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Modeling and analysis of colored petri net based on the semi-tensor product of matrices

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Abstract This paper applies the model petri net method based on the semi-tensor product of matrices to colored petri net. Firstly, we establish the marking evolution equation for colored petri net by using the semi-tensor product of matrices. Then we define the concept of controllability and the control-marking adjacency matrix for colored petri net. Based on the marking evolution equation and control-marking adjacency matrix, we give the necessary and sufficient condition of reachability and controllability for colored petri net. The algorithm to verify the reachability of colored petri net is given, and we analyze the computational complexity of the algorithm. Finally, an example is given to illustrate the effectiveness of the proposed theory. The significance of the paper lies in the application of the model petri net method based on the semi-tensor product of matrices to colored petri net. This is a convenient way of verifying whether one marking is reachable from another one as well as finding all firing sequences between any two reachable markings. Additionally, the method lays the foundations for the analysis of other properties of colored petri net.

Keywords colored petri net, reachability, controllability, marking evolution equation, semi-tensor product of matrices

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1 Introduction

Colored petri net is a graphic-oriented design language for designing, analyzing, simulating, and verifying systems. It is particularly suited to systems with communication, synchronization, and resource sharing properties. It has been widely used in the verification of communication protocols [1], flexible manufacturing systems [2,3], integrated circuit chips [4], automatic processing systems [5]. Refs. [6–8] introduced the concept, definition, analysis method and various applications of colored petri net in detail.

Reachability plays an important role in the analysis of other properties of petri net, and some other properties are based on the analysis of reachability. In the analysis of petri nets, the evolution behavior of petri nets can be seen clearly from the state space reachability graph. The authors of [9] use the ordinary petri net to model a software system and generate test cases from its reachability graph. However, the state space reachability graph is not suitable for the mathematical description and theoretical analysis of the dynamic behavior of petri nets. Place invariants can be used to verify the reachability property of petri net [7], but they cannot provide the required conditions to analyze other properties of petri net.

In recent years, a new matrix product, called the semi-tensor product of matrices, has been proposed by Cheng [10], it is a generalization of the ordinary product of matrices. The semi-tensor product of

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matrices preserves the properties of the ordinary product of matrices. Using the semi-tensor product of matrices, the discrete event system can be conveniently transformed into linear form, thus allowing classical control theory and methods to be applied for the analysis and control of discrete event systems. Today, the semi-tensor product of matrices has been successfully used in many fields and has solved some difficult control problems. In previous literatures [11–20], the semi-tensor product of matrices has been used in the Boolean network to solve the problem of linear representation and a series of control problems of Boolean networks. In [21, 22], the semi-tensor product of matrices was applied to finite automata, and the control problem of finite automata was studied. Additionally, the semi-tensor product of matrices can be used in game theory [23–26], fuzzy control problems [27].

In the field of petri nets, the semi-tensor product of matrices has significant advantages for the representation of the structural properties of petri nets. Ref. [28] uses the semi-tensor product of matrices to calculate the siphon of petri nets. Moreover, the semi-tensor product of matrices can also be used to describe the dynamic behavior of petri nets. The authors of [29] used the semi-tensor product of matrices to describe the dynamic behavior of ordinary petri net; they obtained a discrete-time linear equation, and then analyzed the petri net using the traditional analysis method. Some conditions for reachability and controllability of ordinary petri net were obtained. In addition, the marking evolution equation can also be used to analyze other properties of petri net.

The novelty of this paper lies in the following. (1) The existing petri net modeling method based on the semi-tensor product of matrices is applied to colored petri nets. In a state transfer process of ordinary petri net, a controlled quantity is a transition, but in a state transfer process of colored petri net, a controlled quantity is a transiton and a binding for some variables. We can treat a control quantity as a transiton and a binding of all variables; some variables play a role and the remaining variables have no effect. The vector form of the control quantity of colored petri nets is obtained by resorting to the semi-tensor product of matrices. Each control vector corresponds to a transition and a binding of all variable. (2) The marking evolution equation of colored petri nets is established, and the marking transfer matrix can be used to analyze the reachability and controllability of colored petri nets. (3) The controllability of colored petri net and the control-marking adjacency matrix are defined. The relationship between the marking transfer matrix and the marking transfer adjacency matrix was established. Next, the reachability and controllability conditions of colored petri net are given. (4) An algorithm verifying whether two markings of colored petri nets can be reached is given, through which all the controlled sequences of two reachable markings can be obtained.

2 Preliminaries

2.1 Notations

- $\mathbb{N}$ is the set of natural numbers.
- $\mathbb{N}^+$ is the set of positive integers.
- $\mathbb{R}^n$ is the set of all vectors of dimension n.
- $\mathcal{M}_{m\times n}$ is the set of $m\times n$ real matrices.
- $M_{(i,j)}$ denotes the (i,j)-th entry of matrix M.
- $\mathrm{Col}_i(M)$ is the i-th column of matrix M.
- $\mathrm{Col}(M)$ is the set of all columns of matrix M.
- $1_k := [\underbrace{1, 1, \ldots, 1}_{k}]$, $\delta_n^0 := [\underbrace{0, 0, \ldots, 0}_{n}]^{\mathrm{T}}$.
- δ_n^k denotes the k-th column of I_n, $1 \leqslant k \leqslant n$.
- $\Delta_n := \{\delta_n^1, \delta_n^2, \ldots, \delta_n^n\}$, $\widetilde{\Delta}_n := \{\delta_n^0\} \bigcup \Delta_n$.
- $L \in \mathcal{M}_{m\times n}$ is a logical matrix (generalised logical matrix) if $\mathrm{Col}(L) \subseteq \Delta_m$(resp.$\mathrm{Col}(L) \subseteq \tilde{\Delta}_m$), we denote the set of $m \times n$ logical matrices (resp.generalised logical matrix) by $\mathcal{L}_{m\times n}$(resp.$\tilde{\mathcal{L}}_{m\times n}$).
- If $L \in \mathcal{L}_{m\times n}$(resp.$\tilde{\mathcal{L}}_{m\times n}$), then it can be expressed as $L = [\delta_m^{i_1}, \delta_m^{i_2}, \ldots, \delta_m^{i_n}]$, and it is abbreviated to $L = \delta_m[i_1, i_2, \ldots, i_n]$, where $i_k \in \{0, 1, \ldots, m\}$, $1 \leqslant k \leqslant n$.

• $\text{Blk}_i(A)$ is the i-th $n\times n$ square block of $A \in \mathcal{M}_{n\times mn}$, $1 \leqslant i \leqslant m$. That is, $A = [\text{Blk}_1(A), \text{Blk}_2(A), \ldots, \text{Blk}_m(A)]$.

2.2 Semi-tensor product (STP) of matrices

In the following, we give some definitions of the semi-tensor product (STP) of matrices.

Definition 1 ([10])**.** Let $A \in \mathcal{M}_{m\times n}$ and $B \in \mathcal{M}_{p\times q}$, then the semi-tensor product of A and B is defined as follows:

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}),$$

where $t = \text{lcm}(n, p)$ is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

Remark 1. When $n = p$, $A \ltimes B = AB$. Thus the STP is a generalization of the conventional matrix product.

Definition 2 ([10])**.** A swap matrix $W_{[m,n]}$ is an $mn \times mn$ matrix defined as follows:

$$W_{[m,n]} = \delta_{mn}[1, m+1, 2m+1, \ldots, (n-1)m+1, 2, m+2, 2m+2, \ldots, (n-1)m+2, \ldots, m, 2m, 3m, \ldots, nm].$$

Lemma 1 ([10])**.** Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two column vectors. Then

$$W_{[m,n]}XY = YX, \quad W_{[n,m]}YX = XY.$$

2.3 Colored petri net (CPN)

In this section, we introduce some concepts about colored petri net [6].

Definition 3 ([6])**.** A multi-set m^* over a non-empty set S, is a function $m^* \in [S \to \mathbb{N}]$. The non-negative integer $m^*(s) \in \mathbb{N}$ is the number of appearances of the element s in the multi-set m^*. We usually represent the multi-set m^* by a formal sum as follows: $\sum_{s\in S} m^*(s)'s$. By S_{MS} we denote the set of all multi-sets over S. The non-negative integers $\{m^*(s)|s \in S\}$ are called the coefficients of the multi-set m^*, and $m^*(s)$ is called the coefficient of s. An element $s \in S$ is said to belong to the multi-set m^* iff $m^*(s) \neq 0$, and we write $s \in m^*$.

To give the abstract definition of colored petri net, it is necessary to fix the concrete syntax in which the net expressions are written.

• The elements are of a type, T. The set of all elements in T is denoted by T itself.

• The type of a variable, v is denoted by $\text{Type}(v)$.

• The type of an expression, expr is denoted by Type(expr).

• The set of variables in an expression, expr is denoted by Var(expr).

• A binding of a set of variables, V associates with each variable $v \in V$ with an element $b(v) \in \text{Type}(v)$.

• The value obtained by evaluating an expression, expr, in a binding b is denoted by $\text{expr}\langle b\rangle$. Var(expr) is required to be a subset of the variables of b, and the evaluation is performed by substituting for each variable $v \in \text{Var(expr)}$ the value $b(v) \in \text{Type}(v)$ determined by the binding.

Definition 4 ([6])**.** A non-hierarchical colored petri net is a tuple CPN=$(\Sigma, P, T, A, N, C, G, E, I)$ satisfying the following requirements:

(1) Σ is a finite set of non-empty types, called color sets.

(2) P is a finite set of places.

(3) T is a finite set of transitions.

(4) A is a finite set of arcs such that $P \bigcap T = P \bigcap A = T \bigcap A = \varnothing$.

