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Global practical tracking for stochastic time-delay nonlinear systems with SISS-like inverse dynamics

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Abstract This paper investigates the practical tracking problem of stochastic delayed nonlinear systems. The powers of the nonlinear terms are relaxed to a certain interval rather than a precisely known point. Based on the Lyapunov-Krasovskii (L-K) functional method and the modified adding a power integrator technique, a new controller is constructed to render the solutions of the considered system to be bounded in probability, and furthermore, the tracking error in sense of the mean square can be made small enough by adjusting some designed parameters. A simulation example is provided to demonstrate the validity of the method in this paper.

Keywords nonlinear systems, stochastic systems, time-varying delay, global practical tracking, SISS-like inverse dynamics

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1 Introduction

Recently, stochastic control has become a popular research issue, and considerable attention has been devoted to it. Many results in stochastic control theory have been studied, which can be viewed as extended versions of deterministic linear systems theory. For instances, in [1], the authors proposed a spectrum theory and a observability criteria for stochastic system. A similar criterion for exact detectability has also been proposed in [2]. As for the controller design problems, many researchers concentrated their attentions on state feedback control [3, 4], output feedback control [5] and so on.

Time-delay is a time interval from a behavior start to its being perceived. It has different meanings in various contexts. For example, in the process of performing fiscal policy, time-delay produced by the actions of government mainly includes: recognition lag, implementation lag, time-lag effect and time lag of monetary policy. Due to the limitation of measurement technology, time-delay phenomenon widely exists in various physical systems. In a practical system, time-delay may destroy the system stability. To study some related stabilization problems of delayed systems, in recent years, typical methods have been raised and employed in [6–8]. Particularly, by establishing a new stability criterion, the robust control problem for nominal delayed systems was considered in [6]. By deriving new sufficient/necessary

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conditions, the stability of delayed networks was studied [7]. Via a matrix transformation method, the output control problem of delayed nonlinear system was addressed in [8].

Owing to the practical interest, the practical tracking control has been studied extensively. For example, ref. [9] solved the practical tracking control problem of the deterministic systems, which was extended to stochastic systems by Li et al. [10–12]. To illustrate the strong robustness to uncertainties, Wu investigated the adaptive tracking problem of stochastic Markovian switching systems and stochastic Hamiltonian systems in [13, 14], respectively. Although, for delayed stochastic nonlinear system with disturbances, the stabilization problems have been extensively studied such as the state feedback stabilization [15], the output feedback stabilization [16], robust H_∞ stabilization [17, 18] and so on. Up to date, no method is effective to solve practical tracking problem, especially, to delayed stochastic system with input-to-state stability-like (SISS-like) inverse dynamics. This paper will make contributions to this difficult problem. The contributions are as follows:

(i) A new L-K functional for solving the practical tracking problem is constructed. The existing results for time-delay systems are focused on the stable controller design [15–18]. Hence, the L-K functionals in those researches are no longer valid for tracking control. Besides, the studied system contains many complicated nonlinear terms which make the construction of the L-K functional nontrivial.

(ii) For delayed stochastic nonlinear systems, we modify the method of adding a power integrator and give a delay-independent tracking control scheme.

(iii) The considered system has SISS-like inverse dynamics, nonlinear drift and diffusion terms, and time-varying delay. So it is more general than the previous literatures [3, 5, 7, 19].

Throughout the paper, we adopt such notations: $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^n$ are refer to a set of real numbers, nonnegative real numbers and real n-component vectors, respectively; $a \wedge b = \min\{a, b\}$; $\mathrm{Tr}\{X\}$ is the trace of matrix X; $\mathcal{C}^n$ is a set of nth differentiable functions; $\mathcal{C}^n_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ is a set of $\mathcal{F}_0$-measurable bounded $\mathbb{R}^n$-valued random variables which is $\mathcal{C}^n$ on $[-\tau, 0]$; $(\Omega, \mathcal{F}, P)$ is a complete probability space; The component is sometimes omitted, whenever no confusion can arise.

2 Problem statement and preliminary results

2.1 Problem statement

Consider the delayed stochastic nonlinear system:

$$\begin{cases} \mathrm{d}\zeta_0 &= f_0\big(\zeta_0, x_1, x_1(t-\eta(t)), t\big)\mathrm{d}t + g_0^\top\big(\zeta_0, x_1, x_1(t-\eta(t)), t\big)\mathrm{d}\omega, \\ \mathrm{d}x_i &= \big(a_i(t)x_{i+1}^{p_i} + f_i\big(\zeta_0, \bar{x}_i, \bar{x}_i(t-\eta(t)), t\big)\big)\mathrm{d}t + g_i^\top\big(\zeta_0, \bar{x}_i, \bar{x}_i(t-\eta(t)), t\big)\mathrm{d}\omega, \quad i = 1, \ldots, n-1, \\ \mathrm{d}x_n &= \big(a_n(t)u^{p_n} + f_n\big(\zeta_0, x, x(t-\eta(t)), t\big)\big)\mathrm{d}t + g_n^\top\big(\zeta_0, x, x(t-\eta(t)), t\big)\mathrm{d}\omega, \\ \ \ y &= x_1, \end{cases} \tag{1}$$

where $\bar{x}_j \triangleq [x_1, \ldots, x_j]^\top \in \mathbb{R}^j$, $j = 1, \ldots, n$, $x \triangleq \bar{x}_n$; $\eta(t)$ denotes the time delay satisfying $0 \leqslant \eta(t) \leqslant \tau$ with τ being a constant; $\zeta_0(t) \in \mathbb{R}^m$ denotes the state of stochastic inverse dynamics; $u(t), y(t) \in \mathbb{R}$ are system input and output; $\{x(\theta) : -\tau \leqslant \theta \leqslant 0\} = \phi \in \mathcal{C}^n_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ is the initial condition; the coefficient $a_j(t)$ satisfies $0 < \epsilon \leqslant a_j(t) \leqslant \bar{\epsilon}$ with ϵ and $\bar{\epsilon}$ being positive constants; suppose that $p_j \geqslant 1$ satisfies $p_j \in \mathbb{R}^+_{\mathrm{odd}} \triangleq \{\frac{m}{n} | m \text{ and } n \text{ are positive odd integers}\}$ and $1 + \sum_{k=1}^{n-1} \frac{1}{p_k \cdots p_{n-1}} - \sum_{k=1}^{j-1} \frac{2}{p_k \cdots p_{j-1}} > 0$. $f_k, g_k, k = 0, \ldots, n$ satisfy locally Lipschitz condition with $f_k(0, 0, 0, t), g_k(0, 0, 0, t)$ being bounded; $\omega \in \mathbb{R}^r$ stands for a Wiener process on $(\Omega, \mathcal{F}, P)$.

For a given reference signal $y_{\mathrm{r}}(t)$, a practical tracking controller

$$u = \Phi(y_{\mathrm{r}}, x_1, \ldots, x_n) \tag{2}$$

will be constructed such that

(H1) System (1) and (2) have a unique strong solution on $[-\tau, \infty)$;

(H2) $\zeta_0, x_1, \ldots, x_n$ are bounded in probability;

(H3) The expectation of $z_0 = y - y_\mathrm{r}$ satisfies

$$\lim_{t\to\infty} \mathrm{E}|z_0|^2 \leqslant \rho,$$

where ρ can be made small enough.

We introduce some assumptions as follows.

Assumption 1. $y_\mathrm{r}(t)$ and its derivative $\dot{y}_\mathrm{r}$ are bounded, that is, $|y_\mathrm{r}(t)| + |\dot{y}_\mathrm{r}(t)| \leqslant M$, with $M > 0$ being a constant.

Assumption 2. The derivative of $\eta(t)$ satisfies $\dot{\eta}(t) \leqslant \bar{\tau} < 1$ with $\bar{\tau} > 0$ being a constant.

Assumption 3. Define ϖ as a ratio of an even integer to an odd integer, satisfying $\varpi \geqslant d_M$, $d_M = \max_{1\leqslant k\leqslant n}\{d_k\}, d_1 = \frac{2-\frac{1}{p_1\cdots p_{n-1}}}{1+\sum_{k=1}^{n-1}\frac{1}{p_k\cdots p_{n-1}}}$, $d_j = \frac{\frac{2}{p_1\cdots p_{j-1}}-\frac{1}{p_1\cdots p_{n-1}}}{1+\sum_{k=1}^{n-1}\frac{1}{p_k\cdots p_{n-1}}-\sum_{k=1}^{j-1}\frac{2}{p_k\cdots p_{j-1}}}, j = 2,\ldots,n$. For each $i = 1,\ldots,n$, there exist nonnegative constants C, l_{i1} and l_{i2}, such that

$$\begin{cases} |f_i| \leqslant C\left(|\zeta_0|^{r_i+\varpi} + \sum_{j=1}^{i}\left(|x_j|^{\frac{r_i+\varpi}{r_j}} + |x_j(t-\eta(t))|^{\frac{r_i+\varpi}{r_j}}\right)\right) + l_{i1}, \\ |g_i| \leqslant C\left(|\zeta_0|^{\frac{2r_i+\varpi}{2}} + \sum_{j=1}^{i}\left(|x_j|^{\frac{2r_i+\varpi}{2r_j}} + |x_j(t-\eta(t))|^{\frac{2r_i+\varpi}{2r_j}}\right)\right) + l_{i2}, \end{cases}$$

where $r_1 = 1, r_{i+1} = \frac{r_i+\varpi}{p_i}$.

Assumption 4. For ζ_0-subsystem, there exists a function $V_0(\zeta_0) \in \mathcal{C}^2$ such that

$$l_1|\zeta_0|^{4\lambda} \leqslant V_0(\zeta_0) \leqslant l_2|\zeta_0|^{4\lambda},$$

$$\mathcal{L}V_0(\zeta_0) \leqslant -l_3|\zeta_0|^{4\lambda} + l_4\big(x_1^{4\lambda} + x_1^{4\lambda}(t-\eta(t))\big),$$

where $l_1,\ldots,l_4$ are positive constants and $\lambda \geqslant \max_{1\leqslant i\leqslant n}\{r_i + \varpi\}$.

Remark 1. These assumptions are reasonable. Assumption 1 is similar to those in [9,19,20] for delay-free systems. Assumption 2 is often used for systems with time-varying delay [6–8,15,16]. Assumption 3 enlarges the scope of nonlinear terms. Compared with [21], we can see that the powers of nonlinear terms in this work can change on a certain interval instead of a point. Furthermore, the time-delay here is time varying rather than a constant [22]. Assumption 4 shows that the stochastic dynamics satisfies an ISS-type property, and it is similar to those in [3,5].

Remark 2. Inverse dynamics as one of the inverse problems usually exists in mechanical systems. It generally means inverse rigid body dynamics or inverse structural dynamics. The previous one is often used to compute forces and moments by using the kinematics or the inertial properties of a body. The latter one is sometimes applied to calculate the inertia forces that result from a structure. From Assumption 4 and the sufficient condition proposed in Theorem A.1 of [23], we obtain that the inverse dynamics of ζ_0-subsystem satisfies this property of SISS.

Remark 3. The existing results only consider stabilization problem. The practical tracking control problem of system (1) remains unsolved.

