda^2})=\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2})$ due to $n=2k$ and $\lambda\in\mathbb{F}_{2^k}$. Thus, Case 2) shows that $(\mathrm{Tr}_1^n(\frac{u}{1+\lambda}),\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2}),\mathrm{Tr}_1^n(\frac{v}{1+\lambda}))\notin\{(0,1,0),(0,1,1)\}$, Case 3) shows that $(\mathrm{Tr}_1^n(\frac{u}{1+\lambda}),\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2}),\mathrm{Tr}_1^n(\frac{v}{1+\lambda}))\notin\{(0,1,0),(1,1,0)\}$, and Case 4) shows that $(\mathrm{Tr}_1^n(\frac{u}{1+\lambda}),\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2}),\mathrm{Tr}_1^n(\frac{v}{1+\lambda}))\notin\{(0,1,0),(1,0,1)\}$. Therefore, if $\lambda\neq 1$, one has that $h(a)=\lambda a^{2^k}+a+v\mathrm{Tr}_1^n(ua)+u\mathrm{Tr}_1^n(va)\neq 0$ for any nonzero $a\in\mathbb{F}_{2^n}$ if and only if the first condition in Theorem 2 is satisfied.

Now we consider the case of $\lambda=1$. First we discuss the number of solutions of $h(a)=\lambda a^{2^k}+a+v\mathrm{Tr}_1^n(ua)+u\mathrm{Tr}_1^n(va)$ under the condition $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(0,0)$. In this case, $h(a)=0$ is equivalent to $a\in\mathbb{F}_{2^k}$. Let $N(u,v)$ denote the number of nonzero $a\in\mathbb{F}_{2^k}$ such that $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(0,0)$, where $(u,v)\in\mathbb{F}_{2^n}^*\times\mathbb{F}_{2^n}^*$. Then, according to the balanced property of the trace function and the fact that $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(\mathrm{Tr}_1^k(a(u+u^{2^k})),\mathrm{Tr}_1^k(a(v+v^{2^k})))$, it can be readily verified that $N(u,v)=2^k-1$ if $u,v\in\mathbb{F}_{2^k}$, $N(u,v)=2^{k-1}-1$ if exactly one of $u,v$ belongs to $\mathbb{F}_{2^k}$, $N(u,v)=2^{k-1}-1$ if $u,v\notin\mathbb{F}_{2^k}$ with $u+v\in\mathbb{F}_{2^k}$ and $N(u,v)=2^{k-2}-1$ if $u,v,u+v\notin\mathbb{F}_{2^k}$ respectively. This implies that $h(a)=0$ under the condition $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(0,0)$ has at least one nonzero solution for any given $u,v\in\mathbb{F}_{2^n}$ if $k>2$, i.e., $f(x)$ cannot be negabent if $\lambda=1$ and $k>2$. The conditions on $u,v\in\mathbb{F}_{2^n}$ such that $f(x)$ is negabent for $k=1,2$ can be easily verified based on a simple discussion. This completes the proof. □

*Proof of Corollary 1:*

We only give the proof for $\lambda\neq 1$ since the proof for $\lambda=1$ is trivial due to Theorem 2. For $\lambda\neq 1$, we first determine the number of ordered pairs $(u,v)$ such that $(\mathrm{Tr}_1^n(\frac{u}{1+\lambda}),\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2}),\mathrm{Tr}_1^n(\frac{v}{1+\lambda}))\in\{(0,0,0),(0,0,1)\}$. Note that $(\mathrm{Tr}_1^n(\frac{u}{1+\lambda}),\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2}),\mathrm{Tr}_1^n(\frac{v}{1+\lambda}))\in\{(0,0,0),(0,0,1)\}$ is equivalent to $(\mathrm{Tr}_1^n(\frac{u}{1+\lambda}),\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2}))=(0,0)$. Clearly, the number of $u\in\mathbb{F}_{2^n}^*$ satisfying $\mathrm{Tr}_1^n(\frac{u}{1+\lambda})=0$ is $2^{n-1}-1$, and for each such $u$, there are $2^{n-1}-2$ $v$'s in $\mathbb{F}_{2^n}^*\setminus\{u\}$ such that $\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2})=0$ (notice that $\lambda x^{2^k}+x$ is a permutation polynomial over $\mathbb{F}_{2^n}$ due to $\lambda\neq 1$, then $\lambda u^{2^k}+u\neq 0$ if $u\neq 0$). Thus, in this case we get $(2^{n-1}-1)(2^{n-1}-2)$ ordered pairs $(u,v)$ such that $f(x)$ is negabent.