(5) N is a node function defined from A into $P \times T \cup T \times P$.

(6) C is a color function defined from P into Σ.

(7) G is a guard function defined from T into expressions such that $\forall t \in T : [\text{Type}(G(t)) = B \wedge \text{Type}(\text{Var}(G(t))) \subseteq \Sigma]$.

(8) E is an arc expression function defined from A into expressions such that $\forall a \in A : [\text{Type}(E(a)) = C(p(a))_{\text{MS}} \wedge \text{Type}(\text{Var}(E(a))) \subseteq \Sigma]$, where $p(a)$ is the place of $N(a)$.

(9) I is an initialization function defined from P into closed expressions such that $\forall p \in P : [\mathrm{Type}(I(p)) = C(p)_{\mathrm{MS}}]$.

Definition 5 ([6]). A binding of a transition t is a function b defined on $\mathrm{Var}(t)$, such that

(1) $\forall v \in \mathrm{Var}(t) : b(v) \in \mathrm{Type}(v)$;

(2) $G(t)\langle b\rangle$, $B(t)$ denotes the set of all bindings for t.

Definition 6 ([6]). A token element is a pair (p, c) where $p \in P$ and $c \in C(p)$, while a binding element is a pair (t, b) where $t \in T$ and $b \in B(t)$. The set of all token elements is denoted by TE while the set of all binding elements is denoted by BE. A marking is a multi-set over TE while a step is a non-empty and finite multi-set over BE. The initial marking M_0 is the marking obtained by evaluating the initialization expressions:

$$\forall (p, c) \in \mathrm{TE} : M_0(p, c) = (I(p))(c).$$

The sets of all markings and steps are denoted by $\mathbb{M}$ and $\mathbb{Y}$, respectively.

Definition 7 ([6]). A step Y is enabled in a marking M iff the following property is satisfied

$$\forall p \in P : \sum_{(t,b)\in\mathbb{Y}} E(p, t)\langle b\rangle \leqslant M(p).$$

Let the step Y be enabled in the marking M. When $(t, b) \in \mathbb{Y}$, we say that t is enabled in M for the binding b. We also say that (t,b) is enabled in M, and so is t. When $(t_1,b_1),(t_2,b_2)\in \mathbb{Y}$ and $(t_1,b_1)\neq(t_2,b_2)$ we say that (t_1,b_1) and (t_2,t_2) are concurrently enabled, and so are t_1 and t_2. When $|Y(t)| \geqslant 2$ we say that t is concurrently enabled by itself. When $Y(t, b) \geqslant 2$ we say that (t,b) is concurrently enabled by itself.

Definition 8 ([6]). When a step Y is enabled in a marking M_1 it may be possible to change the marking M_1 to another marking M_2, defined by

$$\forall p \in P : M_2(p) = \left(M_1(p) - \sum_{(t,b)\in\mathbb{Y}} E(p, t)\langle b\rangle\right) + \sum_{(t,b)\in\mathbb{Y}} E(t, p)\langle b\rangle.$$

The first sum is called the removed tokens while the second is called the added tokens. Moreover, we say that M_2 is directly reachable from M_1 by the occurrence of the step Y, which we also denote $M_1[Y > M_2$.

Definition 9 ([6]). A finite occurrence sequence is a sequence of markings and steps,

$$M_1[Y_1 > M_2[Y_2 > M_3 \cdots M_n[Y_n > M_{n+l},$$

such that $n \in \mathbb{N}$, and $M_i[Y_i > M_{i+1}$ for all $i \in 1, \ldots, n$. The marking M_1 is called the start marking of the occurrence sequence, while the marking M_{n+l} is called the end marking. The non-negative integer n is the number of steps in the occurrence sequence, or the length of it.

Definition 10 ([6]). A marking M'' is reachable from a marking M' iff there exists a finite occurrence sequence having M' as start marking and M'' as end marking, that is to say iff for some $n \in \mathbb{N}$ there exists a sequence of steps $Y_1Y_2 \cdots Y_n$ such that

$$M^{'}[Y_1Y_1 \cdots Y_n > M^{''}.$$

Then, we also say that M'' is reachable from M' in n steps. The set of markings which are reachable from M' is denoted by $[M' >$.

3 Matrix representation of colored petri net

Assume that the initial state of a colored petri net is M_1, the marking set of $\langle \mathrm{CPN}, M_1\rangle$ is denoted as $\mathbb{X} = \{M_1, M_2, \ldots, M_s\}$, the transition set is denoted as $T = \{t_1, t_2, t_3, \ldots, t_m\}$, the variable set is

$V = \{v_1, v_1, v_1, \ldots, v_n\}$, and the domain of v_i is $D(v_i) = \{b(v_i)_1, b(v_i)_2, b(v_i)_3, \ldots, b(v_i)_{l_i}\}$, $i = 1, \ldots n$. We define $h = l_1 \times l_2 \times l_3 \times \cdots \times l_n$. Assume that B is a binding for all variables, $B = (v_1 = b(v_1)_{j_1}, v_2 = b(v_2)_{j_2}, \ldots, v_n = b(v_n)_{j_n})$, $j_k = 1, \ldots, l_k$, $k = 1, \ldots, n$. The marking at step $t+1$ is based on the marking at step t and the occurrence of step t. So, we have the following function:

$$f(M_i(t), t_k B_r(V)) = \begin{cases} M_j(t+1), & M_i[t_k B_r(V) > M_j, \\ 0, & \text{otherwise.} \end{cases}$$

Denote $\delta_s^i \sim M_i$ $(1 \leqslant i \leqslant s)$, $\delta_m^x \sim t_x$ $(1 \leqslant x \leqslant m)$. For variable v_i, $\delta_{l_i}^k \backsim b(v_i)_k, k = 1, \ldots, l_i$, we call δ_s^i the vector form of M_i, δ_m^x the vector form of t_x, and $\delta_{l_i}^k$ the vector form of $b(v_i)_k$. Thus, $\mathbb{X} \backsim \Delta_s = \{\delta_s^1, \delta_s^2, \ldots, \delta_s^s\}$, $T \backsim \Delta_m = \{\delta_m^1, \delta_m^2, \ldots, \delta_m^m\}$, and $D(v_i) \backsim \Delta_{l_i} = \{\delta_{l_i}^1, \delta_{l_i}^2, \ldots, \delta_{l_i}^{l_i}\}$.

Assume that the firing transition is t_x, and the variables have the following values $B(v_1 = b(v_1)_{j_1}, v_2 = b(v_2)_{j_2}, \ldots, v_n = b(v_n)_{j_n})$, $j_k = 1, \ldots, l_k$, $k = 1, \ldots, n$, then the controlled quantity can be expressed as

$$u = \delta_m^x \ltimes \delta_{l_1}^{j_1} \ltimes \delta_{l_2}^{j_2} \ltimes \cdots \ltimes \delta_{l_n}^{j_n}.$$

Theorem 1. Given a colored petri net $\langle \text{CPN}, M_1 \rangle$, the dynamics of $\langle \text{CPN}, M_1 \rangle$ can be expressed as

$$x(t+1) = Lu(t)x(t),$$

where $x(t)$ and $u(t)$ denote the markings and controlled quantity of $\langle \text{CPN}, M_1 \rangle$ at step t. $L (L \in \tilde{\mathcal{L}}_{s \times smh})$ is called the control-marking transfer matrix of $\langle \text{CPN}, M_1 \rangle$. The equation is the marking evolution equation of $\langle \text{CPN}, M_1 \rangle$.

Proof. Since

$$f(M_i(t), t_x B_r(V)) = \begin{cases} M_j(t+1), & M_i[t_x B_r(V) > M_j, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $x = 1, \ldots, m$, $r = 1, \ldots, h$.

This can be expressed as

$$f(\delta_s^i, \delta_m^x \ltimes \delta_{l_1}^{j_1} \ltimes \delta_{l_2}^{j_2} \ltimes \cdots \ltimes \delta_{l_n}^{j_n}) = \begin{cases} \delta_s^j, & \delta_s^i[\delta_m^x \ltimes \delta_{l_1}^{j_1} \ltimes \delta_{l_2}^{j_2} \ltimes \cdots \ltimes \delta_{l_n}^{j_n} > \delta_s^j, \\ \delta_s^0, & \text{otherwise.} \end{cases} \tag{2}$$

Denote $g = m \times l_1 \times l_2 \times l_3 \times \cdots \times l_n$. Then the controlled quantity can be expressed as δ_g^y, $y = 1, \ldots, g$. We have

$$f(\delta_s^i, \delta_g^y) = \begin{cases} \delta_s^j, & \delta_s^i[\delta_g^y > \delta_s^j, \\ \delta_s^0, & \text{otherwise.} \end{cases} \tag{3}$$

We define the generalized logical matrix is as follows:

$$L = [L_1, L_2, L_3, \ldots, L_g], \tag{4}$$

where $L_y = [\delta_s^{y_1}, \delta_s^{y_2}, \delta_s^{y_3}, \ldots, \delta_s^{y_s}]$, $\delta_s^{y_i} = f(\delta_s^i, \delta_g^y)$, $i = 1, \ldots, s$, $y = 1, \ldots, g$.

Assume $x(t)$ is the marking at the t-th step, $u(t)$ is the controlled quantity at t-th step. Thus, the theorem holds.

Example 1. Dining philosophers. Five Chinese philosophers are sitting around a circular table. In the middle of the table there is a delicious dish of rice, and between each pair of philosophers there is a single chopstick. Each philosopher alternates between thinking and eating. To eat, the philosopher needs two chopsticks, and he is only allowed to use the two which are situated next to him (on his left and right). The sharing of chopsticks prevents two neighbors from eating at the same time.