2.2 Preliminary results

The considered stochastic system is described by

$$\mathrm{d}x = f\big(x, x(t-\eta(t)), t\big)\mathrm{d}t + g\big(x, x(t-\eta(t)), t\big)\mathrm{d}\omega, \quad \forall t \geqslant 0, \tag{3}$$

where $x(t)$ and $\eta(t)$ denote the system state and the time delay, respectively. The system initial condition is $\{x(\theta) : -\tau \leqslant \theta \leqslant 0\} = \phi \in \mathcal{C}^n_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$. $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^{n\times r}$ satisfy locally Lipschitz condition with $f(0,0,t)$ and $g(0,0,t)$ being bounded.

The definition and several lemmas play a crucial role in later control process and theoretical analysis.

Definition 1 ([24]). For $V(x,t)\in\mathcal{C}^{2,1}$ associated with system (3), $\mathcal{L}$ is defined as $\mathcal{L}V=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f+\frac{1}{2}\mathrm{Tr}\{g^{\top}\frac{\partial^2 V}{\partial x^2}g\}$, where $\frac{1}{2}\mathrm{Tr}\{g^{\top}\frac{\partial^2 V}{\partial x^2}g\}$ is the Hessian term.

Lemma 1 ([22]). If the continuous function $h:[m,n]\rightarrow\mathbb{R}$ is monotone and $h(m)=0$, then $|\int_m^n h(t)\mathrm{d}t|\leqslant|h(n)||n-m|$.

Lemma 2 ([22]). For $x_1,x_2\in\mathbb{R},h(x_1,\,x_2)>0$ and $g(x_1,\,x_2)>0$, one has

$$|h(x_1,\,x_2)x_1^m x_2^n|\leqslant g(x_1,\,x_2)|x_1|^{m+n}+\frac{n}{m+n}\left(\frac{m}{(m+n)g(x_1,\,x_2)}\right)^{\frac{m}{n}}|h(x_1,\,x_2)|^{\frac{m+n}{n}}|x_2|^{m+n}.$$

Lemma 3 ([16]). $|x_1^n-x_2^n|\geqslant 2^{1-n}|x_1-x_2|^n$ when $n\geqslant 1$, and $|x_1^n-x_2^n|\leqslant 2^{1-n}|x_1-x_2|^n$ when $0<n\leqslant 1$.

Lemma 4 ([25]). $|x_1+x_2|^n\leqslant|x_1|^n+|x_2|^n$ when $0\leqslant n<1$, and $|x_1+x_2|^n\leqslant 2^{n-1}(|x_1|^n+|x_2|^n)$ when $n\geqslant 1$.

Lemma 5 ([26]). For $0<n<p<\infty$, let $L^p(\Omega;\mathbb{R}^n)$ be a set of $\mathbb{R}^n$-valued random variables with $E|X|^p<\infty$. The inequality $(E|X|^n)^{\frac{1}{n}}\leqslant(E|X|^p)^{\frac{1}{p}}$ holds.

Lemma 6. For system (3), if there exists $V(x,t)\in\mathcal{C}^{2,1}(\mathbb{R}^n\times[-\tau;\infty);\mathbb{R}^+)$ satisfying

$$\alpha_1(|x|)\leqslant V(x,t)\leqslant\alpha_2\bigg(\sup_{-\tau\leqslant s\leqslant 0}|x(t+s)|\bigg),\tag{4}$$

$$\mathcal{L}V(x,t)\leqslant-\rho_1 V(x,t)+\rho_2,\tag{5}$$

then

(i) there is a unique strong solution with initial condition $\phi\in C^n_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$;

(ii) there holds the following inequality:

$$\mathrm{E}[V(x,t)]\leqslant\mathrm{e}^{-\rho_1(t-t_0)}V(x(t_0),t_0)+\frac{\rho_2}{\rho_1}\big(1-\mathrm{e}^{-\rho_1(t-t_0)}\big),$$

where $\alpha_1(\cdot),\alpha_2(\cdot)\in\mathcal{K}_\infty$, and constants $\rho_1,\rho_2>0$.

Proof. (i) Considering the inequality (5) and the definition of $V(\cdot)$, we have

$$\mathcal{L}V(x,t)\leqslant-\rho_1 V(x,t)+\rho_2\leqslant\rho_2.$$

Similar to the proofs in [24, 27], system (3) has a unique solution on $[-\tau,\infty)$.

(ii) For integer $r\geqslant 1$, the stopping time $\Gamma_{\mathrm{r}}=\inf\{t:t\geqslant 0,|x|\geqslant r\}$. Since system (3) has a unique solution, one gets $P(\Gamma_\infty=\infty)=1$. By the proof of Theorem A.1 in [27], one gets the Itô formula

$$\mathrm{E}[V(x(t_{\mathrm{r}}),t_{\mathrm{r}})]-\mathrm{E}[V(x(t_0),t_0)]=\mathrm{E}\bigg[\int_{t_0}^{t_{\mathrm{r}}}\mathcal{L}V(x(s),s)\mathrm{d}s\bigg],\tag{6}$$

where $t_{\mathrm{r}}=t\wedge\Gamma_{\mathrm{r}}$. From (6), it follows that

$$\begin{aligned}&\mathrm{E}\left[\mathrm{e}^{\rho_1 t_{\mathrm{r}}}V(x(t_{\mathrm{r}}),t_{\mathrm{r}})\right]-\mathrm{E}\left[\mathrm{e}^{\rho_1 t_0}V(x(t_0),t_0)\right]\\&\leqslant\mathrm{E}\bigg[\int_{t_0}^{t_{\mathrm{r}}}\mathrm{e}^{\rho_1 s}\big(-\rho_1 V(x(s),s)+\rho_2\big)\mathrm{d}s\bigg]+\mathrm{E}\bigg[\int_{t_0}^{t_{\mathrm{r}}}\rho_1\mathrm{e}^{\rho_1 s}V(x(s),s)\mathrm{d}s\bigg]\\&=\mathrm{E}\bigg[\int_{t_0}^{t_{\mathrm{r}}}\rho_2\mathrm{e}^{\rho_1 s}\mathrm{d}s\bigg]=\mathrm{E}\bigg[\frac{\rho_2}{\rho_1}\big(\mathrm{e}^{\rho_1 t_{\mathrm{r}}}-\mathrm{e}^{\rho_1 t_0}\big)\bigg],\end{aligned}$$

which further gives

$$\mathrm{E}\left[\mathrm{e}^{\rho_1 t_{\mathrm{r}}}V(x(t_{\mathrm{r}}),t_{\mathrm{r}})\right]\leqslant\mathrm{e}^{\rho_1 t_0}V(x(t_0),t_0)+\mathrm{E}\bigg[\frac{\rho_2}{\rho_1}\big(\mathrm{e}^{\rho_1 t_{\mathrm{r}}}-\mathrm{e}^{\rho_1 t_0}\big)\bigg].$$

Setting $r\rightarrow\infty$ and considering $t_{\mathrm{r}}=t\wedge\Gamma_{\mathrm{r}}$, $P(\Gamma_\infty=\infty)=1$, we have

$$\mathrm{E}[V(x,t)]\leqslant\mathrm{e}^{-\rho_1(t-t_0)}V(x(t_0),t_0)+\frac{\rho_2}{\rho_1}\big(1-\mathrm{e}^{-\rho_1(t-t_0)}\big).$$

This completes the proof.

To facilitate the practical tracking control, the coordinate transformations are introduced,

$$\begin{aligned} z_1 &= (x_1 - y_{\rm r})^{\frac{\mu}{r_1}}, \\ z_i &= x_i^{\frac{\mu}{r_i}} - \alpha_i^{\frac{\mu}{r_i}}, \quad \alpha_i^{p_{i-1}} = -\beta_{i-1}^{\frac{r_i p_{i-1}}{\mu}} z_{i-1}^{\frac{r_i p_{i-1}}{\mu}}, \quad i = 2, \ldots, n, \\ u &= x_{n+1}, \quad \alpha_{n+1}^{p_n} = -\beta_n^{\frac{r_{n+1} p_n}{\mu}} z_n^{\frac{r_{n+1} p_n}{\mu}}, \end{aligned} \tag{7}$$

where $\beta_1, \ldots, \beta_n > 0$ to be specified later and $\mu \in R^+_{\rm odd}$ satisfies $\max_{1\leqslant i\leqslant n}\{2r_i\} \leqslant \mu \leqslant r_n + \varpi$. Define

$$\begin{aligned} V_i &= \sum_{j=1}^{i}(n-j+1)W_j + \sum_{j=1}^{i} U_j + c_0 V_0, \quad W_i = \int_{t-\eta(t)}^{t} {\rm e}^{s-t} z_i^{\frac{4\lambda}{\mu}}(s){\rm d}s, \quad i = 1, \ldots, n, \\ U_1 &= \frac{r_1}{4\lambda - \varpi} z_1^{\frac{4\lambda-\varpi}{\mu}}, \quad U_k = \int_{\alpha_k}^{x_k} \left(s^{\frac{\mu}{r_k}} - \alpha_k^{\frac{\mu}{r_k}}\right)^{\frac{4\lambda-\varpi-r_k}{\mu}} {\rm d}s, \quad k = 2, \ldots, n, \end{aligned} \tag{8}$$

where $0 < c_0 \leqslant \frac{(1-\bar{\tau}){\rm e}^{-\tau}}{3l_4} \cdot 2^{\frac{-4\lambda+r_1}{r_1}}$.

To design a delay-independent tracking controller, the following propositions are given. For clarity, the proofs are placed in appendices.

Proposition 1. The following equation holds

$$\alpha_i^{p_{i-1}} = -\beta_{i-1}^{\frac{r_i p_{i-1}}{\mu}} \left(x_{i-1}^{\frac{\mu}{r_{i-1}}} + \beta_{i-2}\left(x_{i-2}^{\frac{\mu}{r_{i-2}}} + \cdots + \beta_2\left(x_2^{\frac{\mu}{r_2}} + \beta_1 z_1\right)\cdots\right)\right)^{\frac{r_i p_{i-1}}{\mu}}, \quad i = 2, \ldots, n+1.$$

Proposition 2. There exist constants $r_{11}, r_{12}, r_{13} > 0$ and small enough positive parameters C_{11}, C_{12} and C_{13} such that

$$\begin{aligned} z_1^{\frac{4\lambda-\varpi-r_1}{\mu}}(f_1 - \dot{y}_{\rm r}) &\leqslant \tfrac{c_0 l_3}{2(n+1)}|\zeta_0|^{4\lambda} + \tfrac{1-\bar{\tau}}{3}{\rm e}^{-\tau} z_1^{\frac{4\lambda}{\mu}}(t-\eta) + r_{11} z_1^{\frac{4\lambda}{\mu}} + C_{11}, \\ c_0 l_4\left(x_1^{\frac{4\lambda}{r_1}} + x_1^{\frac{4\lambda}{r_1}}(t-\eta)\right) &\leqslant \tfrac{1-\bar{\tau}}{3}{\rm e}^{-\tau} z_1^{\frac{4\lambda}{\mu}}(t-\eta) + r_{12} z_1^{\frac{4\lambda}{\mu}} + C_{12}, \\ \tfrac{4\lambda-\varpi-\mu}{2r_1}{\rm Tr}\left\{g_1^{\top} z_1^{\frac{4\lambda-\varpi-2r_1}{\mu}} g_1\right\} &\leqslant \tfrac{c_0 l_3}{2(n+1)}|\zeta_0|^{4\lambda} + \tfrac{1-\bar{\tau}}{3}{\rm e}^{-\tau} z_1^{\frac{4\lambda}{\mu}}(t-\eta) + r_{13} z_1^{\frac{4\lambda}{\mu}} + C_{13}. \end{aligned}$$

Proposition 3. The infinitesimal generator of U_i along (1) satisfies

$$\mathcal{L}U_i \leqslant -\frac{1}{9} z_{i-1}^{\frac{4\lambda}{\mu}} + \frac{c_0 l_3}{n+1}|\zeta_0|^{4\lambda} + a_i z_i^{\frac{A_i}{\mu}} x_{i+1}^{p_i} + \sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}} + (1-\bar{\tau}){\rm e}^{-\tau}\sum_{j=1}^{i} z_j^{\frac{4\lambda}{\mu}}(t-\eta) + \sum_{j=1}^{9}\left(r_{ij} z_i^{\frac{4\lambda}{\mu}} + C_{ij}\right),$$

with constants $r_{ij} > 0$ and the small enough design parameter $C_{ij} > 0$, $i = 2, \ldots, n$.