Next we count the number of the pairs $(u,v)$ such that $(\mathrm{Tr}_1^n(\frac{u}{1+\lambda}),\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2}),\mathrm{Tr}_1^n(\frac{v}{1+\lambda}))\in\{(1,0,0),(1,1,1)\}$, which is equivalent to counting the number of the pairs $(u,v)$ satisfying $\mathrm{Tr}_1^n(\frac{u}{1+\lambda})=1$ and $\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u)v}{1+\lambda^2})+\mathrm{Tr}_1^n(\frac{v}{1+\lambda})=\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u+1+\lambda)v}{1+\lambda^2})=0$. Similar as above, for this case the number of $u\in\mathbb{F}_{2^n}^*$ satisfying $\mathrm{Tr}_1^n(\frac{u}{1+\lambda})=1$ is $2^{n-1}$, and for each such $u$ (notice that $u\neq 1$ since $\mathrm{Tr}_1^n(\frac{1}{1+\lambda})=0$), there are $2^{n-1}-2$ $v$'s in $\mathbb{F}_{2^n}^*\setminus\{u\}$ such that $\mathrm{Tr}_1^n(\frac{(\lambda u^{2^k}+u+1+\lambda)v}{1+\lambda^2})=0$ (notice that $\lambda x^{2^k}+x+1+\lambda$ is a permutation polynomial over $\mathbb{F}_{2^n}$ due to $\lambda\neq 1$, thus $\lambda u^{2^k}+u+1+\lambda\neq 0$ since $u\neq 1$), i.e., we have $2^{n-1}(2^{n-1}-2)$ ordered pairs $(u,v)$ such that $f(x)$ is negabent. □

*Proof of Theorem 3:*

According to Theorem 1, it is sufficient to prove that

$$f(x)+f(x+a)+\mathrm{Tr}_1^n(ax)=\mathrm{Tr}_1^n\Big((\mathrm{Tr}_1^n(va)u+\mathrm{Tr}_1^n(ua)v+a)x\Big)+\mathrm{Tr}_1^n(ua)\mathrm{Tr}_1^n(va)$$

is balanced for all nonzero $a\in\mathbb{F}_{2^n}$, which is equivalent to show that $\mathrm{Tr}_1^n(va)u+\mathrm{Tr}_1^n(ua)v+a\neq 0$ for all nonzero $a$. Let $h(a)=\mathrm{Tr}_1^n(va)u+\mathrm{Tr}_1^n(ua)v+a$, we have

1) $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(0,0)$: For this case, $h(a)=0$ has the only solution $a=0$.

2) $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(0,1)$: In this case, $h(a)=0$ has the only solution $a=u$ if and only if $\mathrm{Tr}_1^n(u)=0$ and $\mathrm{Tr}_1^n(uv)=1$.

3) $(\mathrm{Tr}_1^n(ua),\mathrm{Tr}_1^n(va))=(1,0)$: Similar as above, for this case $h(a)=0$ has the only solution $a=v$ if and only if $\mathrm{Tr}_1^n(uv)=1$ and $\mathrm{Tr}_1^n(v)=0$.

4) $(\mathrm{Tr}_1^n(ua), \mathrm{Tr}_1^n(va)) = (1,1)$: In this case, $a = u + v$ is the only solution to $\mathrm{Tr}_1^n(va)u + \mathrm{Tr}_1^n(ua)v + a = 0$ if and only if $\mathrm{Tr}_1^n(u(u+v)) = 1$ and $\mathrm{Tr}_1^n(v(u+v)) = 1$.

Based on Cases 1)-4), it can be seen that $\mathrm{Tr}_1^n(va)u + \mathrm{Tr}_1^n(ua)v + a \neq 0$ for all nonzero $a$ if and only if one of the two conditions in Theorem 3 is satisfied. □

*Proof of Theorem 4:*

According to Theorem 1, we only need to show that $f(x) + f(x+a) + \mathrm{Tr}_1^n(ax)$ is balanced for all nonzero $a \in \mathbb{F}_{2^n}$ if $\mathrm{Tr}_1^n(v) = 0$. A direct calculation gives

$$\begin{aligned} f(x) + f(x+a) + \mathrm{Tr}_1^n(ax) &= \mathrm{Tr}_1^n(a^{2^k}x + ax^{2^k}) + \mathrm{Tr}_1^n(a)\mathrm{Tr}_1^n(vx) + \mathrm{Tr}_1^n(va)\mathrm{Tr}_1^n(x) + \mathrm{Tr}_1^n(ax) \\ &\quad + \mathrm{Tr}_1^n(a^{2^k+1}) + \mathrm{Tr}_1^n(a)\mathrm{Tr}_1^n(va) \\ &= \mathrm{Tr}_1^n((a^{2^k} + a^{2^{-k}} + a)x) + \mathrm{Tr}_1^n((v\mathrm{Tr}_1^n(a))x) + \mathrm{Tr}_1^n(\mathrm{Tr}_1^n(va)x) \\ &\quad + \mathrm{Tr}_1^n(a^{2^k+1}) + \mathrm{Tr}_1^n(a)\mathrm{Tr}_1^n(va). \end{aligned}$$