The philosopher system is modelled by the colored petri net, as shown in Figure 1. The PH color set represents the philosophers, while the CS color set represents the chopsticks. The function Chopsticks maps the relation of each philosopher to the two chopsticks next to him.

The definition is as follows:

val $n = 5$;

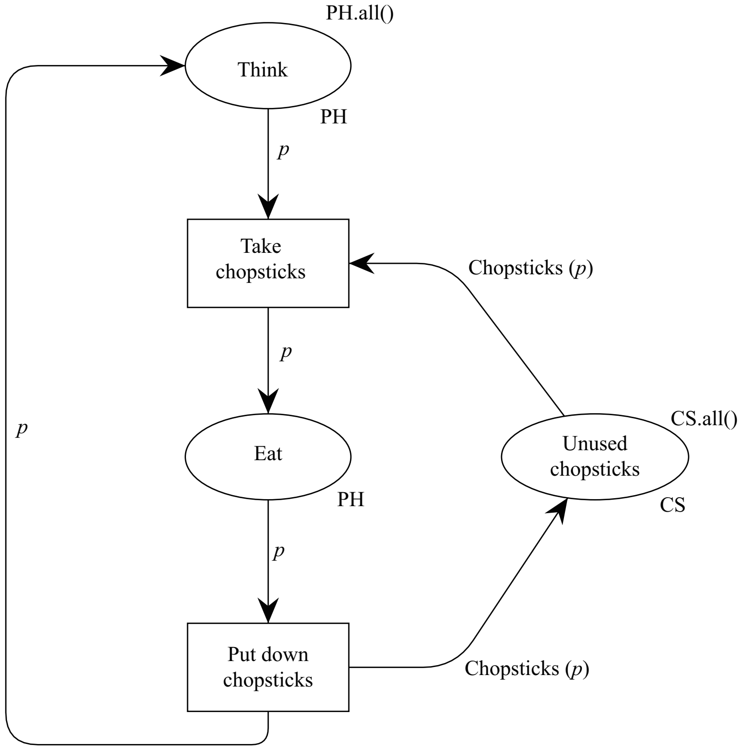

Figure 1 The colored petri net of dining philosophers.

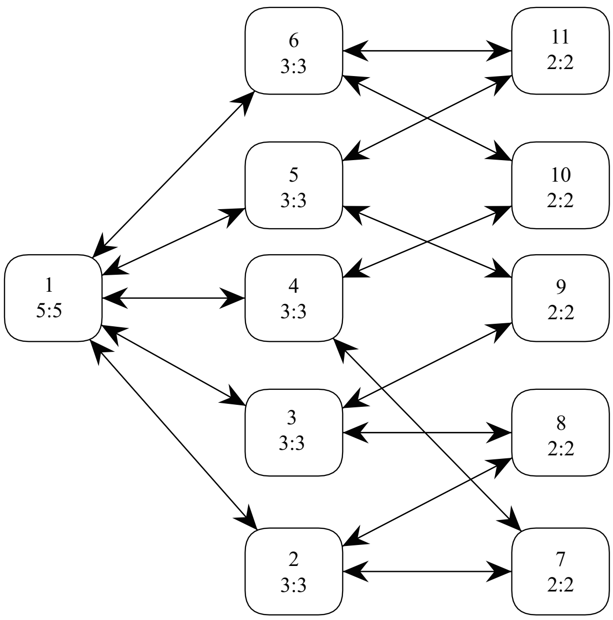

Figure 2 The state space of the colored petri net of dining philosophers.

colset PH = index ph with $1, \ldots, n$;
colset CS = index cs with $1, \ldots, n$;
var p : PH;
fun Chopsticks(ph(i)) $= 1'\mathrm{cs}(i) + +1'\mathrm{cs}$ (if $i = n$ then 1, else $i + 1$).
The state space is shown in Figure 2.
The markings of the state space are as follows.
[1] Think: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(2) + +1'\mathrm{ph}(3) + +1'\mathrm{ph}(4) + +1'\mathrm{ph}(5)$, Eat: empty, Unused chopsticks:

$1'\mathrm{cs}(1) + +1'\mathrm{cs}(2) + +1'\mathrm{cs}(3) + +1'\mathrm{cs}(4) + +1'\mathrm{cs}(5)$.

[2] Think: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(3) + +1'\mathrm{ph}(4) + +1'\mathrm{ph}(5)$, Eat: $1'\mathrm{ph}(2)$, Unused chopsticks: $1'\mathrm{cs}(1) + +1'\mathrm{cs}(4) + +1'\mathrm{cs}(5)$.

[3] Think: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(2) + +1'\mathrm{ph}(3) + +1'\mathrm{ph}(4)$, Eat: $1'\mathrm{ph}(5)$, Unused chopsticks: $1'\mathrm{cs}(2) + +1'\mathrm{cs}(3) + +1'\mathrm{cs}(4)$.

[4] Think: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(2) + +1'\mathrm{ph}(3) + +1'\mathrm{ph}(5)$, Eat: $1'\mathrm{ph}(4)$, Unused chopsticks: $1'\mathrm{cs}(1) + +1'\mathrm{cs}(2) + +1'\mathrm{cs}(3)$.

[5] Think: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(2) + +1'\mathrm{ph}(4) + +1'\mathrm{ph}(5)$, Eat: $1'\mathrm{ph}(3)$, Unused chopsticks: $1'\mathrm{cs}(1) + +1'\mathrm{cs}(2) + +1'\mathrm{cs}(5)$.

[6] Think: $1'\mathrm{ph}(2) + +1'\mathrm{ph}(3) + +1'\mathrm{ph}(4) + +1'\mathrm{ph}(5)$, Eat: $1'\mathrm{ph}(1)$, Unused chopsticks: $1'\mathrm{cs}(3) + +1'\mathrm{cs}(4) + +1'\mathrm{cs}(5)$.

[7] Think: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(3) + +1'\mathrm{ph}(5)$, Eat: $1'\mathrm{ph}(2) + +1'\mathrm{ph}(4)$, Unused chopsticks: $1'\mathrm{cs}(1)$.

[8] Think: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(3) + +1'\mathrm{ph}(4)$, Eat: $1'\mathrm{ph}(2) + +1'\mathrm{ph}(5)$, Unused chopsticks: $1'\mathrm{cs}(4)$.

[9] Think: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(2) + +1'\mathrm{ph}(4)$, Eat:$1'\mathrm{ph}(3) + +1'\mathrm{ph}(5)$, Unused chopsticks: $1'\mathrm{cs}(2)$.

[10] Think: $1'\mathrm{ph}(2) + +1'\mathrm{ph}(3) + +1'\mathrm{ph}(5)$, Eat: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(4)$, Unused chopsticks: $1'\mathrm{cs}(3)$.

[11] Think: $1'\mathrm{ph}(2) + +1'\mathrm{ph}(4) + +1'\mathrm{ph}(5)$, Eat: $1'\mathrm{ph}(1) + +1'\mathrm{ph}(3)$, Unused chopsticks: $1'\mathrm{cs}(5)$.

The ways in which states transfer from one to another are as follows.

[1] $1 \to 2$ Take chopsticks $p = \mathrm{ph}(2)$,
[2] $1 \to 3$ Take chopsticks $p = \mathrm{ph}(5)$,
[3] $1 \to 4$ Take chopsticks $p = \mathrm{ph}(4)$,
[4] $1 \to 5$ Take chopsticks $p = \mathrm{ph}(3)$,
[5] $1 \to 6$ Take chopsticks $p = \mathrm{ph}(1)$,
[6] $2 \to 1$ Put down chopsticks $p = \mathrm{ph}(2)$,
[7] $2 \to 7$ Take chopsticks $p = \mathrm{ph}(5)$,
[8] $2 \to 8$ Take chopsticks $p = \mathrm{ph}(4)$,
[9] $3 \to 1$ Put down chopsticks $p = \mathrm{ph}(5)$,
[10] $3 \to 7$ Take chopsticks $p = \mathrm{ph}(2)$,
[11] $3 \to 9$ Take chopsticks $p = \mathrm{ph}(3)$,
[12] $4 \to 1$ Put down chopsticks $p = \mathrm{ph}(4)$,
[13] $4 \to 10$ Take chopsticks $p = \mathrm{ph}(1)$,
[14] $4 \to 8$ Take chopsticks $p = \mathrm{ph}(2)$,
[15] $5 \to 1$ Put down chopsticks $p = \mathrm{ph}(3)$,
[16] $5 \to 11$ Take chopsticks $p = \mathrm{ph}(1)$,
[17] $5 \to 9$ Take chopsticks $p = \mathrm{ph}(5)$,
[18] $6 \to 1$ Put down chopsticks $p = \mathrm{ph}(1)$,
[19] $6 \to 11$ Take chopsticks $p = \mathrm{ph}(3)$,
[20] $6 \to 10$ Take chopsticks $p = \mathrm{ph}(4)$,
[21] $7 \to 2$ Put down chopsticks $p = \mathrm{ph}(5)$,
[22] $7 \to 3$ Put down chopsticks $p = \mathrm{ph}(2)$,
[23] $8 \to 4$ Put down chopsticks $p = \mathrm{ph}(2)$,
[24] $8 \to 2$ Put down chopsticks $p = \mathrm{ph}(4)$,
[25] $9 \to 3$ Put down chopsticks $p = \mathrm{ph}(3)$,
[26] $9 \to 5$ Put down chopsticks $p = \mathrm{ph}(5)$,
[27] $10 \to 4$ Put down chopsticks $p = \mathrm{ph}(1)$,
[28] $10 \to 6$ Put down chopsticks $p = \mathrm{ph}(4)$,
[29] $11 \to 5$ Put down chopsticks $p = \mathrm{ph}(1)$,
[30] $11 \to 6$ Put down chopsticks $p = \mathrm{ph}(3)$.