3 Practical tracking control design

3.1 Construction procedure of the controller

Now, we construct a tracking controller (2) by a recursive design procedure.

Step 1. From system (1) and transformation (7), a simple calculation gives

$${\rm d}z_1 = \frac{\mu}{r_1} z_1^{\frac{\mu-r_1}{\mu}}({\rm d}x_1 - {\rm d}y_{\rm r}) = \frac{\mu}{r_1} z_1^{\frac{\mu-r_1}{\mu}}\left((a_1 x_2^{p_1} + f_1 - \dot{y}_{\rm r}){\rm d}t + g_1^{\top}{\rm d}\omega\right). \tag{9}$$

Using Definition 1, Assumptions 2, 4, (8) and (9), we have

$$\mathcal{L}V_1 \leqslant -c_0 l_3|\zeta_0|^{4\lambda} - n(1-\bar{\tau}){\rm e}^{-\tau} z_1^{\frac{4\lambda}{\mu}}(t-\eta) - nW_1 + a_1 z_1^{\frac{4\lambda-\varpi-r_1}{\mu}}\left(x_2^{p_1} - \alpha_2^{p_1}\right) + a_1 z_1^{\frac{4\lambda-\varpi-r_1}{\mu}}\alpha_2^{p_1}$$

$$+nz_1^{\frac{4\lambda}{\mu}} + z_1^{\frac{4\lambda-\varpi-r_1}{\mu}}(f_1 - \dot{y}_{\rm r}) + c_0 l_4 \left(x_1^{\frac{4\lambda}{r_1}} + x_1^{\frac{4\lambda}{r_1}}(t-\eta) \right) + \frac{4\lambda-\varpi-\mu}{2r_1}{\rm Tr}\left\{ g_1^\top z_1^{\frac{4\lambda-\varpi-2r_1}{\mu}} g_1 \right\},$$

from which and Proposition 2, it follows

$$\mathcal{L}V_1 \leqslant -\frac{nc_0l_3}{n+1}|\zeta_0|^{4\lambda} - (n-1)(1-\bar{\tau}){\rm e}^{-\tau} z_1^{\frac{4\lambda}{\mu}}(t-\eta) - nW_1 + a_1 z_1^{\frac{4\lambda-\varpi-r_1}{\mu}}\left(x_2^{p_1} - \alpha_2^{p_1}\right)$$
$$+a_1 z_1^{\frac{4\lambda-\varpi-r_1}{\mu}}\alpha_2^{p_1} + (n + r_{11} + r_{12} + r_{13}) z_1^{\frac{4\lambda}{\mu}} + C_1, \tag{10}$$

where $C_1 \triangleq C_{11} + C_{12} + C_{13}$. Similar to the selection process in [21], C_{11}, C_{12} and C_{13} do not depend on any parameter. Choosing $\alpha_2^{p_1} = -\beta_1^{\frac{r_2p_1}{\mu}} z_1^{\frac{r_2p_1}{\mu}}$ with $\beta_1 \geqslant (\frac{1}{\epsilon}(2n + r_{11} + r_{12} + r_{13}))^{\frac{\mu}{r_2p_1}}$ and substituting it into (10), there holds

$$\mathcal{L}V_1 \leqslant -nz_1^{\frac{4\lambda}{\mu}} - \frac{nc_0l_3}{n+1}|\zeta_0|^{4\lambda} - (n-1)(1-\bar{\tau}){\rm e}^{-\tau} z_1^{\frac{4\lambda}{\mu}}(t-\eta) - nW_1 + a_1 z_1^{\frac{4\lambda-\varpi-r_1}{\mu}}\left(x_2^{p_1} - \alpha_2^{p_1}\right) + C_1.$$

Step i ($i = 2,\ldots,n$). At Step $i-1$, assume that there exists a L-K functional $V_{i-1} \in \mathcal{C}^{2,1}$, a virtual control $\alpha_i^{p_{i-1}} = -\beta_{i-1}^{\frac{r_ip_{i-1}}{\mu}} z_{i-1}^{\frac{r_ip_{i-1}}{\mu}}$ and a small enough constant $C_{i-1} > 0$, such that

$$\mathcal{L}V_{i-1} \leqslant -(n-i+2)\sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}} - \frac{(n-i+2)c_0l_3}{n+1}|\zeta_0|^{4\lambda} - (n-i+1)(1-\bar{\tau}){\rm e}^{-\tau}\sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}}(t-\eta)$$
$$-\sum_{j=1}^{i-1}(n-j+1)W_j + a_{i-1} z_{i-1}^{\frac{4\lambda-\varpi-r_{i-1}}{\mu}}\left(x_i^{p_{i-1}} - \alpha_i^{p_{i-1}}\right) + C_{i-1}. \tag{11}$$

Combining Definition 1, Assumptions 2, (8) and (11), we have

$$\mathcal{L}V_i \leqslant -(n-i+2)\sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}} - \frac{(n-i+2)c_0l_3}{n+1}|\zeta_0|^{4\lambda} - (n-i+1)(1-\bar{\tau}){\rm e}^{-\tau}\sum_{j=1}^{i} z_j^{\frac{4\lambda}{\mu}}(t-\eta)$$
$$-\sum_{j=1}^{i}(n-j+1)W_j + (n-i+1)z_i^{\frac{4\lambda}{\mu}} + a_{i-1} z_{i-1}^{\frac{4\lambda-\varpi-r_{i-1}}{\mu}}\left(x_i^{p_{i-1}} - \alpha_i^{p_{i-1}}\right) + C_{i-1} + \mathcal{L}U_i. \tag{12}$$

By Lemmas 2, 3, $x_i^{p_{i-1}} - \alpha_i^{p_{i-1}} = (x_i^{\frac{\mu}{r_i}})^{\frac{r_ip_{i-1}}{\mu}} - (\alpha_i^{\frac{\mu}{r_i}})^{\frac{r_ip_{i-1}}{\mu}}$ and $0 < a_{i-1} \leqslant \bar{\epsilon}$, we can find $r_{i0} > 0$ satisfying

$$a_{i-1} z_{i-1}^{\frac{4\lambda-\varpi-r_{i-1}}{\mu}}\left(x_i^{p_{i-1}} - \alpha_i^{p_{i-1}}\right) \leqslant \bar{\epsilon}\, 2^{1-\frac{r_ip_{i-1}}{\mu}}|z_{i-1}|^{\frac{4\lambda-\varpi-r_{i-1}}{\mu}}|z_i|^{\frac{r_ip_{i-1}}{\mu}} \leqslant \frac{1}{9} z_{i-1}^{\frac{4\lambda}{\mu}} + r_{i0} z_i^{\frac{4\lambda}{\mu}}. \tag{13}$$

Substituting Proposition 3 and (13) into (12), it is not hard to arrive at

$$\mathcal{L}V_i \leqslant -(n-i+1)\sum_{j=1}^{i} z_j^{\frac{4\lambda}{\mu}} - \frac{(n-i+1)c_0l_3}{n+1}|\zeta_0|^{4\lambda} - (n-i)(1-\bar{\tau}){\rm e}^{-\tau}\sum_{j=1}^{i} z_j^{\frac{4\lambda}{\mu}}(t-\eta) + a_i z_i^{\frac{4\lambda-\varpi-r_i}{\mu}}\alpha_{i+1}^{p_i}$$
$$-\sum_{j=1}^{i}(n-j+1)W_j + \left(2n-2i+2+r_{i0}+\sum_{j=1}^{9} r_{ij}\right) z_i^{\frac{4\lambda}{\mu}} + a_i z_i^{\frac{4\lambda-\varpi-r_i}{\mu}}(x_{i+1}^{p_i} - \alpha_{i+1}^{p_i}) + C_i, \tag{14}$$

where $C_i \triangleq C_{i-1} + \sum_{j=1}^{9} C_{ij}$ is a small enough design parameter. If we choose $\alpha_{i+1}^{p_i} = -\beta_i^{\frac{r_{i+1}p_i}{\mu}} z_i^{\frac{r_{i+1}p_i}{\mu}}$ with $\beta_i \geqslant (\frac{1}{\epsilon}(2n-2i+2+r_{i0}+\sum_{j=1}^{9} r_{ij}))^{\frac{\mu}{r_{i+1}p_i}}$, then it follows from (14) that

$$\mathcal{L}V_i \leqslant -(n-i+1)\sum_{j=1}^{i} z_j^{\frac{4\lambda}{\mu}} - \frac{(n-i+1)c_0l_3}{n+1}|\zeta_0|^{4\lambda} - (n-i)(1-\bar{\tau}){\rm e}^{-\tau}\sum_{j=1}^{i} z_j^{\frac{4\lambda}{\mu}}(t-\eta)$$
$$-\sum_{j=1}^{i}(n-j+1)W_j + a_i z_i^{\frac{4\lambda-\varpi-r_i}{\mu}}(x_{i+1}^{p_i} - \alpha_{i+1}^{p_i}) + C_i,$$

which shows that Eq. (11) still holds for Step i. At Step n, we choose

$$u=\alpha_{n+1}=-\beta_n^{\frac{r_{n+1}}{\mu}}\left(x_n^{\frac{\mu}{r_n}}+\beta_{n-1}\left(x_{n-1}^{\frac{\mu}{r_{n-1}}}+\cdots+\beta_2\left(x_2^{\frac{\mu}{r_2}}+\beta_1(x_1-y_{\rm r})^{\frac{\mu}{r_1}}\right)\cdots\right)\right)^{\frac{r_{n+1}}{\mu}}, \tag{15}$$

where the system initial condition $\phi\in\mathcal{C}^n_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$. It can be deduced that

$$\mathcal{L}V_n\leqslant-\sum_{j=1}^n z_j^{\frac{4\lambda}{\mu}}-\frac{c_0l_3}{n+1}|\zeta_0|^{4\lambda}-\sum_{j=1}^n(n-j+1)W_j+C_n, \tag{16}$$

with $C_n=\sum_{k=1}^3C_{1k}+\sum_{l=2}^n\sum_{j=1}^9C_{lj}$ being a small enough design parameter.