This shows that $f(x) + f(x+a) + \mathrm{Tr}_1^n(ax)$ is balanced if and only if $a^{2^k} + a^{2^{-k}} + a + v\mathrm{Tr}_1^n(a) + \mathrm{Tr}_1^n(va) \neq 0$, i.e., $a + a^{2^k} + a^{2^{2k}} + v^{2^k}\mathrm{Tr}_1^n(a) + \mathrm{Tr}_1^n(va) \neq 0$. Notice that $a + a^{2^k} + a^{2^{2k}}$ is a permutation of $\mathbb{F}_{2^n}$ due to Lemma 4. Let $g(a) = a + a^{2^k} + a^{2^{2k}} + v^{2^k}\mathrm{Tr}_1^n(a) + \mathrm{Tr}_1^n(va)$ and $h(a)$ be the compositional inverse of $a + a^{2^k} + a^{2^{2k}}$, then we have

1) $(\mathrm{Tr}_1^n(a), \mathrm{Tr}_1^n(va)) = (0,0)$: For this case, $g(a) = 0$ has the only solution $a = 0$.

2) $(\mathrm{Tr}_1^n(a), \mathrm{Tr}_1^n(va)) = (0,1)$: In this case, $g(a) = 0$ means that $a + a^{2^k} + a^{2^{2k}} = 1$, i.e., $a = h(1) = 1$. However, $\mathrm{Tr}_1^n(va) = \mathrm{Tr}_1^n(v) = 0$, which shows that $g(a) = 0$ has no solution in this case.

3) $(\mathrm{Tr}_1^n(a), \mathrm{Tr}_1^n(va)) = (1,0)$: In this case, $g(a) = 0$ is reduced to $a + a^{2^k} + a^{2^{2k}} = v^{2^k}$, i.e., $a = h(v^{2^k})$. However, by Lemma 4, $\mathrm{Tr}_1^n(a) = \mathrm{Tr}_1^n(h(v^{2^k})) = \mathrm{Tr}_1^n(v^{2^k}) = \mathrm{Tr}_1^n(v) = 0$. This shows that $g(a) = 0$ has no solution in this case.

4) $(\mathrm{Tr}_1^n(a), \mathrm{Tr}_1^n(va)) = (1,1)$: Similar as above, $g(a) = 0$ implies that $a + a^{2^k} + a^{2^{2k}} = 1 + v^{2^k}$, i.e., $a = h(1 + v^{2^k})$. Note that $\mathrm{Tr}_1^n(1) = 0$ since $n$ is even. From Lemma 4, $\mathrm{Tr}_1^n(a) = \mathrm{Tr}_1^n(h(1 + v^{2^k})) = \mathrm{Tr}_1^n(1 + v^{2^k}) = \mathrm{Tr}_1^n(v) = 0$, which shows that $g(a) = 0$ has no solution in this case.

From the above Cases 1)-4), we can see that $a + a^{2^k} + a^{2^{2k}} + v^{2^k}\mathrm{Tr}_1^n(a) + \mathrm{Tr}_1^n(va) \neq 0$ for all $a \in \mathbb{F}_{2^n}^*$ if $\mathrm{Tr}_1^n(v) = 0$. □

*Proof of Theorem 5:*

According to Theorem 1, it is enough to prove that $f(x) + f(x+a) + \mathrm{Tr}_1^n(ax)$ is balanced for all nonzero $a \in \mathbb{F}_{2^n}$ for the $v \in \mathbb{F}_{2^n}$ satisfying $\mathrm{Tr}_1^n(v) = 0$. Note that

$$\begin{aligned} f(x) + f(x+a) + \mathrm{Tr}_1^n(ax) &= \mathrm{Tr}_1^n(\lambda(a^{2^k}x + ax^{2^k})) + \mathrm{Tr}_1^n(a)\mathrm{Tr}_1^n(vx) + \mathrm{Tr}_1^n(va)\mathrm{Tr}_1^n(x) + \mathrm{Tr}_1^n(ax) \\ &\quad + \mathrm{Tr}_1^n(\lambda a^{2^k+1}) + \mathrm{Tr}_1^n(a)\mathrm{Tr}_1^n(va) \\ &= \mathrm{Tr}_1^n((\lambda a^{2^k} + (\lambda a)^{2^{-k}} + a)x) + \mathrm{Tr}_1^n(v\mathrm{Tr}_1^n(a)x) + \mathrm{Tr}_1^n(\mathrm{Tr}_1^n(va)x) \\ &\quad + \mathrm{Tr}_1^n(\lambda a^{2^k+1}) + \mathrm{Tr}_1^n(a)\mathrm{Tr}_1^n(va). \end{aligned}$$