Denote take chopsticks $\sim \delta_2^1$, put down chopsticks $\sim \delta_2^2$, $M_i \sim \delta_{11}^i$, $p \sim v_1$, $D(v_1) = \{\mathrm{ph}(1), \mathrm{ph}(2), \mathrm{ph}(3), \mathrm{ph}(4), \mathrm{ph}(5)\}$, and $\mathrm{ph}(i) \sim \delta_5^i$, $i = 1, 2, \ldots, 5$. Then the controlled quantity can be expressed as $u = \delta_2^l \ltimes \delta_5^i$, $l = 1, 2$, $i = 1, 2, \ldots, 5$. $g = 2 \times 5 = 10$.

$$
\begin{aligned}
L_1 &= [\delta_{11}^{1_1}, \delta_{11}^{1_2}, \delta_{11}^{1_3}, \ldots, \delta_{11}^{1_{11}}] = \delta_{11}[6\,0\,0\,10\,11\,0\,0\,0\,0\,0\,0],\\
L_2 &= [\delta_{11}^{2_1}, \delta_{11}^{2_2}, \delta_{11}^{2_3}, \ldots, \delta_{11}^{2_{11}}] = \delta_{11}[2\,0\,7\,8\,0\,0\,0\,0\,0\,0\,0],\\
L_3 &= [\delta_{11}^{3_1}, \delta_{11}^{3_2}, \delta_{11}^{3_3}, \ldots, \delta_{11}^{3_{11}}] = \delta_{11}[5\,0\,9\,0\,0\,11\,0\,0\,0\,0\,0],\\
L_4 &= [\delta_{11}^{4_1}, \delta_{11}^{4_2}, \delta_{11}^{4_3}, \ldots, \delta_{11}^{4_{11}}] = \delta_{11}[4\,8\,0\,0\,0\,10\,0\,0\,0\,0\,0],\\
L_5 &= [\delta_{11}^{5_1}, \delta_{11}^{5_2}, \delta_{11}^{5_3}, \ldots, \delta_{11}^{5_{11}}] = \delta_{11}[3\,7\,0\,0\,9\,0\,0\,0\,0\,0\,0],\\
L_6 &= [\delta_{11}^{6_1}, \delta_{11}^{6_2}, \delta_{11}^{6_3}, \ldots, \delta_{11}^{6_{11}}] = \delta_{11}[0\,0\,0\,0\,0\,1\,0\,0\,0\,4\,5],\\
L_7 &= [\delta_{11}^{7_1}, \delta_{11}^{7_2}, \delta_{11}^{7_3}, \ldots, \delta_{11}^{7_{11}}] = \delta_{11}[0\,1\,0\,0\,0\,0\,3\,4\,0\,0\,0],\\
L_8 &= [\delta_{11}^{8_1}, \delta_{11}^{8_2}, \delta_{11}^{8_3}, \ldots, \delta_{11}^{8_{11}}] = \delta_{11}[0\,0\,0\,0\,1\,0\,0\,0\,3\,0\,6],\\
L_9 &= [\delta_{11}^{9_1}, \delta_{11}^{9_2}, \delta_{11}^{9_3}, \ldots, \delta_{11}^{9_{11}}] = \delta_{11}[0\,0\,0\,1\,0\,0\,0\,2\,0\,6\,0],\\
L_{10} &= [\delta_{11}^{10_1}, \delta_{11}^{10_2}, \delta_{11}^{10_3}, \ldots, \delta_{11}^{10_{11}}] = \delta_{11}[0\,0\,1\,0\,0\,0\,2\,0\,5\,0\,0].
\end{aligned}
$$

Then

$$L = [L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}].$$

The marking evolution equation is

$$x(t+1) = Lu(t)x(t).$$

If the current marking is δ_{11}^3, the controlled quantity is $\delta_2^1 \ltimes \delta_5^3$, $L \ltimes \delta_2^1 \ltimes \delta_5^3 \ltimes \delta_{11}^3 = \delta_{11}^9$, and the next marking is δ_{11}^9.

Whereas, if the current marking is δ_{11}^5, the controlled quantity is $\delta_2^1 \ltimes \delta_5^2$, $L \ltimes \delta_2^1 \ltimes \delta_5^2 \ltimes \delta_{11}^5 = \delta_{11}^0$, and the next marking is δ_{11}^5.

4 The reachability and controllability of colored petri net

4.1 Verify the reachability and controllability of colored petri net

Definition 11. Given a colored petri net $\langle \mathrm{CPN}, M_1 \rangle$, the marking set is $\mathbb{X} = \{M_1, M_2, \ldots, M_s\}$. $\langle \mathrm{CPN}, M_1 \rangle$ is called controllable at M_1, iff $[M_1^{'} >= \mathbb{X}$. $\langle \mathrm{CPN}, M_1 \rangle$ is controllable, iff it is controllable at every marking of $\mathbb{X}$.

Theorem 2 ([30])**.** Assume that $\ltimes_1^t \delta_{m_i}^{j_i} = \delta_{m_1 \times m_2 \times \cdots \times m_t}^l$, then $\delta_{m_i}^{j_i} = S_i \ltimes \delta_{m_1 \times m_2 \times \cdots \times m_t}^l, i = 1, 2, \ldots, t$, where

$$\begin{cases} S_1 = I_{m_1} \otimes 1_{m_2 \times \cdots \times m_t}, \\ S_2 = [\underbrace{I_{m_2} \otimes 1_{m_3 \times \cdots \times m_t}, \ldots, I_{m_2} \otimes 1_{m_3 \times \cdots \times m_t}}_{m_1}], \\ S_3 = [\underbrace{I_{m_3} \otimes 1_{m_4 \times \cdots \times m_t}, \ldots, I_{m_3} \otimes 1_{m_4 \times \cdots \times m_t}}_{m_1 \times m_2}], \\ \quad \vdots \\ S_{t-1} = [\underbrace{I_{m_{t-1}} \otimes 1_{m_t}, \ldots, I_{m_{t-1}} \otimes 1_{m_t}}_{m_1 \times \cdots \times m_{t-2}}], \\ S_t = [\underbrace{I_{m_t}, \ldots, I_{m_t}}_{m_1 \times \cdots \times m_{t-1}}]. \end{cases}$$

In this subsection, we will verify the reachability and controllability of colored petri net based on the marking evolution equation.

Theorem 3. Given a colored petri net and its marking set $\mathbb{X} = \{M_1, M_2, \ldots, M_s\}$, $M_1 = \delta_s^1$ is the initial marking, $M_d = \delta_s^d$ is the target marking. The transition set is $\{t_1, t_2, \ldots, t_m\}$, the variable set is $V = \{v_1, v_2, v_3, \ldots, v_n\}$, the domain of v_i is $D(v_i) = \{b(v_i)_1, b(v_i)_2, b(v_i)_3, \ldots, b(v_i)_{l_i}\}$, $i = 1, \ldots, n$. $g = m \times l_1 \times l_2 \times l_3 \times \cdots \times l_n$, and thus the controlled quantity could be expressed as δ_g^y, $y = 1, \ldots, g$. Then

(1) M_d is reachable from M_1 by t step iff $\exists i \in \{1, 2, \ldots, g^t\}$, such that

$$\delta_s^d = \mathrm{Col}_i((LW_{[s,mh]})^t \delta_s^1). \tag{5}$$

Additionally, the firing sequences are $\ltimes_{j=1}^t \delta_g^{i_j} = \delta_{g^t}^i$, and each $\delta_g^{i_j}$ can be obtained by decomposing the semi-tensor product of matrices. From this, we can obtain the firing transition and the binding for all variables.

(2) A colored petri net $\langle \mathrm{CPN}, M_1 \rangle$ is controllable at $M_j \sim \delta_s^j$, iff $\forall M_i = \delta_s^i \in \mathbb{X}$, $\exists c_i \in \mathbb{N}^+$, such that

$$\delta_s^i \in \mathrm{Col}((LW_{[s,mh]})^{c_i} \delta_s^j), \quad 1 \leqslant i \leqslant s. \tag{6}$$

Proof. (1)

$$\begin{aligned} x(t+1) &= Lu(t)x(t) \\ &= LW_{[s,mh]}x(t)u(t) \\ &= (LW_{[s,mh]})^2 x(t-1)u(t-1)u(t) \\ &\vdots \\ &= (LW_{[s,mh]})^t x(1)u(1)u(2)\cdots u(t) \\ &= ((LW_{[s,mh]})^t \delta_s^1) \ltimes_{j=1}^t u(j). \end{aligned}$$

(existence). Assume that M_d is reachable from M_1 by t steps, then there exists a firing sequence of at least length t, $u(1)u(2)\cdots u(t)$, such that

$$\delta_s^d = ((LW_{[s,mh]})^t \delta_s^1) \ltimes_{j=1}^t u(j),$$

then we have

$$\delta_s^d \in \mathrm{Col}((LW_{[s,mh]})^t \delta_s^1).$$

Then there exists a positive integer $i \in \{1,2,\ldots,(mh)^t\}$ such that $\delta_s^d = \mathrm{Col}_i((LW_{[s,mh]})^t \delta_s^1)$, the firing sequence can be obtained by decomposing the semi-tensor product of matrices $\delta_{(mh)^t}^i = \ltimes_{j=1}^t \delta_{mh}^{i_j}$.