By Assumption 4, the following inequality holds

$$-\frac{c_0l_3}{n+1}|\zeta_0|^{4\lambda}\leqslant-\frac{c_0l_3}{l_2(n+1)}V_0. \tag{17}$$

Noting $0<\frac{r_i}{\mu}<1, 0<\frac{r_1}{4\lambda-\varpi}<1$ and using Lemmas 1 and 3, we obtain

$$\begin{aligned}\sum_{j=1}^n U_j&\leqslant\frac{r_1}{4\lambda-\varpi}z_1^{\frac{4\lambda-\varpi}{\mu}}+\sum_{j=2}^n|z_i|^{\frac{4\lambda-\varpi-r_i}{\mu}}\left|\left(x_i^{\frac{\mu}{r_i}}\right)^{\frac{r_i}{\mu}}-\left(\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{r_i}{\mu}}\right|\\&\leqslant\frac{r_1}{4\lambda-\varpi}z_1^{\frac{4\lambda-\varpi}{\mu}}+2\sum_{j=2}^n z_i^{\frac{4\lambda-\varpi}{\mu}}\leqslant2\sum_{j=1}^n z_i^{\frac{4\lambda-\varpi}{\mu}}.\end{aligned} \tag{18}$$

Let $0<C_n\leqslant\min\{(\frac{l_3}{l_2(n+1)})^2,1\}$, by using Lemma 2, we have

$$2\sum_{j=1}^n z_i^{\frac{4\lambda-\varpi}{\mu}}\leqslant C_n^{-\frac12}\sum_{j=1}^n z_i^{\frac{4\lambda}{\mu}}+\frac{n\varpi}{2\lambda}\left(\frac{4\lambda-\varpi}{2\lambda C_n^{-\frac12}}\right)^{\frac{4\lambda-\varpi}{\varpi}}. \tag{19}$$

Combing (18) and (19), it follows that

$$-\sum_{j=1}^n z_i^{\frac{4\lambda}{\mu}}\leqslant-C_n^{\frac12}\sum_{j=1}^n U_j+C_n^{\frac{2\lambda}{\varpi}}\frac{n\varpi}{2\lambda}\left(\frac{4\lambda-\varpi}{2\lambda}\right)^{\frac{4\lambda-\varpi}{\varpi}}. \tag{20}$$

Substituting (17) and (20) into (16) yields

$$\begin{aligned}\mathcal{L}V_n&\leqslant-C_n^{\frac12}\sum_{j=1}^n U_j-\frac{c_0l_3}{l_2(n+1)}V_0-\sum_{j=1}^n(n-j+1)W_j+C_n^{\frac{2\lambda}{\varpi}}\frac{n\varpi}{2\lambda}\left(\frac{4\lambda-\varpi}{2\lambda}\right)^{\frac{4\lambda-\varpi}{\varpi}}+C_n\\&\leqslant-\rho_1V_n+\rho_2,\end{aligned} \tag{21}$$

where ρ_1 and ρ_2 are defined as $\rho_1=C_n^{\frac12}$, $\rho_2=C_n^{\frac{2\lambda}{\varpi}}\frac{n\varpi}{2\lambda}\left(\frac{4\lambda-\varpi}{2\lambda}\right)^{\frac{4\lambda-\varpi}{\varpi}}+C_n$.

3.2 Main results

We summarize the major consequences of this paper.

Theorem 1. If Assumptions 1–4 are satisfied, then, for system (1), a practical tracking controller (15) is designed which satisfies properties (H1)–(H3).

Proof. Firstly, it follows from Assumption 4, Lemma 3 and Proposition 1 that

$$\begin{aligned}c_0V_0+\sum_{j=1}^n U_j&=c_0V_0+\frac{r_1}{4\lambda-\varpi}|z_1|^{\frac{4\lambda-\varpi}{\mu}}+\sum_{j=2}^n\int_{\alpha_j}^{x_j}\left|s^{\frac{\mu}{r_j}}-\alpha_j^{\frac{\mu}{r_j}}\right|^{\frac{4\lambda-\varpi-r_j}{\mu}}{\rm d}s\\&\geqslant c_0l_1|\zeta_0|^{4\lambda}+\frac{r_1}{4\lambda-\varpi}|z_1|^{\frac{4\lambda-\varpi}{\mu}}+\sum_{j=2}^n\int_{\alpha_j}^{x_j}\left(2^{1-\frac{\mu}{r_j}}|s-\alpha_j|^{\frac{\mu}{r_j}}\right)^{\frac{4\lambda-\varpi-r_j}{\mu}}{\rm d}s,\end{aligned}$$

which implies that $c_0V_0+\sum_{j=1}^n U_j$ is a positive definite function.

Subsequently, by Assumption 4, Lemma 1 and Proposition 1, it follows that

$$c_0V_0+\sum_{j=1}^n U_j \leqslant c_0l_2|\zeta_0|^{4\lambda}+\frac{r_1}{4\lambda-\varpi}|z_1|^{\frac{4\lambda-\varpi}{\mu}}+\sum_{j=2}^{n-1}|z_j|^{\frac{4\lambda-\varpi-r_j}{\mu}}|x_j-\alpha_j|.$$

Therefore, $c_0V_0+\sum_{j=1}^n U_j$ is positive definite and radially unbounded. Applying Lemma 4.3 in [28], for $Y(t)\triangleq[\zeta_0^\top(t),z_1(t),x_2(t),\ldots,x_n(t)]^\top$, there exist $\alpha_1(Y(t)),\alpha_2(Y(t))\in\mathcal{K}_\infty$ such that

$$\alpha_1(|Y(t)|)\leqslant c_0V_0+\sum_{j=1}^n U_j\leqslant \alpha_2(|Y(t)|)\leqslant \alpha_2\left(\sup_{-\tau\leqslant s\leqslant 0}|Y(t+s)|\right). \tag{22}$$

From (22) and the definition of V_n, it follows that

$$V_n(t,Y(t))\geqslant\alpha_1(|Y(t)|),\quad W_j\geqslant 0,\ j=1,\ldots,n. \tag{23}$$

Using Lemma 1, (7) and Proposition 1, there is a function $\alpha_3(\cdot)\in\mathcal{K}_\infty$ satisfying

$$\begin{aligned}\sum_{j=1}^n(n-j+1)W_j &\leqslant \sup_{-\tau\leqslant s\leqslant 0}\left(\sum_{j=1}^n(n-j+1)\tau\left|x_j^{\frac{\mu}{r_j}}(t+s)-\alpha_j^{\frac{\mu}{r_j}}(t+s)\right|^{\frac{4\lambda}{\mu}}\right)\\ &\triangleq \alpha_3\left(\sup_{-\tau\leqslant s\leqslant 0}|Y(t+s)|\right).\end{aligned} \tag{24}$$

From (22)–(24), we know that

$$\alpha_1(|Y(t)|)\leqslant V_n\leqslant\alpha_4\left(\sup_{-\tau\leqslant s\leqslant 0}|Y(t+s)|\right), \tag{25}$$

where $\alpha_4(\sup_{-\tau\leqslant s\leqslant 0}|Y(t+s)|)\in\mathcal{K}_\infty$.

Considering Lemma 6, inequalities (21) and (25) show that the property (H1) holds and

$$\mathrm{E}[V_n(Y(t),t)]\leqslant \mathrm{e}^{-\rho_1(t-t_0)}V_n(Y(t_0),t_0)+\rho_1^{-1}\rho_2\big(1-\mathrm{e}^{-\rho_1(t-t_0)}\big)\leqslant V_n(Y(t_0),t_0)+\rho_1^{-1}\rho_2. \tag{26}$$

From the boundedness of ρ_1,ρ_2, then $V_n(Y(t),t)$ is bounded in probability, which further indicates that the property (H2) is fulfilled. Finally, using the inequality $\frac{r_1}{4\lambda-\varpi}|z_1|^{\frac{4\lambda-\varpi}{\mu}}\leqslant V_n(Y(t),t)$, and noting $\frac{4\lambda-\varpi}{\mu}>\frac{2r_1}{\mu}$, Lemma 5 and (26), one has

$$\begin{aligned}\mathrm{E}|z_0|^2 &= \mathrm{E}|z_1|^{\frac{2r_1}{\mu}}\leqslant\left(\mathrm{E}\left[|z_1|^{\frac{4\lambda-\varpi}{\mu}}\right]\right)^{\frac{2r_1}{4\lambda-\varpi}}\leqslant\left(\frac{4\lambda-\varpi}{r_1}\mathrm{E}[V_n(Y(t),t)]\right)^{\frac{2r_1}{4\lambda-\varpi}}\\ &\leqslant\left(\frac{4\lambda-\varpi}{r_1}\left(\mathrm{e}^{-\rho_1(t-t_0)}V_n(Y(t_0),t_0)+\rho_1^{-1}\rho_2\big(1-\mathrm{e}^{-\rho_1(t-t_0)}\big)\right)\right)^{\frac{2r_1}{4\lambda-\varpi}}.\end{aligned} \tag{27}$$

Taking the limit on both sides of (27), it follows that $\lim_{t\to\infty}\mathrm{E}|z_0|^2\leqslant\rho$ with $\rho=(\frac{\rho_2(4\lambda-\varpi)}{\rho_1r_1})^{\frac{2r_1}{4\lambda-\varpi}}$.

Noticing $\rho_1=C_n^{\frac{1}{2}}$, $\rho_2=C_n^{\frac{2\lambda}{\varpi}}\frac{n\varpi}{2\lambda}\big(\frac{4\lambda-\varpi}{2\lambda}\big)^{\frac{4\lambda-\varpi}{\varpi}}+C_n$, it follows that

$$\begin{aligned}\rho &= \left(\frac{\rho_2(4\lambda-\varpi)}{\rho_1r_1}\right)^{\frac{2r_1}{4\lambda-\varpi}}=\left(\frac{(C_n^{\frac{2\lambda}{\varpi}}\frac{n\varpi}{2\lambda}\big(\frac{4\lambda-\varpi}{2\lambda}\big)^{\frac{4\lambda-\varpi}{\varpi}}+C_n)(4\lambda-\varpi)}{C_n^{\frac{1}{2}}r_1}\right)^{\frac{2}{4\lambda-\varpi}}\\ &=\left(C_n^{\frac{4\lambda-\varpi}{2\varpi}}\frac{n\varpi}{r_1}\left(\frac{4\lambda-\varpi}{2\lambda}\right)^{\frac{4\lambda}{\varpi}}+C_n^{\frac{1}{2}}\frac{4\lambda-\varpi}{r_1}\right)^{\frac{2}{4\lambda-\varpi}}.\end{aligned}$$

Since n,r_1,λ,ϖ are constants, $4\lambda-\varpi>0$, $C_n=\sum_{k=1}^3C_{1k}+\sum_{l=2}^n\sum_{j=1}^9C_{lj}\leqslant\min\{\big(\frac{l_3}{l_2(n+1)}\big)^2,1\}$, and $C_{1k},C_{lj},k=1,\ldots,3,l=2,\ldots,n,j=1,\ldots,9$ are designed to be independent of any parameter (see the work in [21] for details). By choosing C_{1k},C_{lj} small enough, the parameter ρ can be adjusted to be small enough. The property (H3) is met. The proof is completed.