Thus, $f(x) + f(x+a) + \mathrm{Tr}_1^n(ax)$ is balanced if and only if

$$\lambda a^{2^k} + (\lambda a)^{2^{-k}} + a + v\mathrm{Tr}_1^n(a) + \mathrm{Tr}_1^n(va) \neq 0. \tag{B2}$$

Raising both sides of (B2) to the $2^k$-th power, we get $\lambda a^{2^{2k}} + \lambda a + a^{2^k} + v^{2^k}\mathrm{Tr}_1^n(a) + \mathrm{Tr}_1^n(va) \neq 0$ due to $\lambda \in \mathbb{F}_{2^k}$. Let $g(a) = \lambda a + a^{2^k} + \lambda a^{2^{2k}} + v^{2^k}\mathrm{Tr}_1^n(a) + \mathrm{Tr}_1^n(va)$. According to Lemma 5, $\lambda a + a^{2^k} + \lambda a^{2^{2k}}$ is a permutation of $\mathbb{F}_{2^n}$ since $\gcd(\lambda + x + \lambda x^2, x^r - 1) = 1$. Let $h(a)$ be the compositional inverse of $\lambda a + a^{2^k} + \lambda a^{2^{2k}}$. Similar as in the proof of Theorem 4, we have

1) $(\mathrm{Tr}_1^n(a), \mathrm{Tr}_1^n(va)) = (0,0)$: For this case, $g(a) = 0$ has the only solution $a = 0$.

2) $(\mathrm{Tr}_1^n(a), \mathrm{Tr}_1^n(va)) = (0,1)$: In this case, $g(a) = 0$ means that $\lambda a + a^{2^k} + \lambda a^{2^{2k}} = 1$, i.e., $a = h(1) = 1$ since $\lambda \cdot 1 + 1^{2^k} + \lambda \cdot 1^{2^{2k}} = 1$. However, $\mathrm{Tr}_1^n(va) = \mathrm{Tr}_1^n(v) = 0$, which shows that $g(a) = 0$ has no solution in this case.

3) $(\mathrm{Tr}_1^n(a), \mathrm{Tr}_1^n(va)) = (1,0)$: In this case, $g(a) = 0$ means that $\lambda a + a^{2^k} + \lambda a^{2^{2k}} = v^{2^k}$, i.e., $a = h(v^{2^k})$. From Lemma 5, $\mathrm{Tr}_1^n(a) = \mathrm{Tr}_1^n(h(v^{2^k})) = \mathrm{Tr}_1^n(v^{2^k}) = \mathrm{Tr}_1^n(v) = 0$, which shows that $g(a) = 0$ has no solution in this case.

4) $(\mathrm{Tr}_1^n(a), \mathrm{Tr}_1^n(va)) = (1,1)$: Similar as above, $g(a) = 0$ implies that $\lambda a + a^{2^k} + \lambda a^{2^{2k}} = 1 + v^{2^k}$, i.e., $a = h(1 + v^{2^k})$. Note that $\mathrm{Tr}_1^n(1) = 0$ due to $n$ is even. Again by Lemma 5, $\mathrm{Tr}_1^n(a) = \mathrm{Tr}_1^n(h(1 + v^{2^k})) = \mathrm{Tr}_1^n(1 + v^{2^k}) = \mathrm{Tr}_1^n(v) = 0$. This implies that $g(a) = 0$ has no solution in this case.

From the above Cases 1)-4), we can see that if $\mathrm{Tr}_1^n(v) = 0$, then $\lambda a + a^{2^k} + \lambda a^{2^{2k}} + v^{2^k}\mathrm{Tr}_1^n(a) + \mathrm{Tr}_1^n(va) \neq 0$ for all nonzero $a \in \mathbb{F}_{2^n}$. □

*Proof of Corollary 3:*

According to Theorem 5, it is sufficient to show that $\gcd(\lambda + x + \lambda x^2, x^3 - 1) = 1$ if $\lambda \neq 1$. Then the result follows from the fact that $\gcd(\lambda + x + \lambda x^2, x^3 - 1) = \gcd(\lambda + x + \lambda x^2, x^2 + x + 1) = \gcd(\lambda(x^2 + x + 1) + (\lambda + 1)x, x^2 + x + 1) = \gcd((\lambda + 1)x, x^2 + x + 1)$. □