(sufficiency). If there exists $i \in \{1,2,\ldots,(mh)^t\}$, such that $\delta_s^d = \mathrm{Col}_i((LW_{[s,mh]})^t \delta_s^1)$, then we can choose a firing sequence of length t, $\ltimes_{j=1}^t u(j) = \delta_{(mh)^t}^i$ such that $\delta_s^d = ((LW_{[s,mh]})^t \delta_s^1)\delta_{(mh)^t}^i$. Then M_d can be reachable from M_1 in t steps. The firing sequence can be obtained by decomposing the semi-tensor product of matrices $\delta_{(mh)^t}^i = \ltimes_{j=1}^t \delta_{mh}^{i_j}$.

(2) By means of part (1) and Definition 11, the result obviously holds.

Corollary 1. $M_d = \delta_s^d$ is reachable from $M_1 = \delta_s^1$, iff $\exists k \in N^+$ $(k \leqslant s)$ such that

$$\delta_s^d \in \left\{ \bigcup_{t=1}^k \mathrm{Col}((LW_{[s,mh]})^t \delta_s^1) \right\}.$$

Definition 12. A controlled quantity of a colored petri net is a transition and a binding for all variables. By making use of the semi-tensor product of matrices, it can be expressed as

$$u = \delta_m^x \ltimes \delta_{l_1}^{j_1} \ltimes \delta_{l_2}^{j_2} \ltimes \cdots \ltimes \delta_{l_n}^{j_n},$$

where m is the number of transitions, l_k is the number of values of v_k, $x = 1,2,\ldots,m$, $k = 1,2,\ldots,n$, $j_k = 1,\ldots,l_k$. When the controlled quantity has an effect on the making, the binding variables may only work partially and the others may not work at all. We call the variables which do not work the irrelevant variables.

Assume that the variable set is $\{P,Q,W\}$, $D(P) = \{p_1,p_2,\ldots,p_{n_1}\}$, $D(Q) = \{q_1,q_2,\ldots,q_{n_2}\}$, $D(W) = \{w_1,w_2,\ldots,w_{n_3}\}$, and the transition set is $\{t_1,t_2,t_3\}$. The firing transition of $1 \to 2$ is t_2 and the variable Q plays a role, while the other variables make no difference. Thus, P and W are irrelevant variables. The number of combinations of the irrelevant variables is $n_1 \times n_3$. A transition and a binding for all variables are a controlled quantity. Thus the number of controlled quantities is $3 \times n_1 \times n_2 \times n_3$.

Definition 13. Given a colored petri net $\langle \mathrm{CPN}, M_1 \rangle$, the marking set is $\mathbb{X} = \{M_1, M_2, \ldots, M_s\}$. The control-marking transfer matrix is $L = [\mathrm{Blk}_1(L), \mathrm{Blk}_2(L), \ldots, \mathrm{Blk}_{mh}(L)]$, where $\mathrm{Blk}_l(L)$ represents the l-th $s \times s$ block of L, $1 \leqslant l \leqslant mh$. The matrix $A = (a_{ij})_{s\times s}$ is the control-marking adjacency matrix of $\langle \mathrm{CPN}, M_1 \rangle$ and is defined

$$A = \sum_{l=1}^{mh} \mathrm{Blk}_l(L),$$

if $a_{ij} = c$, there are c different ways to drive M_j to M_i.

Next, we will discuss whether $(A^t)_{ij}$ has a similar meaning.

Theorem 4. Given a colored petri net $\langle \mathrm{CPN}, M_1 \rangle$ and its control-marking adjacency matrix $A = (a_{ij})_{s\times s}$, $A^t = (a^t_{ij})_{s\times s}$ and $A^1 = A$. If $a^t_{ij} > 0$, then M_j is reachable from M_i in t steps. Additionally, if $a^t_{ij} = c \neq 0$, then there are c different ways of length t in which can drive M_j to M_i.

Proof. We prove the result by mathematical induction. When $t = 1$ the conclusion is obvious. Assume that the conclusion is true for $t = k$, that is to say if $a^k_{ij} > 0$, then M_i is reachable from M_j in k steps. Additionaly, if $a^k_{ij} = c \neq 0$, then there are c different ways of length t which can drive M_j to M_i.

Now, let us consider the case of $t = k + 1$, because a control path of length $k + 1$ which drive M_j to M_i can be decomposed into a control path of length k which drives M_j to M_l and a controlled quantity of length 1 which drive M_l to M_i. Now that M_l has s choices, we know that the number of control paths of length $k + 1$ which drive M_j to M_i is

$$\sum_{l=1}^{s} a_{il} a^k_{lj} = (AA^k)_{ij} = (A^{k+1})_{ij} = a^{k+1}_{ij}.$$

If $a^{k+1}_{ij} > 0$, then M_j is reachable from M_i by $k + 1$ steps. If $a^{k+1}_{ij} = c$, there exist c different control paths of length $k + 1$ that could drive M_j to M_i. Therefore, when $t = k + 1$ the theorem also holds. Based on mathematical induction, the conclusion holds.

Remark 2. M_j may be reachable from M_i in t steps by different state paths. Assume that the number of state paths is z, and r^x_d is the number of combinations of the irrelevant variables at the d-th step of the x-th kind of state path, $x = 1, 2, \ldots, z$, $d = 1, 2, \ldots, t$, then $r^x_1 \times r^x_2 \times \cdots \times r^x_t$ is the number of ways to the control the x-th kind of state path, $a^t_{ij} = \sum_{x=1}^{z} r^x_1 \times r^x_2 \times \cdots \times r^x_t$.

The matrices $\{A^t/t = 1, 2, \ldots\}$ include all the information of reachable markings of $\langle \mathrm{CPN}, M_1 \rangle$. By the Cayley-Hamilton theorem of linear algebra [31], we can known that if $(A^t)_{(i,j)} = 0$, $\forall t \leqslant s$, then $(A^t)_{(i,j)} = 0$, $\forall t \in \mathbb{N}^+$. So we can only consider $\{A^t/t \leqslant s\}$. From the results discussed above, we have the following results about the reachability and controllability of colored petri nets.

Corollary 2. Given a colored petri net $\langle \mathrm{CPN}, M_1 \rangle$ with the marking set $\mathbb{X} = \{M_1, M_2, \ldots, M_s\}$, the control-marking adjacency matrix of the colored petri net is given by the above definition. Then

(1) M_i is reachable from M_j, iff $\exists k \in \mathbb{N}^+$ $(k \leqslant s)$ such that

$$\sum_{t=1}^{k} (A^t)_{(i,j)} > 0;$$

(2) A colored petri net $\langle \mathrm{CPN}, M_1 \rangle$ is controllable at M_j, iff $\exists k \in \mathbb{N}^+$ $(k \leqslant s)$ such that

$$\sum_{t=1}^{k} \mathrm{Col}_j(A^t) > 0;$$

(3) The colored petri net $\langle \mathrm{CPN}, M_1 \rangle$ is controllable, iff $\exists k \in \mathbb{N}^+$ $(k \leqslant s)$ such that

$$\sum_{t=1}^{k} (A^t) > 0.$$

4.2 Design of algorithm and complexity analysis

We design an algorithm to verify the reachability of colored petri nets, as shown in Algorithm 1. Additionally, the computational complexity of the method is also discussed.

In the worst case, the complexities of calculating matrices L and $(LW_{[s,mh]})^t \delta^i_s$ $(t \leqslant s)$ are $o(s^2mh)$ and $o(s(mh)^s)$ respectively. So the overall computational complexity is $o(s^2mh + s(mh)^s)$, where h is the number of the combinations of all the variables.

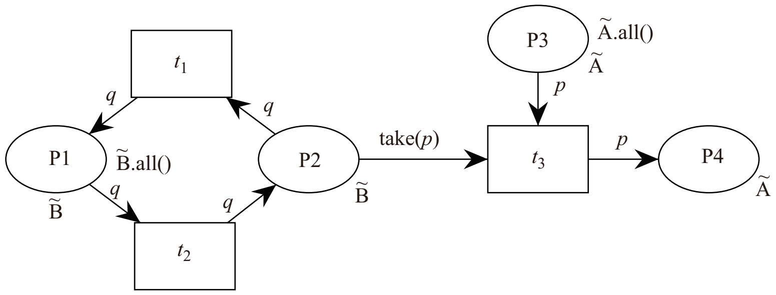

Figure 3 The colored petri net used to verify the reachability and controllability.

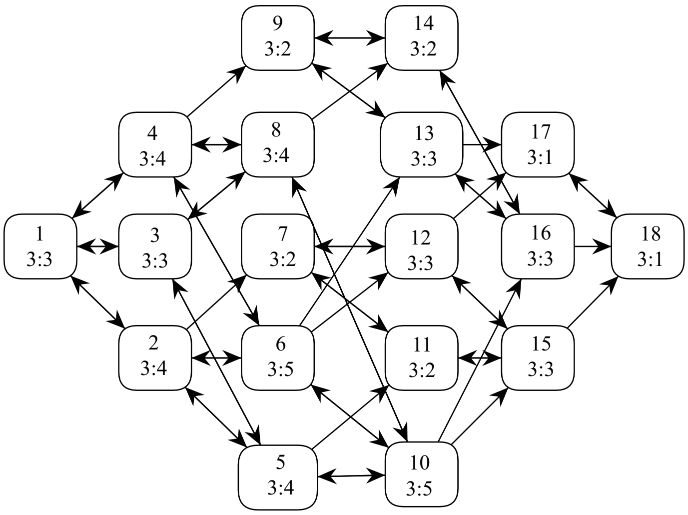

Figure 4 The state space of the colored petri net used to verify the reachability and controllability.