Remark 4. For the deterministic systems [22, 29], $V(x,t)$ is $\mathcal{C}^1$. However for the stochastic systems [3, 5, 9, 19, 20], due to the existence of Hessian term, the L-K functional $V(x,t)$ must be $\mathcal{C}^2$ with respect to x and $\mathcal{C}^1$ with respect to t. Therefore, in this paper, how to choose a well-defined and meaningful L-K functional is a difficult task. To solve this problem, let $\mu \in R^+_{\rm odd}$ and suppose $\max_{1\leqslant i\leqslant n}\{2r_i\} \leqslant \mu \leqslant r_n + \varpi, \lambda \geqslant \max_{1\leqslant i\leqslant n}\{r_i+\varpi\}$, then $\frac{4\lambda-\varpi-r_i-2\mu}{\mu} \geqslant 1, \frac{\mu-2r_i}{r_i} \geqslant 0$. In this case, $\frac{\partial^2 U_i}{\partial z_1^2}$ and $\frac{\partial^2 U_i}{\partial x_j^2}, j=2,\ldots,i-1$ are well defined, which ensure that $U_i, i=2,\ldots,n$ is $\mathcal{C}^2$ with respect to $[z_1,x_2,\ldots,x_n]^\top$. By the definition of V_n and $V_0(\zeta_0) \in \mathcal{C}^2$, we know that V_n is $\mathcal{C}^2$ with respect to $[\zeta_0,z_1,x_2,\ldots,x_n]^\top$ and $\mathcal{C}^1$ with respect to t.

In fact, when system (1) has no SISS-like inverse dynamics, we can also design the practical tracking controller. The stochastic system is described as

$$\begin{cases} {\rm d}x_i &= \big(a_i x_{i+1}^{p_i} + f_i(\bar{x}_i, \bar{x}_i(t-\eta)), t\big){\rm d}t + g_i^\top(\bar{x}_i,\bar{x}_i(t-\eta),t){\rm d}\omega, \quad i=1,\ldots,n-1,\\ {\rm d}x_n &= \big(a_n u^{p_n} + f_n(x,x(t-\eta)), t\big){\rm d}t + g_n^\top(x,x(t-\eta),t){\rm d}\omega,\\ \ \ y &= x_1, \end{cases} \tag{28}$$

where the vector $[x,x(t-\eta)]^\top \in \mathbb{R}^{2n}$; $f_j(\cdot), g_j(\cdot), j=1,\ldots,n$ satisfy locally Lipschitz condition and other parameters are defined as in (1). Assumption 5 is imposed on system (28).

Assumption 5. Let C, l_{i1} and $l_{i2}, i=1,\ldots,n$ be nonnegative constants, then

$$\begin{cases} |f_i| \leqslant C\displaystyle\sum_{j=1}^{i}\left(|x_j|^{\frac{r_i+\varpi}{r_j}} + |x_j(t-\eta)|^{\frac{r_i+\varpi}{r_j}}\right) + l_{i1},\\ |g_i| \leqslant C\displaystyle\sum_{j=1}^{i}\left(|x_j|^{\frac{2r_i+\varpi}{2r_j}} + |x_j(t-\eta)|^{\frac{2r_i+\varpi}{2r_j}}\right) + l_{i2}. \end{cases} \tag{29}$$

Theorem 2. For the reference trajectory $y_{\rm r}$ to be traced, suppose that Assumptions 1, 2 and 5 hold, then there is a tracking controller similar to (15), such that the nonlinear system (28) has a unique strong solution which is bounded in probability and the mean square of z_0 converges to an arbitrary neighborhood of zero.

Proof. Introducing the same coordinate transformation (7), an L-K functional is chosen as

$$\overline{V}_n = \sum_{j=1}^{n}(n-j+1)\int_{t-\eta}^{t} {\rm e}^{s-t} z_j^{\frac{4\lambda}{\mu}}(s){\rm d}s + \frac{r_1}{4\lambda-\varpi} z_1^{\frac{4\lambda-\varpi}{\mu}} + \sum_{j=2}^{n}\int_{\alpha_j}^{x_j}\left(s^{\frac{\mu}{r_j}} - \alpha_j^{\frac{\mu}{r_j}}\right)^{\frac{4\lambda-\varpi-r_j}{\mu}}{\rm d}s.$$

Based on the similar design procedure to Theorem 1, one gets

$$\bar{\alpha}_1(|\bar{Y}|) \leqslant \overline{V}_n \leqslant \bar{\alpha}_2\left(\sup_{-\tau\leqslant s\leqslant 0}|\bar{Y}(t+s)|\right), \quad \mathcal{L}\overline{V}_n \leqslant -\bar{\rho}_1\overline{V}_n + \bar{\rho}_2,$$

where $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$, $\bar{Y} \triangleq [y_{\rm r}, x_1, \ldots, x_n]^\top$ and $\bar{\rho}_1, \bar{\rho}_2 > 0$. Similar to Theorem 1, Theorem 2 can be shown easily.

4 Example

The above method is used to the following example:

$$\begin{cases} {\rm d}\zeta_0 &= \big(-5\zeta_0 + x_1(t-\eta(t))\big){\rm d}t + \zeta_0{\rm d}\omega,\\ {\rm d}x_1 &= x_2^3{\rm d}t + 0.2x_1^2(t-\eta(t)){\rm d}\omega,\\ {\rm d}x_2 &= \big(u^3 + \sin(\zeta_0)\sin(x_1(t-\eta(t)))x_2^3(t-\eta(t))\big){\rm d}t + \sin(x_2)x_2^2{\rm d}\omega,\\ \ \ y &= x_1, \end{cases} \tag{30}$$

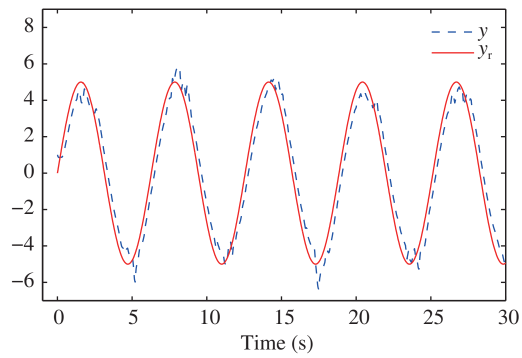

Figure 1 (Color online) Trajectories of y and $y_{\rm r}$.

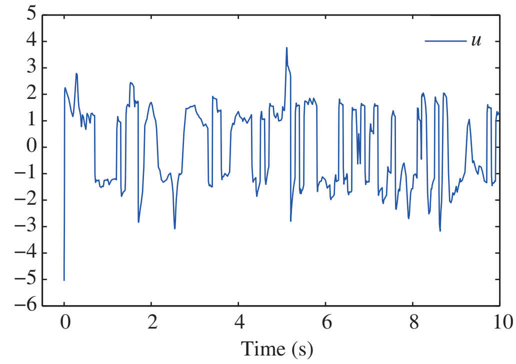

Figure 2 (Color online) The trajectory of input u.

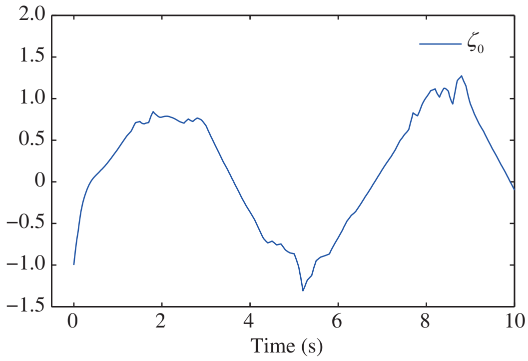

Figure 3 (Color online) The trajectory of the inverse dynamics ζ_0.

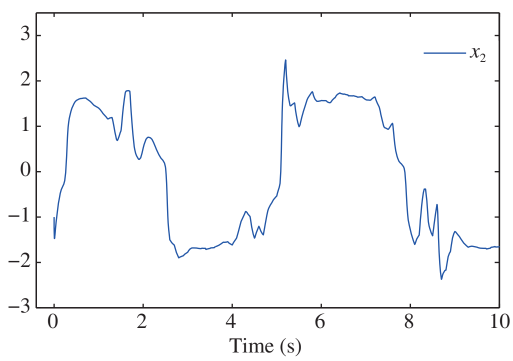

Figure 4 (Color online) The trajectory of the state x_2.

where the reference signal $y_{\rm r} = 5\sin(t)$ and $\eta(t) = 0.1 + 0.1\sin(t)$. By simple calculation, Assumptions 1 and 2 hold. Taking $\varpi = 2$, from $p_1 = p_2 = 3, r_1 = 1, r_{i+1} = \frac{r_i+\varpi}{p_i}, i = 1, 2$, one gets $r_2 = r_3 = 1$. By choosing $\mu = \lambda = 3$, Assumption 3 holds with $f_1 = 0, |f_2| \triangleq |\sin(\zeta_0)\sin(x_1(t-\eta))x_2^3(t-\eta)| \leqslant |x_2(t-\eta)|^3, |g_1| \triangleq 0.2|x_1(t-\eta)|^2$, and $|g_2| \triangleq |\sin(x_2)x_2^2| \leqslant |x_2|^2$. If we choose $V_0 = \zeta_0^{12}$, it can be seen that $\mathcal{L}V_0 \leqslant -49\zeta_0^{12} + x_1^{12}(t-\eta)$, which shows that Assumption 4 holds.

Following the design procedure above, we can construct a tracking controller

$$u = -3.5\big(x_2^3 + 1.6(x_1 - \sin(t))^3\big)^{\frac{1}{3}}.$$

In the simulation, the design parameters are selected as $\zeta_0(t_0) = -1$, $x_1(t_0) = 1$, $x_2(t_0) = -1, t_0 \in [-0.2, 0]$. From Figures 1–4, the effectiveness of the design procedure is verified.

Remark 5. Tracking control design has been widely studied both in the theoretical research and in the practical application. As demonstrated in [19], a practical example satisfying system (28) is investigated. Via the coordinate conversion, the underactuated system with weak coupling is transformed into the stochastic system:

$$\begin{aligned} &{\rm d}x_1 = x_2{\rm d}t, \quad {\rm d}x_2 = a_2x_3^3{\rm d}t + f_2{\rm d}t, \quad {\rm d}x_3 = x_4{\rm d}t, \\ &{\rm d}x_4 = (a_4u + f_4){\rm d}t + g_4{\rm d}\omega, \quad y = x_1, \end{aligned}$$

where $x_1, \ldots, x_4$ are the system states, y is the system output, $y_{\rm r} = 0.2\sin(t)$ is the signal to be tracked, a_2, a_4 are constants, the time-delay is equal to zero, $p_1 = \cdots = p_4 = 1$, and f_2, f_4, g_4 are smooth functions.