*Proof of Corollary 4:*

The result follows from that if $r=4$, then $\gcd(\lambda+x+\lambda x^2, x^4-1)=\gcd(\lambda+x+\lambda x^2, x-1)=1$ for any $\lambda\in\mathbb{F}_{2^k}^*$. □

*Proof of Corollary 5:*

According to Theorem 5, we need to determine the condition on $\lambda$ such that $\gcd(\lambda+x+\lambda x^2, x^5-1)=1$. Notice that $\gcd(\lambda+x+\lambda x^2, x^5-1)=\gcd(\lambda+x+\lambda x^2, x^4+x^3+x^2+x+1)$. By a simple calculation, we have $x^4+x^3+x^2+x+1=(1+\mu x+x^2)(\mu^2+\mu+(\mu+1)x+x^2)+(\mu^2+\mu+1)(\mu x+1)$, where $\mu=\lambda^{-1}$. This leads to $\gcd(\lambda+x+\lambda x^2, x^4+x^3+x^2+x+1)=\gcd(1+\mu x+x^2, x^4+x^3+x^2+x+1)=\gcd(1+\mu x+x^2, (\mu^2+\mu+1)(\mu x+1))=1$ if and only of $\mu^2+\mu+1\neq 0$. □

## Appendix C

*Proof of Theorem 6:*

Since $k$ is odd, then $f(x)=x^2+x+1$ is irreducible over $\mathbb{F}_{2^k}$ as it is irreducible over $\mathbb{F}_2$. Let $\omega$ be a root of $f(x)$. Then $\mathbb{F}_{2^n}=\mathbb{F}_{2^k}[\omega]$, i.e., each $x\in\mathbb{F}_{2^n}$ can be uniquely represented as $x_0+x_1\omega$, where $x_i\in\mathbb{F}_{2^k}$. Then

$$x^d=(x_0+x_1\omega)^d=x_0^4+x_1^4+x_1x_0^3+x_0x_1^3+(x_0^2x_1^2+x_0x_1^3+x_1^4)\omega \tag{C1}$$

and

$$\begin{aligned}(x+a)^d \;=\; & (x_0+a_0)^4+(x_1+a_1)^4+(x_1+a_1)(x_0+a_0)^3+(x_0+a_0)(x_1+a_1)^3\\ & +((x_0+a_0)^2(x_1+a_1)^2+(x_0+a_0)(x_1+a_1)^3+(x_1+a_1)^4)\omega,\end{aligned} \tag{C2}$$

where $a=a_0+a_1\omega$.

Note that $\mathrm{Tr}_k^{2k}(1)=0$ and $\mathrm{Tr}_k^{2k}(\omega)=\omega+\omega^{2^k}=1$ since $k$ is odd and $\omega$ is a root of $x^2+x+1$. Let $\lambda=\lambda_0+\lambda_1\omega$. Then from (C1), (C2) and $\mathrm{Tr}_k^{2k}(ax)=a_0x_1+a_1x_0+a_1x_1$, we have

$$\begin{aligned}&\mathrm{Tr}_k^{2k}(\lambda x^d+\lambda(x+a)^d+ax)\\ =\;&\lambda_1x_0^2a_1^2+\lambda_0x_0a_1^3+\lambda_0x_0^2a_1^2+\lambda_1a_1x_0^3+\lambda_0a_0x_1^3+a_1x_0+\lambda_1x_1x_0a_0^2+\lambda_1x_1x_0^2a_0+\lambda_1a_1x_0a_0^2\\ &+\lambda_1a_1x_0^2a_0+x_1a_1+\lambda_0a_0x_1^2a_1+\lambda_0x_0x_1^2a_1+\lambda_0x_0x_1a_1^2+\lambda_0a_0x_1a_1^2+\lambda_0a_0a_1^3+\lambda_0a_1^4\\ &+\lambda_1a_0^4+\lambda_1x_1a_0^3+\lambda_1a_1a_0^3+\lambda_1a_0^2x_1^2+\lambda_1a_0^2a_1^2+\lambda_0a_0^2x_1^2+\lambda_0a_0^2a_1^2+a_0x_1=G(x_0,x_1).\end{aligned} \tag{C3}$$

Suppose that $\lambda_1\neq 0$. We will show that for each $\lambda=\lambda_0+\lambda_1\omega$ with $\lambda_1\neq 0$, there exists at least one nonzero $a=a_0+a_1\omega\in\mathbb{F}_{2^n}$ such that $\mathrm{Tr}_1^n(\lambda x^d+\lambda(x+a)^d+ax)=\mathrm{Tr}_1^k(G(x_0,x_1))$ is not balanced. We consider this in three cases.

Case (i) $\lambda_1\neq 0, \lambda_0^2+\lambda_1^2+\lambda_0\lambda_1+1\neq 0$.