Algorithm 1 Verifying the reachability of colored petri net

Consider a colored petri net $\langle \mathrm{CPN}, M_1 \rangle$ with m transitions, n variables, and s markings, and where $M_i = \delta_s^i$ and $M_j = \delta_s^j$ are two markings. We can verify whether M_j is reachable from M_i by t steps by the following method.

Step 1: Construct the marking evolution equation of $\langle \mathrm{CPN}, M_1 \rangle$.

Step 2: Calculate the matrix $(LW_{[s,mh]})^t \delta_s^i$. If $\delta_s^j \notin \mathrm{Col}((LW_{[s,mh]})^t \delta_s^i)$ then δ_s^j is not reachable from δ_s^i in t steps.

Step 3: Find all k such that $\delta_s^j = \mathrm{Col}_k((LW_{[s,mh]})^t \delta_s^i)$. Now $\delta_{(mh)^t}^k = \ltimes_{j=1}^t \delta_{mh}^{k_j}$ and we can obtain k_j, $(j = 1, 2, \ldots, t)$ by decomposing the semi-tensor product of matrices. Thus, we can find the firing sequence $\delta_{mh}^{k_1}, \delta_{mh}^{k_2}, \delta_{mh}^{k_3}, \ldots, \delta_{mh}^{k_t}$. For each $\delta_{mh}^{k_j}, (j = 1, 2, \ldots, t)$, we can also obtain the firing transition and the binding for all variables by decomposing the semi-tensor product of matrices.

Example 2. Verifying the reachability and controllability of a colored petri net.

As shown in Figure 3, the definition of the colored petri net is as follows:

val $n = 2$;

val $m = 3$;

colset $\tilde{\mathrm{A}} =$ index a with $1, \ldots, n$;

colset $\tilde{\mathrm{B}} =$ index b with $1, \ldots, m$;

var $p : \tilde{\mathrm{A}}$;

var $q : \tilde{\mathrm{B}}$;

fun $\mathrm{take}(a(i)) = 1'b(i)$.

Using the algorithm to construct the state space of the colored petri net which is shown in Figure 4, we obtained the state space and the ways in which states transfer from one to another. The markings of the state space are as follows:

[1] $\mathrm{P1} : 1'b(1) + +1'b(2) + +1'b(3),\ \mathrm{P2} : \mathrm{empty},\ \mathrm{P3} : 1'a(1) + +1'a(2),\ \mathrm{P4} : \mathrm{empty}.$

[2] $\mathrm{P1} : 1'b(1) + +1'b(3),\ \mathrm{P2} : 1'b(2),\ \mathrm{P3} : 1'a(1) + +1'a(2),\ \mathrm{P4} : \mathrm{empty}.$

[3] $\mathrm{P1}: 1'b(1) + +1'b(2),\ \mathrm{P2}: 1'b(3),\ \mathrm{P3}: 1'a(1) + +1'a(2),\ \mathrm{P4}: \text{empty}.$
[4] $\mathrm{P1}: 1'b(2) + +1'b(3),\ \mathrm{P2}: 1'b(1),\ \mathrm{P3}: 1'a(1) + +1'a(2),\ \mathrm{P4}: \text{empty}.$
[5] $\mathrm{P1}: 1'b(1),\ \mathrm{P2}: 1'b(2) + +1'b(3),\ \mathrm{P3}: 1'a(1) + +1'a(2),\ \mathrm{P4}: \text{empty}.$
[6] $\mathrm{P1}: 1'b(3),\ \mathrm{P2}: 1'b(1) + +1'b(2),\ \mathrm{P3}: 1'a(1) + +1'a(2),\ \mathrm{P4}: \text{empty}.$
[7] $\mathrm{P1}: 1'b(1) + +1'b(3),\ \mathrm{P2}: \text{empty},\ \mathrm{P3}: 1'a(1),\ \mathrm{P4}: 1'a(2).$
[8] $\mathrm{P1}: 1'b(2),\ \mathrm{P2}: 1'b(1) + +1'b(3),\ \mathrm{P3}: 1'a(1) + +1'a(2),\ \mathrm{P4}: \text{empty}.$
[9] $\mathrm{P1}: 1'b(2) + +1'b(3),\ \mathrm{P2}: \text{empty},\ \mathrm{P3}: 1'a(2),\ \mathrm{P4}: 1'a(1).$
[10] $\mathrm{P1}: \text{empty},\ \mathrm{P2}: 1'b(1) + +1'b(2) + +1'b(3),\ \mathrm{P3}: 1'a(1) + +1'a(2),\ \mathrm{P4}: \text{empty}.$
[11] $\mathrm{P1}: 1'b(1),\ \mathrm{P2}: 1'b(3),\ \mathrm{P3}: 1'a(1),\ \mathrm{P4}: 1'a(2).$
[12] $\mathrm{P1}: 1'b(3),\ \mathrm{P2}: 1'b(1),\ \mathrm{P3}: 1'a(1),\ \mathrm{P4}: 1'a(2).$
[13] $\mathrm{P1}: 1'b(3),\ \mathrm{P2}: 1'b(2),\ \mathrm{P3}: 1'a(2),\ \mathrm{P4}: 1'a(1).$
[14] $\mathrm{P1}: 1'b(2),\ \mathrm{P2}: 1'b(3),\ \mathrm{P3}: 1'a(2),\ \mathrm{P4}: 1'a(1).$
[15] $\mathrm{P1}: \text{empty},\ \mathrm{P2}: 1'b(1) + +1'b(3),\ \mathrm{P3}: 1'a(1),\ \mathrm{P4}: 1'a(2).$
[16] $\mathrm{P1}: \text{empty},\ \mathrm{P2}: 1'b(2) + +1'b(3),\ \mathrm{P3}: 1'a(2),\ \mathrm{P4}: 1'a(1).$
[17] $\mathrm{P1}: 1'b(3),\ \mathrm{P2}: \text{empty},\ \mathrm{P3}: \text{empty},\ \mathrm{P4}: 1'a(1) + +1'a(2).$
[18] $\mathrm{P1}: \text{empty},\ \mathrm{P2}: 1'b(3),\ \mathrm{P3}: \text{empty},\ \mathrm{P4}: 1'a(1) + +1'a(2).$

The ways in which states transfer from one to another are as follows:

[1] $1 \to 2\ \ t_2\ q = b(2)$, **[2]** $1 \to 3\ \ t_2\ q = b(3)$, **[3]** $1 \to 4\ \ t_2\ q = b(1)$, **[4]** $2 \to 5\ \ t_2\ q = b(3)$,
[5] $2 \to 6\ \ t_2\ q = b(1)$, **[6]** $2 \to 7\ \ t_3\ p = a(2)$, **[7]** $2 \to 1\ \ t_1\ q = b(2)$, **[8]** $3 \to 8\ \ t_2\ q = b(1)$,
[9] $3 \to 5\ \ t_2\ q = b(2)$, **[10]** $3 \to 1\ \ t_1\ q = b(3)$, **[11]** $4 \to 8\ \ t_2\ q = b(3)$, **[12]** $4 \to 6\ \ t_2\ q = b(2)$,
[13] $4 \to 9\ \ t_3\ p = a(1)$, **[14]** $4 \to 1\ \ t_1\ q = b(1)$, **[15]** $5 \to 10\ \ t_2\ q = b(1)$, **[16]** $5 \to 11\ \ t_3\ p = a(2)$,
[17] $5 \to 2\ \ t_1\ q = b(3)$, **[18]** $5 \to 3\ \ t_1\ q = b(2)$, **[19]** $6 \to 10\ \ t_2\ q = b(3)$, **[20]** $6 \to 12\ \ t_3\ p = a(2)$,
[21] $6 \to 13\ \ t_3\ p = a(1)$, **[22]** $6 \to 4\ \ t_1\ q = b(2)$, **[23]** $6 \to 2\ \ t_1\ q = b(1)$, **[24]** $7 \to 12\ \ t_2\ q = b(1)$,
[25] $7 \to 11\ \ t_2\ q = b(3)$, **[26]** $8 \to 10\ \ t_2\ q = b(2)$, **[27]** $8 \to 14\ \ t_3\ p = a(1)$, **[28]** $8 \to 4\ \ t_1\ q = b(3)$,
[29] $8 \to 3\ \ t_1\ q = b(1)$, **[30]** $9 \to 14\ \ t_2\ q = b(3)$, **[31]** $9 \to 13\ \ t_2\ q = b(2)$, **[32]** $10 \to 15\ \ t_3\ p = a(2)$,
[33] $10 \to 16\ \ t_3\ p = a(1)$, **[34]** $10 \to 8\ \ t_1\ q = b(2)$, **[35]** $10 \to 5\ \ t_1\ q = b(1)$, **[36]** $10 \to 6\ \ t_1\ q = b(3)$,
[37] $11 \to 15\ \ t_2\ q = b(1)$, **[38]** $11 \to 7\ \ t_1\ q = b(3)$, **[39]** $12 \to 15\ t_2\ q = b(3)$, **[40]** $12 \to 17\ t_3\ p = a(1)$,
[41] $12 \to 7\ \ t_1\ q = b(1)$, **[42]** $13 \to 16\ \ t_2\ q = b(3)$, **[43]** $13 \to 17\ t_3\ p = a(2)$, **[44]** $13 \to 9\ \ t_1\ q = b(2)$,
[45] $14 \to 16\ \ t_2\ q = b(2)$, **[46]** $14 \to 9\ \ t_1\ q = b(3)$, **[47]** $15 \to 18\ t_3\ p = a(1)$, **[48]** $15 \to 12\ \ t_1\ q = b(3)$,
[49] $15 \to 11\ \ t_1\ q = b(1)$, **[50]** $16 \to 18\ \ t_3\ p = a(2)$, **[51]** $16 \to 13\ t_1\ q = b(3)$, **[52]** $16 \to 14\ \ t_1\ q = b(2)$,
[53] $17 \to 18\ \ t_2\ q = b(3)$, **[54]** $18 \to 17\ \ t_1\ q = b(3)$.