5 Conclusion

For stochastic nonlinear systems (1) and (28), this paper discusses the problem of practical tracking control. Before the control design, a useful theoretical tool is presented to achieve some meaningful theoretical results, that is, the closed-loop system has a unique strong solution and the solutions are bounded in probability. Since the conventional quadratic Lyapunov function does no longer suit to the design procedure, a class of new L-K functionals are constructed flexibly. A modified version of adding a power integrator is also introduced. Finally, an example is utilized to demonstrate the feasibility of our design scheme and excellent tracking performance. There still exist some problems for further investigation, such as how to solve the control problem when the considered system contains unknown parameters or some of the system states cannot be measured. Future work will contribute to adaptive tracking control model and adaptive output feedback design of system (1).

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Conflict of interest The authors declare that they have no conflict of interest.

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Appendix A Proof of Proposition 1

By transformation (7), it can be deduced that

$$\begin{aligned}\alpha_i^{p_{i-1}} &= -\beta_{i-1}^{\frac{r_ip_{i-1}}{\mu}} z_{i-1}^{\frac{r_ip_{i-1}}{\mu}} = -\beta_{i-1}^{\frac{r_ip_{i-1}}{\mu}}\left(x_{i-1}^{\frac{\mu}{r_{i-1}}} - \alpha_{i-1}^{\frac{\mu}{r_{i-1}}}\right)^{\frac{r_ip_{i-1}}{\mu}}\\ &= -\beta_{i-1}^{\frac{r_ip_{i-1}}{\mu}}\left(x_{i-1}^{\frac{\mu}{r_{i-1}}} + \beta_{i-2}z_{i-2}\right)^{\frac{r_ip_{i-1}}{\mu}} = -\beta_{i-1}^{\frac{r_ip_{i-1}}{\mu}}\left(x_{i-1}^{\frac{\mu}{r_{i-1}}} + \beta_{i-2}\left(x_{i-2}^{\frac{\mu}{r_{i-2}}} - \alpha_{i-2}^{\frac{\mu}{r_{i-2}}}\right)\right)^{\frac{r_ip_{i-1}}{\mu}}\\ &= -\beta_{i-1}^{\frac{r_ip_{i-1}}{\mu}}\left(x_{i-1}^{\frac{\mu}{r_{i-1}}} + \beta_{i-2}\left(x_{i-2}^{\frac{\mu}{r_{i-2}}} + \cdots + \beta_2\left(x_2^{\frac{\mu}{r_2}} + \beta_1 z_1\right)\cdots\right)\right)^{\frac{r_ip_{i-1}}{\mu}}.\end{aligned}$$

Appendix B Proof of Proposition 2

By defining $\delta_{i0} = \max\{1, 2^{\frac{r_i+\varpi-r_1}{r_1}}\}$, $i = 1,\ldots,n$ and using Assumption 1, Lemma 4 and (7), it yields that

$$|x_1|^{\frac{r_i+\varpi}{r_1}} \leqslant \delta_{i0}\left(|z_1|^{\frac{r_i+\varpi}{\mu}} + M^{\frac{r_i+\varpi}{r_1}}\right), \quad |x_k|^{\frac{r_i+\varpi}{r_k}} \leqslant |z_k|^{\frac{r_i+\varpi}{\mu}} + |\beta_{k-1}z_{k-1}|^{\frac{r_i+\varpi}{\mu}},$$

from which and Assumptions 1–3, one gets

$$\begin{aligned}|f_i| \leqslant\; & C|\zeta_0|^{r_i+\varpi} + \delta_{i0}C\left(|z_1|^{\frac{r_i+\varpi}{\mu}} + |z_1(t-\eta)|^{\frac{r_i+\varpi}{\mu}}\right) + 2\delta_{i0}CM^{\frac{r_i+\varpi}{r_1}} + l_{i1}\\ & + \sum_{k=2}^{i} C\left(|z_k|^{\frac{r_i+\varpi}{\mu}} + |\beta_{k-1}z_{k-1}|^{\frac{r_i+\varpi}{\mu}} + |z_k(t-\eta)|^{\frac{r_i+\varpi}{\mu}} + |\beta_{k-1}z_{k-1}(t-\eta)|^{\frac{r_i+\varpi}{\mu}}\right)\\ \leqslant\; & \widehat{C}_{i1}\left(1 + |\zeta_0|^{r_i+\varpi} + \sum_{k=1}^{i}\left(|z_k|^{\frac{r_i+\varpi}{\mu}} + |z_k(t-\eta)|^{\frac{r_i+\varpi}{\mu}}\right)\right),\end{aligned} \tag{B1}$$

where $\widehat{C}_{i1} \triangleq \max_{2\leqslant k\leqslant i-1}\{C, C(\delta_{i0} + \beta_1^{\frac{r_i+\varpi}{\mu}}), C\beta_k^{\frac{r_i+\varpi}{\mu}}, 2\delta_{i0}CM^{\frac{r_i+\varpi}{r_1}} + l_{i1}\}$. For simplicity, let $\sum_{k=2}^{1} x_k = 0$ for all x_k.

Similarly, define $\delta_{i2} \triangleq \max\{1, 2^{\frac{2r_i+\varpi-2r_1}{2r_1}}\}$. By $0 < \frac{\mu-2r_k}{\mu} < 1$, one has

$$|x_1|^{\frac{2r_i+\varpi}{2r_1}} = |z_1^{\frac{r_1}{\mu}} + y_r|^{\frac{2r_i+\varpi}{2r_1}} \leqslant \delta_{i2}\left(|z_1|^{\frac{2r_i+\varpi}{2\mu}} + M^{\frac{2r_i+\varpi}{2r_1}}\right), \quad |x_k|^{\frac{2r_i+\varpi}{2r_k}} \leqslant |z_k|^{\frac{2r_i+\varpi}{2\mu}} + |\beta_{k-1}z_{k-1}|^{\frac{2r_i+\varpi}{2\mu}}.$$

Then one can find two positive constants $\widehat{C}_{i2}$ and $\widehat{C}_{i3}$, such that

$$\begin{aligned}|g_i| \leqslant\; & C|\zeta_0|^{\frac{2r_i+\varpi}{2}} + \delta_{i2}C\left(|z_1|^{\frac{2r_i+\varpi}{2\mu}} + |z_1(t-\eta)|^{\frac{2r_i+\varpi}{2\mu}}\right) + 2\delta_{i2}CM^{\frac{2r_i+\varpi}{2r_1}} + l_{i2}\\ & + \sum_{k=2}^{i} C\left(|z_k|^{\frac{2r_i+\varpi}{2\mu}} + |\beta_{k-1}z_{k-1}|^{\frac{2r_i+\varpi}{2\mu}} + |z_k(t-\eta)|^{\frac{2r_i+\varpi}{2\mu}} + |\beta_{k-1}z_{k-1}(t-\eta)|^{\frac{2r_i+\varpi}{2\mu}}\right)\\ \leqslant\; & \widehat{C}_{i2}\left(1 + |\zeta_0|^{\frac{2r_i+\varpi}{2}} + \sum_{k=1}^{i}\left(|z_k|^{\frac{2r_i+\varpi}{2\mu}} + |z_k(t-\eta)|^{\frac{2r_i+\varpi}{2\mu}}\right)\right),\end{aligned} \tag{B2}$$

$$|g_i|^2 \leqslant \widehat{C}_{i3}\left(1+|\zeta_0|^{2r_i+\varpi}+\sum_{k=1}^{i}\left(|z_k|^{\frac{2r_i+\varpi}{\mu}}+|z_k(t-\eta)|^{\frac{2r_i+\varpi}{\mu}}\right)\right). \tag{B3}$$

From Assumption 1, Lemma 2 and (B1), there exist a small enough parameter $C_{11}>0$ and a constant $r_{11}>0$ such that

$$\begin{aligned}z_1^{\frac{4\lambda-\varpi-r_1}{\mu}}(f_1-\dot{y}_{\rm r}) &\leqslant |z_1|^{\frac{4\lambda-\varpi-r_1}{\mu}}\left(\widehat{C}_{11}\left(1+|\zeta_0|^{r_1+\varpi}+|z_1|^{\frac{r_1+\varpi}{\mu}}+|z_1(t-\eta)|^{\frac{r_1+\varpi}{\mu}}\right)+M\right)\\ &\leqslant \frac{c_0l_3}{2(n+1)}|\zeta_0|^{4\lambda}+\frac{1-\bar{\tau}}{3}\mathrm{e}^{-\tau}z_1^{\frac{4\lambda}{\mu}}(t-\eta)+r_{11}z_1^{\frac{4\lambda}{\mu}}+C_{11}.\end{aligned} \tag{B4}$$

Similar to the proof of (B4), one obtains

$$\begin{aligned}c_0l_4\left(x_1^{\frac{4\lambda}{r_1}}+x_1^{\frac{4\lambda}{r_1}}(t-\eta)\right) &\leqslant c_0l_4\left(\left(z_1^{\frac{r_1}{\mu}}+y_{\rm r}\right)^{\frac{4\lambda}{r_1}}+\left(z_1^{\frac{r_1}{\mu}}(t-\eta)+y_{\rm r}(t-\eta)\right)^{\frac{4\lambda}{r_1}}\right)\\ \leqslant c_0l_4 2^{\frac{4\lambda-r_1}{r_1}}\left(z_1^{\frac{4\lambda}{\mu}}+z_1^{\frac{4\lambda}{\mu}}(t-\eta)\right)&+c_0l_4(2M)^{\frac{4\lambda}{r_1}} \leqslant \frac{1-\bar{\tau}}{3}\mathrm{e}^{-\tau}z_1^{\frac{4\lambda}{\mu}}(t-\eta)+r_{12}z_1^{\frac{4\lambda}{\mu}}+C_{12},\end{aligned}$$

where C_{12} is a small enough design parameter and $r_{12}>0$. By Lemma 2, (7) and (B3), one can find a constant $r_{13}>0$ and a small enough design parameter $C_{13}>0$, such that

$$\begin{aligned}\frac{4\lambda-\varpi-\mu}{2r_1}\mathrm{Tr}\left\{g_1^\top z_1^{\frac{4\lambda-\varpi-2r_1}{\mu}}g_1\right\} &\leqslant \frac{4\lambda-\varpi-\mu}{2r_1}|z_1|^{\frac{4\lambda-\varpi-2r_1}{\mu}}|g_1|^2\\ &\leqslant \frac{4\lambda-\varpi-\mu}{2r_1}|z_1|^{\frac{4\lambda-\varpi-2r_1}{\mu}}\widehat{C}_{13}\left(1+|\zeta_0|^{2r_1+\varpi}+|z_1|^{\frac{2r_1+\varpi}{\mu}}+|z_1(t-\eta)|^{\frac{2r_1+\varpi}{\mu}}\right)\\ &\leqslant \frac{c_0l_3}{2(n+1)}|\zeta_0|^{4\lambda}+\frac{1-\bar{\tau}}{3}\mathrm{e}^{-\tau}z_1^{\frac{4\lambda}{\mu}}(t-\eta)+r_{13}z_1^{\frac{4\lambda}{\mu}}+C_{13}.\end{aligned}$$