In this case, let $a_1=0$ and $a_0\neq 0$. Then

$$\begin{aligned}&\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(G(x_0,x_1))}\\ =\;&\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(\lambda_1x_1x_0^2a_0+\lambda_1x_1x_0a_0^2+\lambda_1a_0^2x_1^2+\lambda_0a_0x_1^3+\lambda_0a_0^2x_1^2+\lambda_1a_0^4+\lambda_1x_1a_0^3+a_0x_1)}\\ =\;&\sum_{x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(\lambda_1a_0^2x_1^2+\lambda_0a_0x_1^3+\lambda_0a_0^2x_1^2+\lambda_1a_0^4+\lambda_1x_1a_0^3+a_0x_1)}\sum_{x_0\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k((\lambda_1x_1a_0+\lambda_1^2x_1^2a_0^4)x_0^2)}\\ =\;&2^k\sum_{x_1=0\text{ or }x_1=(\lambda_1a_0^3)^{-1}}(-1)^{\mathrm{Tr}_1^k(\lambda_1a_0^2x_1^2+\lambda_0a_0x_1^3+\lambda_0a_0^2x_1^2+\lambda_1a_0^4+\lambda_1x_1a_0^3+a_0x_1)}\\ =\;&2^k\big((-1)^{\mathrm{Tr}_1^k(\lambda_1a_0^4)}+(-1)^{\mathrm{Tr}_1^k(\lambda_1a_0^2t^2+\lambda_0a_0t^3+\lambda_0a_0^2t^2+\lambda_1a_0^4+\lambda_1ta_0^3+a_0t)}\big),\end{aligned} \tag{C4}$$

where $t=(\lambda_1a_0^3)^{-1}$. By (C4), if there exists $a_0\in\mathbb{F}_q^*$ such that $\mathrm{Tr}_1^k(\lambda_1a_0^2t^2+\lambda_0a_0t^3+\lambda_0a_0^2t^2+\lambda_1ta_0^3+a_0t)=0$, then $\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(G(x_0,x_1))}=(-1)^{\mathrm{Tr}_1^k(\lambda_1a_0^4)}\cdot 2^{k+1}\neq 0$, i.e., $\mathrm{Tr}_1^k(G(x_0,x_1))$ is not balanced for such $a_0\in\mathbb{F}_q^*$. Since $t=(\lambda_1a_0^3)^{-1}$, we have $\mathrm{Tr}_1^k(\lambda_1a_0^2t^2+\lambda_0a_0t^3+\lambda_0a_0^2t^2+\lambda_1ta_0^3+a_0t)=\mathrm{Tr}_1^k(\frac{\lambda_1^2+\lambda_0^2+1+\lambda_0\lambda_1}{\lambda_1^4}(a_0^8)^{-1}+1)$, which implies that there exists $a_0\neq 0$ such that $\mathrm{Tr}_1^k(\frac{\lambda_1^2+\lambda_0^2+1+\lambda_0\lambda_1}{\lambda_1^4}(a_0^8)^{-1})+1=0$ if $\lambda\in\mathbb{F}_{2^n}$ satisfying $\lambda_0^2+\lambda_1^2+\lambda_0\lambda_1+1\neq 0$ and $\lambda_1\neq 0$.

Case (ii) $\lambda_1\neq 0, \lambda_0^2+\lambda_1^2+\lambda_0\lambda_1+1=0$ and $\lambda_0\neq 0$.

In this case, let $a_0=0$ and $a_1\neq 0$. Then

$$\begin{aligned}&\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(G(x_0,x_1))}\\ =\;&\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(\lambda_0x_0x_1^2a_1+(\lambda_0a_1^2x_0+a_1)x_1+\lambda_1a_1x_0^3+(\lambda_1a_1^2+\lambda_0a_1^2)x_0^2+(\lambda_0a_1^3+a_1)x_0+\lambda_0a_1^4)}\\ =\;&\sum_{x_0\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(\lambda_1a_1x_0^3+(\lambda_1a_1^2+\lambda_0a_1^2)x_0^2+(\lambda_0a_1^3+a_1)x_0+\lambda_0a_1^4)}\sum_{x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k((\lambda_0x_0a_1+\lambda_0^2a_1^4x_0^2+a_1^2)x_1^2)}\\ =\;&2^k\sum_{x_0=y_1\text{ or }x_0=y_2}(-1)^{\mathrm{Tr}_1^k(\lambda_1a_1x_0^3+(\lambda_1a_1^2+\lambda_0a_1^2)x_0^2+(\lambda_0a_1^3+a_1)x_0+\lambda_0a_1^4)},\end{aligned} \tag{C5}$$

where $y_1$ and $y_2$ are the two roots of $\lambda_0x_0a_1+\lambda_0^2a_1^4x_0^2+a_1^2=0$ ($x_0$ as the indeterminate variable) under the condition $\mathrm{Tr}_1^k(a_1)=0$. Thus, $y_1+y_2=\frac{1}{\lambda_0a_1^3}$ and $y_1y_2=\frac{1}{\lambda_0^2a_1^2}$. By (C5), if there exists $a_1\in\mathbb{F}_q^*$ such that $\mathrm{Tr}_1^k(a_1)=0$ and