The marking set is $\mathbb{X} = \{M_1, M_2, \ldots, M_{18}\}$ and the transition set is $T = \{t_1, t_2, t_3\}$. The variables are $V = \{p, q\}$, $D(p) = \{a_1, a_2\}$ and $D(q) = \{b_1, b_2, b_3\}$. We denote $\delta_{18}^i \sim M_i$ $(1 \leqslant i \leqslant 18)$ and $\delta_3^x \sim t_x$ $(1 \leqslant x \leqslant 3)$. To variable p, $\delta_2^k \sim b(p)_k,\ k = 1, 2$ and for variable q, $\delta_3^k \sim b(q)_k,\ k = 1, 2, 3$.

Each controlled quantity can be expressed as $u = \delta_3^l \ltimes \delta_2^{i_1} \ltimes_3^{i_2}$, $l = 1, 2, 3,\ i_1 = 1, 2,\ i_2 = 1, 2, 3$. Then, the controlled quantity is $\delta_{3\times2\times3}^y,\ y = 1, \ldots, 18$.

$$L_1 = [\delta_{18}^{1_1}, \delta_{18}^{1_2}, \delta_{18}^{1_3}, \ldots, \delta_{18}^{1_{18}}] = \delta_{18}[0\,0\,0\,1\,0\,2\,0\,3\,0\,5\,0\,7\,0\,0\,11\,0\,0\,0],$$
$$L_2 = [\delta_{18}^{2_1}, \delta_{18}^{2_2}, \delta_{18}^{2_3}, \ldots, \delta_{18}^{2_{18}}] = \delta_{18}[0\,1\,0\,0\,3\,4\,0\,0\,0\,8\,0\,0\,9\,0\,0\,14\,0\,0],$$
$$L_3 = [\delta_{18}^{3_1}, \delta_{18}^{3_2}, \delta_{18}^{3_3}, \ldots, \delta_{18}^{3_{18}}] = \delta_{18}[0\,0\,1\,0\,2\,0\,0\,4\,0\,6\,7\,0\,0\,9\,12\,13\,0\,17],$$
$$L_4 = [\delta_{18}^{4_1}, \delta_{18}^{4_2}, \delta_{18}^{4_3}, \ldots, \delta_{18}^{4_{18}}] = \delta_{18}[0\,0\,0\,1\,0\,2\,0\,3\,0\,5\,0\,7\,0\,0\,11\,0\,0\,0],$$
$$L_5 = [\delta_{18}^{5_1}, \delta_{18}^{5_2}, \delta_{18}^{5_3}, \ldots, \delta_{18}^{5_{18}}] = \delta_{18}[0\,1\,0\,0\,3\,4\,0\,0\,0\,8\,0\,0\,9\,0\,0\,14\,0\,0],$$
$$L_6 = [\delta_{18}^{6_1}, \delta_{18}^{6_2}, \delta_{18}^{6_3}, \ldots, \delta_{18}^{6_{18}}] = \delta_{18}[0\,0\,1\,0\,2\,0\,0\,4\,0\,6\,7\,0\,0\,9\,12\,13\,0\,17],$$
$$L_7 = [\delta_{18}^{7_1}, \delta_{18}^{7_2}, \delta_{18}^{7_3}, \ldots, \delta_{18}^{7_{18}}] = \delta_{18}[4\,6\,8\,0\,10\,0\,12\,0\,0\,0\,15\,0\,0\,0\,0\,0\,0\,0],$$
$$L_8 = [\delta_{18}^{8_1}, \delta_{18}^{8_2}, \delta_{18}^{8_3}, \ldots, \delta_{18}^{8_{18}}] = \delta_{18}[2\,0\,5\,6\,0\,0\,0\,10\,13\,0\,0\,0\,0\,16\,0\,0\,0\,0],$$
$$L_9 = [\delta_{18}^{9_1}, \delta_{18}^{9_2}, \delta_{18}^{9_3}, \ldots, \delta_{18}^{9_{18}}] = \delta_{18}[3\,5\,0\,8\,0\,10\,11\,0\,14\,0\,0\,15\,16\,0\,0\,0\,18\,0],$$

$$L_{10} = [\delta_{18}^{10_1}, \delta_{18}^{10_2}, \delta_{18}^{10_3}, \ldots, \delta_{18}^{10_{18}}] = \delta_{18}[4\,6\,8\,0\,10\,0\,12\,0\,0\,0\,15\,0\,0\,0\,0\,0\,0\,0],$$
$$L_{11} = [\delta_{18}^{11_1}, \delta_{18}^{11_2}, \delta_{18}^{11_3}, \ldots, \delta_{18}^{11_{18}}] = \delta_{18}[2\,0\,5\,6\,0\,0\,0\,10\,13\,0\,0\,0\,0\,16\,0\,0\,0\,0],$$
$$L_{12} = [\delta_{18}^{12_1}, \delta_{18}^{12_2}, \delta_{18}^{12_3}, \ldots, \delta_{18}^{12_{18}}] = \delta_{18}[3\,5\,0\,8\,0\,10\,11\,0\,14\,0\,0\,15\,16\,0\,0\,0\,18\,0],$$
$$L_{13} = [\delta_{18}^{13_1}, \delta_{18}^{13_2}, \delta_{18}^{13_3}, \ldots, \delta_{18}^{13_{18}}] = \delta_{18}[0\,0\,0\,9\,0\,13\,0\,14\,0\,16\,0\,17\,0\,0\,18\,0\,0\,0],$$
$$L_{14} = [\delta_{18}^{14_1}, \delta_{18}^{14_2}, \delta_{18}^{14_3}, \ldots, \delta_{18}^{14_{18}}] = \delta_{18}[0\,0\,0\,9\,0\,13\,0\,14\,0\,16\,0\,17\,0\,0\,18\,0\,0\,0],$$
$$L_{15} = [\delta_{18}^{15_1}, \delta_{18}^{15_2}, \delta_{18}^{15_3}, \ldots, \delta_{18}^{15_{18}}] = \delta_{18}[0\,0\,0\,9\,0\,13\,0\,14\,0\,16\,0\,17\,0\,0\,18\,0\,0\,0],$$
$$L_{16} = [\delta_{18}^{16_1}, \delta_{18}^{16_2}, \delta_{18}^{16_3}, \ldots, \delta_{18}^{16_{18}}] = \delta_{18}[0\,7\,0\,0\,11\,12\,0\,0\,0\,15\,0\,0\,17\,0\,0\,18\,0\,0],$$
$$L_{17} = [\delta_{18}^{17_1}, \delta_{18}^{17_2}, \delta_{18}^{17_3}, \ldots, \delta_{18}^{17_{18}}] = \delta_{18}[0\,7\,0\,0\,11\,12\,0\,0\,0\,15\,0\,0\,17\,0\,0\,18\,0\,0],$$
$$L_{18} = [\delta_{18}^{18_1}, \delta_{18}^{18_2}, \delta_{18}^{18_3}, \ldots, \delta_{18}^{18_{18}}] = \delta_{18}[0\,7\,0\,0\,11\,12\,0\,0\,0\,15\,0\,0\,17\,0\,0\,18\,0\,0].$$

Since

$$L = [L_1, L_2, L_3, \ldots, L_{18}],$$

$L = \delta_{18}$[0 0 0 1 0 2 0 3 0 5 0 7 0 0 11 0 0 0 0 0 1 0 0 3 4 0 0 0 8 0 0 9 0 0 14 0 0 0 0 0 1 0 2 0 0 4 0 6 7 0 0 9 12 13 0 17 0 0 0 1 0
2 0 3 0 5 0 7 0 0 11 0 0 0 0 0 1 0 0 3 4 0 0 0 8 0 0 9 0 0 14 0 0 0 0 0 1 0 2 0 0 4 0 6 7 0 0 9 12 13 0 17 4 6 8 0 10 0 12 0 0 0 15 0 0
0 0 0 0 0 2 0 5 6 0 0 0 10 13 0 0 0 0 16 0 0 0 0 3 5 0 8 0 10 11 0 14 0 0 15 16 0 0 0 18 0 4 6 8 0 10 0 12 0 0 0 15 0 0 0 0 0 0 0
2 0 5 6 0 0 0 10 13 0 0 0 0 16 0 0 0 0 3 5 0 8 0 10 11 0 14 0 0 15 16 0 0 0 18 0 0 0 0 9 0 13 0 14 0 16 0 17 0 0 18 0 0 0 0 0 0
9 0 13 0 14 0 16 0 17 0 0 18 0 0 0 0 0 0 9 0 13 0 14 0 16 0 17 0 0 18 0 0 0 0 7 0 0 11 12 0 0 0 15 0 0 17 0 0 18 0 0 0 7 0 0 11
12 0 0 0 15 0 0 17 0 0 18 0 0 0 7 0 0 11 12 0 0 0 15 0 0 17 0 0 18 0 0].

The marking evolution equation can be expressed as

$$x(t+1) = Lu(t)x(t).$$

Assuming that the current marking is δ_{18}^3 and the firing transition is t_2, $q = b(2)$, then the controlled quantities are $\delta_3^2 \ltimes \delta_2^1 \ltimes \delta_3^2$ or $\delta_3^2 \ltimes \delta_2^2 \ltimes \delta_3^2$, $L \ltimes \delta_3^2 \ltimes \delta_2^1 \ltimes \delta_3^2 \ltimes \delta_{18}^3 = \delta_{18}^5$, $L \ltimes \delta_3^2 \ltimes \delta_2^2 \ltimes \delta_3^2 \ltimes \delta_{18}^3 = \delta_{18}^5$, and the results are the same. So, when the current marking is δ_{18}^3, the transition is t_2, $q = b(2)$, and the value of p has no effect on the result.