Appendix C Proof of Proposition 3

For simplicity, we define $A_i=4\lambda-\varpi-r_i, B_i=4\lambda-\varpi-\mu-r_i$ and $D_i=4\lambda-\varpi-2\mu-r_i$. Then the function U_i can be rewritten as $U_i=\int_{\alpha_i}^{x_i}(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}})^{\frac{A_i}{\mu}}\mathrm{d}s$, $i=2,\ldots,n$. By Definition 1 and (9), one has

$$\begin{aligned}\mathcal{L}U_i = &\frac{\partial U_i}{\partial z_1}\frac{\mu}{r_1}z_1^{\frac{\mu-r_1}{\mu}}(a_1x_2^{p_1}+f_1-\dot{y}_{\rm r})+\sum_{j=2}^{i-1}\frac{\partial U_i}{\partial x_j}(a_jx_{j+1}^{p_j}+f_j)+\frac{\partial U_i}{\partial x_i}(a_ix_{i+1}^{p_i}+f_i)\\ &+\frac{1}{2}\frac{\partial^2U_i}{\partial z_1^2}\left|\frac{\mu}{r_1}z_1^{\frac{\mu-r_1}{\mu}}g_1\right|^2+\frac{1}{2}\sum_{j=2}^{i-1}\frac{\partial^2U_i}{\partial x_j^2}|g_j|^2+\frac{1}{2}\frac{\partial^2U_i}{\partial x_i^2}|g_i|^2+\sum_{j=2}^{i}\frac{\partial^2U_i}{\partial z_1\partial x_j}\left|\frac{\mu}{r_1}z_1^{\frac{\mu-r_1}{\mu}}g_1\right||g_j|\\ &+\frac{1}{2}\sum_{k,j=2,k\neq j}^{i-1}\frac{\partial^2U_i}{\partial x_k\partial x_j}|g_k||g_j|+\sum_{j=2}^{i-1}\frac{\partial^2U_i}{\partial x_i\partial x_j}|g_i||g_j|.\end{aligned} \tag{C1}$$

Next, we estimate the terms in (C1).

Term 1. By Lemmas 1, 3 and (7),

$$\left|\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{B_i}{\mu}}\mathrm{d}s\right| \leqslant |z_i|^{\frac{B_i}{\mu}}|x_i-\alpha_i| \leqslant 2^{\frac{\mu-r_i}{\mu}}|z_i|^{\frac{4\lambda-\varpi-\mu}{\mu}}. \tag{C2}$$

By the definitions of U_i, A_i, B_i and Proposition 1, the partial derivative $\frac{\partial U_i}{\partial z_1}$ is given by

$$\frac{\partial U_i}{\partial z_1}=-\frac{A_i}{\mu}\frac{\partial \alpha_i^{\frac{\mu}{r_i}}}{\partial z_1}\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{B_i}{\mu}}\mathrm{d}s=\frac{A_i}{\mu}\beta_{i-1}\cdots\beta_1\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{B_i}{\mu}}\mathrm{d}s. \tag{C3}$$

Taking the absolute value on both sides of (C3), by (C2), we have

$$\left|\frac{\partial U_i}{\partial z_1}\right| \leqslant \frac{A_i}{\mu}2^{\frac{\mu-r_i}{\mu}}\beta_{i-1}\cdots\beta_1|z_i|^{\frac{4\lambda-\varpi-\mu}{\mu}}. \tag{C4}$$

From $0<a_1\leqslant\bar{\epsilon}$, $|x_2|^{p_1}\leqslant|z_2|^{\frac{r_1+\varpi}{\mu}}+|\beta_1z_1|^{\frac{r_1+\varpi}{\mu}}$ and (B1), it follows that

$$\begin{aligned}|a_1x_2^{p_1}+f_1-\dot{y}_{\rm r}| &\leqslant \bar{\epsilon}\left(|z_2|^{\frac{r_1+\varpi}{\mu}}+|\beta_1z_1|^{\frac{r_1+\varpi}{\mu}}\right)+\widehat{C}_{11}\left(1+|\zeta_0|^{r_1+\varpi}+|z_1|^{\frac{r_1+\varpi}{\mu}}+|z_1(t-\eta)|^{\frac{r_1+\varpi}{\mu}}\right)+M\\ &\leqslant \widehat{C}_{14}\left(1+|\zeta_0|^{r_1+\varpi}+|z_1|^{\frac{r_1+\varpi}{\mu}}+|z_1(t-\eta)|^{\frac{r_1+\varpi}{\mu}}+|z_2|^{\frac{r_1+\varpi}{\mu}}\right)\end{aligned} \tag{C5}$$

with $\widehat{C}_{14}=\max\{\bar{\epsilon},\bar{\epsilon}\beta_1^{\frac{r_1+\varpi}{\mu}}+\widehat{C}_{11},\widehat{C}_{11}+M\}$. By using Lemma 2, (C4) and (C5), the first term of (C1) is estimated

$$\left|\frac{\partial U_i}{\partial z_1}\frac{\mu}{r_1}z_1^{\frac{\mu-r_1}{\mu}}(a_1x_2^{p_1}+f_1-\dot{y}_{\rm r})\right| \leqslant \frac{A_i}{r_1}2^{\frac{\mu-r_i}{\mu}}\beta_{i-1}\cdots\beta_1|z_i|^{\frac{4\lambda-\varpi-\mu}{\mu}}|z_1|^{\frac{\mu-r_1}{\mu}}|a_1x_2^{p_1}+f_1-\dot{y}_{\rm r}|$$

$$\leqslant \frac{c_0 l_3}{9(n+1)}|\zeta_0|^{4\lambda}+\frac{1}{9}\left(z_1^{\frac{4\lambda}{\mu}}+z_2^{\frac{4\lambda}{\mu}}\right)+\frac{1-\bar{\tau}}{9}\mathrm{e}^{-\tau}z_1^{\frac{4\lambda}{\mu}}(t-\eta)+r_{i1}z_i^{\frac{4\lambda}{\mu}}+C_{i1}.$$

Term 2. Similar to the proofs of (C3) and (C4), it can be verified that

$$\frac{\partial U_i}{\partial x_j}=-\frac{A_i}{\mu}\frac{\partial \alpha_i^{\frac{\mu}{r_i}}}{\partial x_j}\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{B_i}{\mu}}\mathrm{d}s=\frac{A_i}{r_j}\beta_{i-1}\cdots\beta_j x_j^{\frac{\mu-r_j}{r_j}}\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{B_i}{\mu}}\mathrm{d}s, \tag{C6}$$

$$\left|\frac{\partial U_i}{\partial x_j}\right|\leqslant \frac{A_i}{r_j}2^{\frac{\mu-r_i}{\mu}}\beta_{i-1}\cdots\beta_j\left(|z_j|^{\frac{\mu-r_j}{\mu}}+|\beta_{j-1}z_{j-1}|^{\frac{\mu-r_j}{\mu}}\right)|z_i|^{\frac{4\lambda-\varpi-\mu}{\mu}},\quad j=2,\ldots,i-1,$$

from which and using Lemma 2, (B1) and $0<a_j\leqslant\bar{\epsilon}$, the second term of (C1) is estimated as

$$\begin{aligned}&\left|\sum_{j=2}^{i-1}\frac{\partial U_i}{\partial x_j}\left(a_j x_{j+1}^{p_j}+f_j\right)\right|\\ &\leqslant \sum_{j=2}^{i-1}\frac{A_i}{r_j}2^{\frac{\mu-r_i}{\mu}}\beta_{i-1}\cdots\beta_j\left(|z_j|^{\frac{\mu-r_j}{\mu}}+|\beta_{j-1}z_{j-1}|^{\frac{\mu-r_j}{\mu}}\right)|z_i|^{\frac{4\lambda-\varpi-\mu}{\mu}}\left(\bar{\epsilon}\left(|z_{j+1}|^{\frac{r_j+\varpi}{\mu}}+|\beta_j z_j|^{\frac{r_j+\varpi}{\mu}}\right)+|f_j|\right)\\ &\leqslant \frac{c_0 l_3}{9(n+1)}|\zeta_0|^{4\lambda}+\frac{1}{9}\sum_{j=1}^{i-1}z_j^{\frac{4\lambda}{\mu}}+\frac{1-\bar{\tau}}{9}\mathrm{e}^{-\tau}\sum_{j=1}^{i-1}z_j^{\frac{4\lambda}{\mu}}(t-\eta)+r_{i2}z_i^{\frac{4\lambda}{\mu}}+C_{i2}.\end{aligned}$$

Term 3. By Lemma 2, (B1) and $\frac{\partial U_i}{\partial x_i}=z_i^{\frac{A_i}{\mu}}$, there holds

$$\begin{aligned}&\frac{\partial U_i}{\partial x_i}\left(a_i x_{i+1}^{p_i}+f_i\right)\leqslant a_i z_i^{\frac{A_i}{\mu}}x_{i+1}^{p_i}+|z_i|^{\frac{A_i}{\mu}}|f_i|\\ &\leqslant \frac{c_0 l_3}{9(n+1)}|\zeta_0|^{4\lambda}+\frac{1}{9}\sum_{j=1}^{i-1}z_i^{\frac{4\lambda}{\mu}}+\frac{1-\bar{\tau}}{9}\mathrm{e}^{-\tau}\sum_{j=1}^{i}z_j^{\frac{4\lambda}{\mu}}(t-\eta)+a_i z_i^{\frac{A_i}{\mu}}x_{i+1}^{p_i}+r_{i3}z_i^{\frac{4\lambda}{\mu}}+C_{i3}.\end{aligned}$$

Term 4. By (C3) and Proposition 1, it follows that

$$\frac{\partial^2 U_i}{\partial z_1^2}=\frac{A_iB_i}{\mu^2}\left(\frac{\partial \alpha_i^{\frac{\mu}{r_i}}}{\partial z_1}\right)^2\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{D_i}{\mu}}\mathrm{d}s=\frac{A_iB_i}{\mu^2}(\beta_{i-1}\cdots\beta_1)^2\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{D_i}{\mu}}\mathrm{d}s.$$

Furthermore, using the inequality

$$\left|\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{D_i}{\mu}}\mathrm{d}s\right|\leqslant 2^{\frac{\mu-r_i}{\mu}}|z_i|^{\frac{4\lambda-\varpi-2\mu}{\mu}}, \tag{C7}$$

we deduce that

$$\left|\frac{\partial^2 U_i}{\partial z_1^2}\right|\leqslant \frac{A_iB_i}{\mu^2}2^{\frac{\mu-r_i}{\mu}}(\beta_{i-1}\cdots\beta_1)^2|z_i|^{\frac{4\lambda-\varpi-2\mu}{\mu}}. \tag{C8}$$

Finally, using Lemma 2, (B3) and (C8),

$$\begin{aligned}\frac{1}{2}\left|\frac{\partial^2 U_i}{\partial z_1^2}\right|\left|\frac{\mu}{r_1}z_1^{\frac{\mu-r_1}{\mu}}g_1\right|^2&\leqslant \frac{A_iB_i}{2r_1^2}2^{\frac{\mu-r_i}{\mu}}(\beta_{i-1}\cdots\beta_1)^2|z_i|^{\frac{4\lambda-\varpi-2\mu}{\mu}}|z_1|^{\frac{2\mu-2r_1}{\mu}}|g_1|^2\\ &\leqslant \frac{c_0 l_3}{9(n+1)}|\zeta_0|^{4\lambda}+\frac{1}{9}z_1^{\frac{4\lambda}{\mu}}+\frac{1-\bar{\tau}}{9}\mathrm{e}^{-\tau}z_1^{\frac{4\lambda}{\mu}}(t-\eta)+r_{i4}z_i^{\frac{4\lambda}{\mu}}+C_{i4}.\end{aligned}$$