$\mathrm{Tr}_1^k(\lambda_1 a_1(y_1^3+y_2^3)+(\lambda_1 a_1^2+\lambda_0 a_1^2)(y_1+y_2)^2+(\lambda_0 a_1^3+a_1)(y_1+y_2))=0$, then $\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(G(x_0,x_1))}=\pm 2^{k+1}\neq 0$, i.e., $\mathrm{Tr}_1^k(G(x_0,x_1))$ is not balanced for such $a_1\in\mathbb{F}_q^*$. By $y_1^3+y_2^3=(y_1+y_2)^3+y_1y_2(y_1+y_2)=\frac{1}{\lambda_0^3}(\frac{1}{a_1^9}+\frac{1}{a_1^5})$, one obtains that

$$\begin{aligned}&\mathrm{Tr}_1^k(\lambda_1 a_1(y_1^3+y_2^3)+(\lambda_1 a_1^2+\lambda_0 a_1^2)(y_1+y_2)^2+(\lambda_0 a_1^3+a_1)(y_1+y_2))\\ =\ &\mathrm{Tr}_1^k((\frac{\lambda_1^2}{\lambda_0^6}+\frac{\lambda_0^2+\lambda_1^2+\lambda_0\lambda_1+1}{\lambda_0^4})\frac{1}{a_1^8}+1)=\mathrm{Tr}_1^k(\frac{\lambda_1^2}{\lambda_0^6}\cdot\frac{1}{a_1^8}+1).\end{aligned}$$

According to Lemma 2, for odd $k>2$, there exists $a_1\in\mathbb{F}_q^*$ such that $\mathrm{Tr}_1^k(\frac{\lambda_1^2}{\lambda_0^6}\cdot\frac{1}{a_1^8}+1)=\mathrm{Tr}_1^k((\frac{\lambda_1^2}{\lambda_0^6})^{-8}\cdot\frac{1}{a_1})+1=0$ and $\mathrm{Tr}_1^k(a_1)=0$. Thus, for any $\lambda\in\mathbb{F}_{2^n}$ such that $\lambda_0^2+\lambda_1^2+\lambda_0\lambda_1+1=0$ and $\lambda_0\lambda_1\neq 0$, there exists $a_1\neq 0$ such that $\mathrm{Tr}_1^k(a_1)=0$ and $\mathrm{Tr}_1^k(\lambda_1 a_1(y_1^3+y_2^3)+(\lambda_1 a_1^2+\lambda_0 a_1^2)(y_1+y_2)^2+(\lambda_0 a_1^3+a_1)(y_1+y_2))=0$. That is, $\mathrm{Tr}_1^k(G(x_0,x_1))$ is not balanced for such $a_1\in\mathbb{F}_q^*$.

Case (iii) $\lambda_1\neq 0,\lambda_0^2+\lambda_1^2+\lambda_0\lambda_1+1=0$ and $\lambda_0=0$.

For this case, $\lambda_1=1$ and $\lambda_0=0$. Let $a_0=a_1\neq 0$. Then

$$\begin{aligned}&\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(G(x_0,x_1))}\\ =\ &\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(a_0^2x_1^2+(x_0^2a_0+x_0a_0^2+a_0^3)x_1+x_0^3a_0+(a_0^3+a_0)x_0+a_0^4)}\\ =\ &\sum_{x_0\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(x_0^3a_0+(a_0^3+a_0)x_0+a_0^4)}\sum_{x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k((a_0+x_0^2a_0+x_0a_0^2+a_0^3)x_1)}\\ =\ &2^k\sum_{x_0=y_1\ \text{or}\ x_0=y_2}(-1)^{\mathrm{Tr}_1^k(x_0^3a_0+(a_0^3+a_0)x_0+a_0^4)},\end{aligned}\tag{C6}$$