If the current marking is δ_{18}^8, the transition is t_1, $p = a(1)$, and $q = b(2)$, then the controlled quantity is $\delta_3^1 \ltimes \delta_2^1 \ltimes \delta_3^2$, $L \ltimes \delta_3^1 \ltimes \delta_2^1 \ltimes \delta_3^2 \ltimes \delta_{18}^8 = \delta_{18}^0$, and the next marking is still δ_{18}^8.

When $t = 1$, $LW_{[18,3\times2\times3]}\delta_{18}^1 \in \mathcal{M}_{18\times18}$, $\mathrm{Col}_7(LW_{[18,18]}\delta_{18}^1) = \mathrm{Col}_{10}(LW_{[18,18]}\delta_{18}^1) = \delta_{18}^4$, $\mathrm{Col}_8(LW_{[18,18]}\delta_{18}^1) = \mathrm{Col}_{11}(LW_{[18,18]}\delta_{18}^1) = \delta_{18}^2$, $\mathrm{Col}_9(LW_{[18,18]}\delta_{18}^1) = \mathrm{Col}_{12}(LW_{[18,18]}\delta_{18}^1) = \delta_{18}^3$, so the markings M_2, M_3, and M_4 are reachable from M_1 by 1 step.

$\delta_{18}^7 = \delta_3^2 \ltimes \delta_2^1 \ltimes \delta_3^1$, $\delta_{18}^{10} = \delta_3^2 \ltimes \delta_2^2 \ltimes \delta_3^1$, so when the firing transition is t_2, $p = a_1$ or $p = a_2$, $q = b_1$, M_1 could reach M_4.

$\delta_{18}^8 = \delta_3^2 \ltimes \delta_2^1 \ltimes \delta_3^2$, $\delta_{18}^{11} = \delta_3^2 \ltimes \delta_2^2 \ltimes \delta_3^2$, so when the firing transition is t_2, $p = a_1$ or $p = a_2$, $q = b_2$, M_1 could reach M_2.

$\delta_{18}^9 = \delta_3^2 \ltimes \delta_2^1 \ltimes \delta_3^3$, $\delta_{18}^{12} = \delta_3^2 \ltimes \delta_2^2 \ltimes \delta_3^3$, so when the firing transition is t_2, $p = a_1$ or $p = a_2$, $q = b_3$, M_1 could reach M_3.

When $t = 2$, $(LW_{[18,3\times2\times3]})^2\delta_{18}^1 \in \mathcal{M}_{18\times324}$,

$$\begin{aligned}
&\mathrm{Col}_{135}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{138}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{152}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 \\
&\quad = \mathrm{Col}_{155}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{189}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{192}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 \\
&\quad = \mathrm{Col}_{206}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{209}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \delta_{18}^5, \\
&\mathrm{Col}_{116}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{119}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{133}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 \\
&\quad = \mathrm{Col}_{136}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{170}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{173}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 \\
&\quad = \mathrm{Col}_{187}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{190}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \delta_{18}^6, \\
&\mathrm{Col}_{142}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{143}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{144}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 \\
&\quad = \mathrm{Col}_{196}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{189}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \mathrm{Col}_{197}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 \\
&\quad = \mathrm{Col}_{198}(LW_{[18,3\times2\times3]})^2\delta_{18}^1 = \delta_{18}^7,
\end{aligned}$$

$$\begin{aligned}&\mathrm{Col}_{117}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\mathrm{Col}_{120}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\mathrm{Col}_{151}(LW_{[18,3\times2\times3]})^2\delta_{18}^1\\&\quad=\mathrm{Col}_{154}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\mathrm{Col}_{171}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\mathrm{Col}_{174}(LW_{[18,3\times2\times3]})^2\delta_{18}^1\\&\quad=\mathrm{Col}_{205}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\mathrm{Col}_{208}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\delta_{18}^8,\\&\mathrm{Col}_{121}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\mathrm{Col}_{122}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\mathrm{Col}_{123}(LW_{[18,3\times2\times3]})^2\delta_{18}^1\\&\quad=\mathrm{Col}_{175}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\mathrm{Col}_{176}(LW_{[18,3\times2\times3]})^2\delta_{18}^1=\mathrm{Col}_{177}(LW_{[18,3\times2\times3]})^2\delta_{18}^1\\&\quad=\delta_{18}^9.\end{aligned}$$

So, M_5, M_6, M_7, M_8 and M_9 are reachable from M_1 by 2 steps. Take M_9 for example. By applying the method of decomposing the semi-tensor product of matrices, we can obtain the following results.

$\delta_{324}^{121}=\delta_3^2\ltimes\delta_2^1\ltimes\delta_3^1\ltimes\delta_3^3\ltimes\delta_2^1\ltimes\delta_3^1$. The firing transition of $u(1)$ is t_2, $p=a_1$, $q=b_1$. The firing transition of $u(2)$ is t_3, $p=a_1$, $q=b_1$.

$\delta_{324}^{122}=\delta_3^2\ltimes\delta_2^1\ltimes\delta_3^1\ltimes\delta_3^3\ltimes\delta_2^1\ltimes\delta_3^2$. The firing transition of $u(1)$ is t_2, $p=a_1$, $q=b_1$. The firing transition of $u(2)$ is t_3, $p=a_1$, $q=b_2$.

$\delta_{324}^{123}=\delta_3^2\ltimes\delta_2^1\ltimes\delta_3^1\ltimes\delta_3^3\ltimes\delta_2^1\ltimes\delta_3^3$. The firing transition of $u(1)$ is t_2, $p=a_1$, $q=b_1$. The firing transition of $u(2)$ is t_3, $p=a_1$, $q=b_3$.

$\delta_{324}^{175}=\delta_3^2\ltimes\delta_2^2\ltimes\delta_3^1\ltimes\delta_3^3\ltimes\delta_2^1\ltimes\delta_3^1$. The firing transition of $u(1)$ is t_2, $p=a_2$, $q=b_1$. The firing transition of $u(2)$ is t_3,$p=a_1$,$q=b_1$.

$\delta_{324}^{176}=\delta_3^2\ltimes\delta_2^2\ltimes\delta_3^1\ltimes\delta_3^3\ltimes\delta_2^1\ltimes\delta_3^2$. The firing transition of $u(1)$ is t_2, $p=a_2$, $q=b_1$. The firing transition of $u(2)$ is t_3, $p=a_1$, $q=b_2$.

$\delta_{324}^{177}=\delta_3^2\ltimes\delta_2^2\ltimes\delta_3^1\ltimes\delta_3^3\ltimes\delta_2^1\ltimes\delta_3^3$. The firing transition of $u(1)$ is t_2, $p=a_2$, $q=b_1$. The firing transition of $u(2)$ is t_3, $p=a_1$, $q=b_3$.

M_9 is reachable from M_1 in 2 steps by one of the above paths. The firing transition of $u(1)$ is t_2, $q=b_1$, and the value of p has no effect on the result. The firing transition of $u(2)$ is t_3, $p=a_1$, and the value of q has no effect on the result.

Using the above conclusions, we can verify whether some given markings are reachable from the initial marking. Consider the markings M_{13}, M_{16}, and M_{17}. Is M_{16} reachable from M_{13} and M_{17}? By a direct computation, we have $\mathrm{Col}_9(LW_{[18,3\times2\times3]}\delta_{18}^{13})=\delta_{18}^{16}$, $\mathrm{Col}_{12}(LW_{[18,3\times2\times3]}\delta_{18}^{13})=\delta_{18}^{16}$, Thus M_{16} is reachable from M_{13}. The firing transition of u is t_2, $p=a_1$ or $p=a_2$, $q=b_3$.

However, there is no a positive integer k ($k\leqslant18$), such that

$$\delta_{18}^{16}\in\left\{\bigcup_{t=1}^{k}\mathrm{Col}((LW_{[18,18]})^t\delta_{18}^{17})\right\},$$

so M_{16} is not reachable from M_{17}.

Next we will verify the controllabililty of $\langle\mathrm{CPN},M_1\rangle$. Because M_1 could reach any markings, $\langle\mathrm{CPN},M_1\rangle$ is controllable at M_1.

$A=\sum_{k=1}^{18}\mathrm{Blk}_l(L)$, by a direct computation, we have $(\sum_{t=1}^{18}(A)^t)_{(1,17)}=0$. So, M_1 is not reachable from M_{17}. Thus $\langle\mathrm{CPN},M_1\rangle$ is not controllable at M_{17}. We can therefore conclude that the colored petri net $\langle\mathrm{CPN},M_1\rangle$ is not controllable.

5 Conclusion

In this paper, we studied colored petri nets. Firstly, the marking evolution equation of colored petri nets was established by using of the semi-tensor product of matrices. The marking evolution equation gives a mathematical model of colored petri nets to analyze their properties. Based on this model, we studied the dynamic properties of colored petri nets, i.e., reachability and controllability. Next, the method of judging the reachability and controllability of colored petri nets was given, and an algorithm was proposed to verify the reachability of colored petri nets. The algorithm found all the firing sequences of two reachable markings, and we analyzed the computational complexity of the proposed method. Finally, an example was given to illustrate the effectiveness of the proposed method in the study of the reachability and controllability of colored petri nets. In addition, the proposed method can be used not only for reachability and controllability analysis of colored petri nets, but also for some other control problems, such as stability and observability. Future work will study the observability and stability of colored petri nets based on the established model.

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