Term 5. Taking the partial derivative of the equation (C6), we have

$$\frac{\partial^2 U_i}{\partial x_j^2}=\frac{A_i(\mu-r_j)}{r_j^2}\beta_{i-1}\cdots\beta_j x_j^{\frac{\mu-2r_j}{r_j}}\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{B_i}{\mu}}\mathrm{d}s+\frac{A_iB_i}{r_j^2}(\beta_{i-1}\cdots\beta_j)^2x_j^{\frac{2\mu-2r_j}{r_j}}\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}}-\alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{D_i}{\mu}}\mathrm{d}s.$$

In addition, considering $|x_k|^{\frac{2\mu-2r_k}{r_k}}\leqslant 2(|z_k|^{\frac{2\mu-2r_k}{\mu}}+|\beta_{k-1}z_{k-1}|^{\frac{2\mu-2r_k}{\mu}})$, $|x_k|^{\frac{\mu-2r_k}{r_k}}\leqslant |z_k|^{\frac{\mu-2r_k}{\mu}}+|\beta_{k-1}z_{k-1}|^{\frac{\mu-2r_k}{\mu}}$, and using (C2) and (C7), we obtain

$$\begin{aligned}\left|\frac{\partial^2 U_i}{\partial x_j^2}\right|\leqslant\ &\frac{A_i(\mu-r_j)}{r_j^2}2^{\frac{\mu-r_i}{\mu}}\beta_{i-1}\cdots\beta_j\left(|z_j|^{\frac{\mu-2r_j}{\mu}}+|\beta_{j-1}z_{j-1}|^{\frac{\mu-2r_j}{\mu}}\right)|z_i|^{\frac{4\lambda-\varpi-\mu}{\mu}}\\ &+\frac{A_iB_i}{r_j^2}2^{\frac{2\mu-r_i}{\mu}}(\beta_{i-1}\cdots\beta_j)^2\left(|z_j|^{\frac{2\mu-2r_j}{\mu}}+|\beta_{j-1}z_{j-1}|^{\frac{2\mu-2r_j}{\mu}}\right)|z_i|^{\frac{4\lambda-\varpi-2\mu}{\mu}}.\end{aligned} \tag{C9}$$

Then, by Lemma 2, (B3) and (C9), the fifth term of (C1) is estimated as follows:

$$\frac{1}{2}\sum_{j=2}^{i-1}\left|\frac{\partial^2 U_i}{\partial x_j^2}\right||g_j|^2\leqslant \frac{c_0 l_3}{9(n+1)}|\zeta_0|^{4\lambda}+\frac{1-\bar{\tau}}{9}\mathrm{e}^{-\tau}\sum_{j=1}^{i-1}z_j^{\frac{4\lambda}{\mu}}(t-\eta)+\frac{1}{9}\sum_{j=1}^{i-1}z_j^{\frac{4\lambda}{\mu}}+r_{i5}z_i^{\frac{4\lambda}{\mu}}+C_{i5}.$$

Term 6. By utilizing $|\frac{\partial^2 U_i}{\partial x_i^2}| = |\frac{A_i}{r_i} x_i^{\frac{\mu-r_i}{r_i}} z_i^{\frac{B_i}{\mu}}| \leqslant \frac{A_i}{r_i}(|z_i|^{\frac{\mu-r_i}{\mu}} + |\beta_{i-1}z_{i-1}|^{\frac{\mu-r_i}{\mu}})|z_i|^{\frac{B_i}{\mu}}$, Lemma 2 and (B3), it yields

$$\begin{aligned}\frac{1}{2}\left|\frac{\partial^2 U_i}{\partial x_i^2}\right||g_i|^2 &\leqslant \frac{A_i}{2r_i}\left(|z_i|^{\frac{\mu-r_i}{\mu}} + |\beta_{i-1}z_{i-1}|^{\frac{\mu-r_i}{\mu}}\right)|z_i|^{\frac{B_i}{\mu}}|g_i|^2 \\ &\leqslant \frac{c_0 l_3}{9(n+1)}|\zeta_0|^{4\lambda} + \frac{1-\bar{\tau}}{9}\mathrm{e}^{-\tau}\sum_{j=1}^{i} z_j^{\frac{4\lambda}{\mu}}(t-\eta) + \frac{1}{9}\sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}} + r_{i6} z_i^{\frac{4\lambda}{\mu}} + C_{i6}.\end{aligned}$$

Term 7. Similar to the proof of Term 5, we can deduce

$$\frac{\partial^2 U_i}{\partial z_1 \partial x_j} = \frac{A_i B_i}{r_j \mu}(\beta_{i-1}\cdots\beta_j)^2\beta_{j-1}\cdots\beta_1 x_j^{\frac{\mu-r_j}{r_j}} \int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}} - \alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{D_i}{\mu}}\mathrm{d}s,$$

$$\left|\frac{\partial^2 U_i}{\partial z_1 \partial x_j}\right| \leqslant \frac{A_i B_i}{r_j \mu} 2^{\frac{\mu-r_i}{\mu}}(\beta_{i-1}\cdots\beta_j)^2\beta_{j-1}\cdots\beta_1\left(|z_j|^{\frac{\mu-r_j}{\mu}} + |\beta_{j-1}z_{j-1}|^{\frac{\mu-r_j}{\mu}}\right)|z_i|^{\frac{4\lambda-\varpi-2\mu}{\mu}}.$$

Utilizing (B2) and Lemma 2, one has

$$\begin{aligned}&\sum_{j=2}^{i}\left|\frac{\partial^2 U_i}{\partial z_1 \partial x_j}\right|\left|\frac{\mu}{r_1} z_1^{\frac{\mu-r_1}{\mu}} g_1\right||g_j| \\ &\leqslant \sum_{j=2}^{i}\frac{A_i B_i}{r_1 r_j}2^{\frac{\mu-r_i}{\mu}}(\beta_{i-1}\cdots\beta_j)^2\beta_{j-1}\cdots\beta_1\left(|z_j|^{\frac{\mu-r_j}{\mu}} + |\beta_{j-1}z_{j-1}|^{\frac{\mu-r_j}{\mu}}\right)|z_i|^{\frac{4\lambda-\varpi-2\mu}{\mu}}|z_1|^{\frac{\mu-r_1}{\mu}}|g_1||g_j| \\ &\leqslant \frac{c_0 l_3}{9(n+1)}|\zeta_0|^{4\lambda} + \frac{1-\bar{\tau}}{9}\mathrm{e}^{-\tau}\sum_{j=1}^{i} z_j^{\frac{4\lambda}{\mu}}(t-\eta) + \frac{1}{9}\sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}} + r_{i7} z_i^{\frac{4\lambda}{\mu}} + C_{i7}.\end{aligned} \tag{C10}$$

Term 8. For $k \neq j, 2 \leqslant k, j \leqslant i-1$, we have

$$\frac{\partial^2 U_i}{\partial x_k \partial x_j} = \frac{A_i B_i}{r_k r_j}\beta_{i-1}\cdots\beta_k\cdot\beta_{i-1}\cdots\beta_j x_k^{\frac{\mu-r_k}{r_k}} x_j^{\frac{\mu-r_j}{r_j}}\int_{\alpha_i}^{x_i}\left(s^{\frac{\mu}{r_i}} - \alpha_i^{\frac{\mu}{r_i}}\right)^{\frac{D_i}{\mu}}\mathrm{d}s.$$

By $0 < \frac{\mu-r_k}{\mu} < 1$, $|x_k|^{\frac{\mu-r_k}{r_k}} \leqslant |z_k|^{\frac{\mu-r_k}{\mu}} + |\beta_{k-1}z_{k-1}|^{\frac{\mu-r_k}{\mu}}$ and (C7), it yields that

$$\begin{aligned}\left|\frac{\partial^2 U_i}{\partial x_k \partial x_j}\right| &\leqslant \frac{A_i B_i}{r_k r_j}2^{\frac{\mu-r_i}{\mu}}\beta_{i-1}\cdots\beta_k\cdot\beta_{i-1}\cdots\beta_j\left(|z_k|^{\frac{\mu-r_k}{\mu}} + |\beta_{k-1}z_{k-1}|^{\frac{\mu-r_k}{\mu}}\right) \\ &\quad\cdot\left(|z_j|^{\frac{\mu-r_j}{\mu}} + |\beta_{j-1}z_{j-1}|^{\frac{\mu-r_j}{\mu}}\right)|z_i|^{\frac{4\lambda-\varpi-2\mu}{\mu}}.\end{aligned} \tag{C11}$$

Using Lemma 2, (B2), (C11) and taking the similar manipulations of (C10), we obtain

$$\frac{1}{2}\sum_{k,j=2,k\neq j}^{i-1}\left|\frac{\partial^2 U_i}{\partial x_k \partial x_j}\right||g_k||g_j| \leqslant \frac{c_0 l_3}{9(n+1)}|\zeta_0|^{4\lambda} + \frac{1}{9}\sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}} + \frac{1-\bar{\tau}}{9}\mathrm{e}^{-\tau}\sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}}(t-\eta) + r_{i8} z_i^{\frac{4\lambda}{\mu}} + C_{i8}.$$

Term 9. For $j < i$, notice

$$\left|\frac{\partial^2 U_i}{\partial x_i \partial x_j}\right| = \left|\frac{A_i}{r_j}\beta_{i-1}\cdots\beta_j x_j^{\frac{\mu-r_j}{r_j}} z_i^{\frac{B_i}{\mu}}\right| \leqslant \frac{A_i}{r_j}\beta_{i-1}\cdots\beta_j\left(|z_j|^{\frac{\mu-r_j}{\mu}} + |\beta_{j-1}z_{j-1}|^{\frac{\mu-r_j}{\mu}}\right)|z_i|^{\frac{B_i}{\mu}},$$

and use Lemma 2 and (B2), the last term of (C1) is estimated as

$$\begin{aligned}\sum_{j=2}^{i-1}\left|\frac{\partial^2 U_i}{\partial x_i \partial x_j}\right||g_i||g_j| &\leqslant \sum_{j=2}^{i-1}\frac{A_i}{r_j}\beta_{i-1}\cdots\beta_j\left(|z_j|^{\frac{\mu-r_j}{\mu}} + |\beta_{j-1}z_{j-1}|^{\frac{\mu-r_j}{\mu}}\right)|z_i|^{\frac{B_i}{\mu}}|g_i||g_j| \\ &\leqslant \frac{c_0 l_3}{9(n+1)}|\zeta_0|^{4\lambda} + \frac{1}{9}\sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}} + \frac{1-\bar{\tau}}{9}\mathrm{e}^{-\tau}\sum_{j=1}^{i-1} z_j^{\frac{4\lambda}{\mu}}(t-\eta) + r_{i9} z_i^{\frac{4\lambda}{\mu}} + C_{i9}.\end{aligned}$$

Substituting Terms 1–9 into (C1) leads to Proposition 3.