where $y_1$ and $y_2$ are the two roots of $a_0+x_0^2a_0+x_0a_0^2+a_0^3=0$ ($x_0$ as the indeterminate variable) under the condition $\mathrm{Tr}_1^k(a_0^{-1})=1$. Thus, $y_1+y_2=a_0$ and $y_1y_2=1+a_0^2$. By (C6), if there exists $a_0\in\mathbb{F}_q^*$ such that $\mathrm{Tr}_1^k(a_0^{-1})=1$ and $\mathrm{Tr}_1^k((y_1^3+y_2^3)a_0+(a_0^3+a_0)(y_1+y_2))=0$, then $\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(G(x_0,x_1))}=\pm 2^{k+1}\neq 0$. That is, $\mathrm{Tr}_1^k(G(x_0,x_1))$ is not balanced for such $a_0\in\mathbb{F}_q^*$. Note that $y_1^3+y_2^3=(y_1+y_2)^3+y_1y_2(y_1+y_2)=a_0^3+(1+a_0^2)a_0=a_0$, then $\mathrm{Tr}_1^k((y_1^3+y_2^3)a_0+(a_0^3+a_0)(y_1+y_2))=\mathrm{Tr}_1^k(a_0^2+(a_0^3+a_0)a_0)=\mathrm{Tr}_1^k(a_0)=0$. Again by Lemma 2, for odd $k>2$, there exists $a_0\in\mathbb{F}_q^*$ such that $\mathrm{Tr}_1^k(a_0)=0$ and $\mathrm{Tr}_1^k(a_0^{-1})=1$. Thus, for $\lambda=\lambda_0+\lambda_1\omega=\omega$, there exists $a_0\neq 0$ such that $\mathrm{Tr}_1^k(a_0^{-1})=1$ and $\mathrm{Tr}_1^k((y_1^3+y_2^3)a_0+(a_0^3+a_0)(y_1+y_2))=0$, which implies that $\mathrm{Tr}_1^k(G(x_0,x_1))$ is not balanced.

From the above Cases (i)-(iii), for each $\lambda=\lambda_0+\lambda_1\omega$ with $\lambda_1\neq 0$, there exists at least one nonzero $a=a_0+a_1\omega\in\mathbb{F}_{2^n}$ such that $\mathrm{Tr}_1^n(\lambda x^d+\lambda(x+a)^d+ax)=\mathrm{Tr}_1^k(G(x_0,x_1))$ is not balanced.

In the following we assume that $\lambda_1=0$ and $\lambda=\lambda_0+\lambda_1\omega=\lambda_0\neq 0$. Let $a_1=0$. Then

$$\begin{aligned}\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(G(x_0,x_1))}&=\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(\lambda_0a_0x_1^3+\lambda_0a_0^2x_1^2+a_0x_1)}\\ &=2^k\sum_{x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(\lambda_0a_0x_1^3+\lambda_0a_0^2x_1^2+a_0x_1)}\\ &=2^k\sum_{x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(\lambda_0a_0x_1^3+(\lambda_0^{2^{k-1}}a_0+a_0)x_1)}.\end{aligned}\tag{C7}$$

Since $k$ is odd, then $\gcd(3,2^k-1)=1$. Let $\lambda_0=r^3,a_0=t^3$, then from (C7), one gets

$$\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(G(x_0,x_1))}=2^k\sum_{x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(x_1^3+(r^{3\cdot 2^{k-1}}+1)r^{-1}t^2x_1)}.$$

Thus, if $\lambda_0=r^3\neq 1$, then $r^{3\cdot 2^{k-1}}+1\neq 0$. We claim that for any $r\in\mathbb{F}_q^*$ and $r\neq 1$, there must exist some $a_0\in\mathbb{F}_q^*$ such that $\sum_{x_0,x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(G(x_0,x_1))}=2^k\sum_{x_1\in\mathbb{F}_q}(-1)^{\mathrm{Tr}_1^k(x_1^3+(r^{3\cdot 2^{k-1}}+1)r^{-1}t^2x_1)}\neq 0$, i.e., $\mathrm{Tr}_1^k(G(x_0,x_1))$ is not balanced. Otherwise, the Walsh-Hadamard transform of $\mathrm{Tr}_1^k(x^3)$ at any point $t\in\mathbb{F}_q$ is zero, which contradicts with Parseval's theorem[1)].

Therefore, if $\mathrm{Tr}_1^n(\lambda x^d)$ is negabent on $\mathbb{F}_{2^n}$, then $\lambda$ has to be in $\mathbb{F}_2$. Zhou and Qu [8,Theorem 6] proved that if $\lambda\in\mathbb{F}_2$, then $\mathrm{Tr}_1^n(\lambda x^d)$ is indeed negabent on $\mathbb{F}_{2^n}$. □

1) Parseval's theorem shows that for any Boolean function $f(x)$ from $\mathbb{F}_{2^k}$ to $\mathbb{F}_2$, its Walsh-Hadamard transform $W_f(u)$ satisfies $\sum_{u\in\mathbb{F}_{2^k}}(W_f(u))^2=2^{2k}$.