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Time-inconsistent stochastic linear quadratic control for discrete-time systems

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Abstract This paper is mainly concerned with the time-inconsistent stochastic linear quadratic (LQ) control problem in a more general formulation for discrete-time systems. The time-inconsistency arises from three aspects: the coefficient matrices depending on the initial pair, the terminal of the cost function involving the initial pair together with the nonlinear terms of the conditional expectation. The main contributions are: firstly, the maximum principle is derived by using variational methods, which forms a flow of forward and backward stochastic difference equations (FBSDE); secondly, in the case of the system state being one-dimensional, the equilibrium control is obtained by solving the FBSDE with feedback gain based on several nonsymmetric Riccati equations; finally, the necessary and sufficient solvability condition for the time-inconsistent LQ control problem is presented explicitly. The key techniques adopted are the maximum principle and the solution to the FBSDE developed in this paper.

Keywords time-inconsistency, open-loop equilibrium control, maximum principle, FBSDE, nonsymmetric Riccati equations

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1 Introduction

As is well known, the classical stochastic control problem is assumed to be time-consistent in the sense that the well-celebrated Bellman optimality principle can be applied. Time consistency means that the optimal control is independent of the initial pair, i.e., if the control is optimal on the full time interval, it is also optimal on any time subinterval. The Bellman optimality principle from the dynamic programming methods serves as a basic tool in seeking the optimal control for the time-consistent stochastic control problems, see [1–5].

However, stochastic control problems may not be time-consistent for real-world systems (or time-inconsistency) in the sense that the optimal control for a specific initial pair on a later time interval may not be optimal for that corresponding initial pair [6,7]. The time-inconsistent control problems have vast potential applications, especially for mathematical finance, such as hyperbolic discounting and mean-variance portfolio selection problem [8–15]. In [16], the mathematical formulation of time-inconsistent

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problems was first proposed. Following [16], the time-inconsistent control problem has been a hot research topic and many significant contributions have been made [17–26].

As studied in previous literatures, there are two main approaches to handling time-inconsistent control problems: pre-committed control method and game theoretic formulation. Pre-commitment means that the initial pair (τ, x) is fixed and we find control to minimize the cost function starting from time τ, disregarding the fact that the control may not be optimal for any $k > \tau$. Though the pre-committed control method is reasonable in some practical situations, the time-inconsistency is actually not considered [18]. On the other hand, the time-inconsistency can be dealt with within the game theoretic framework. To be more specific, the control u_k for $k \geqslant \tau$ can be viewed as the kth player, which controls the system only at time k. This formulation indicates that the terminal pair (k, X_k) is the initial pair for $(k+1)$th player; each player tries to find an optimal control for his/her own problem, i.e., the control u_k is only optimal at time k in the infinitesimal sense [27]. This kind of control is termed as "equilibrium control". A precise definition of equilibrium control was given in [18,19]. In this paper, the time-inconsistent control problem is investigated in a dynamic manner from the game theoretic perspective. Instead of seeking the "optimal" control, we aim to find the open-loop "equilibrium" control by solving the essentially cooperative stochastic differential game.

The time-inconsistent stochastic LQ control problem for continuous-time systems was studied in [17,28]. The definition of open-loop equilibrium control was presented. The maximum principle was derived, and the equilibrium control was obtained by decoupling a flow of forward and backward stochastic difference equations (FBSDE) in the case of coefficient matrices being deterministic and the system state being one-dimensional. Finally, a mean-variance portfolio selection example was provided to verify the main results.

For the discrete-time systems, time-inconsistent control problems were studied in [22] and [24–26, 29]; some significant contributions have been made in the past several years. A time-inconsistent control problem with indefinite weighting matrices was considered in [22, 24], while the mean-field time-inconsistent control problem for discrete time systems was investigated in [26, 29].

We would like to point out that the time-inconsistent control problem to be solved in this paper is more general than that in [22, 24–26, 29]. To be specific, the time-inconsistency only arises from the coefficient matrices and weighting matrices depending on the initial time in [22, 24–26, 29], while, the terminal terms in the cost function involve the initial state together with the matrices depending on the initial time result in time-inconsistency in our framework, which brings essential difficulties in exploring the solvability conditions. Another thing to note is that the FBSDE obtained in [22] cannot be decoupled, i.e., the explicit equilibrium control is not obtained. However, the open-loop equilibrium control is derived in a more general formulation investigated in this paper, which is based on several difference equations and nonsymmetric Riccati difference equations.

Upon the above analysis, the time-inconsistent control problem for discrete-time systems remains to be studied, although some progress has been made in previous literature. The objectives to be achieved in this paper can be concluded as: 1) To explore the maximum principle for the considered general time-inconsistent equilibrium control problem; 2) To provide the explicit necessary and sufficient solvability conditions for the time-inconsistent control problem and obtain the explicit equilibrium control.

Firstly, by using the variational method, we derive the necessary conditions (i.e., maximum principle) of the equilibrium control for the general time-inconsistent equilibrium control problem, which is the key tool in solving the time-inconsistent control problem. Secondly, the solution to an FBSDE composed of costate and system dynamics is derived using the induction method. Finally, the open-loop equilibrium control is designed based on nonsymmetric Riccati equations in the case of the system state being one-dimensional. The necessary and sufficient solvability condition is thus obtained. It should be noted that the necessary and sufficient solvability condition developed in this paper is based on the positive definiteness of Υ_k, which can be easily verified, while this has never been obtained in the aforementioned studies like [17, 22, 28] and so forth. The main tools used are the maximum principle and the solution to the FBSDE.

The following notations will be used throughout this paper: $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; Real symmetric matrix $A > 0$ ($\geqslant 0$) means A is positive definite (positive semi-definite). $\mathbb{N}$ is used to indicate $\{0, 1, \dots, N\}$ and $\mathbb{N}_l = \{l, l+1, \dots, N\}$. B' is the transpose of the real matrix B, and C^{-1} represents the inverse of matrix C. $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_k\}_{k=0}^N\}$ is a complete probability space with natural filtration $\mathcal{F}_k$ generated by $\{x_0, w_0, \dots, w_k\}$. $E_l[\cdot] = E[\cdot|\mathcal{F}_l]$ denotes the conditional mathematical expectation with respect to $\mathcal{F}_l$ and $\mathcal{F}_{-1} = \{\phi, \Omega\}$ with ϕ being the empty set.

The remainder of this paper is organized as follows. In Section 2, the time-inconsistent equilibrium control problem for discrete-time systems is formulated. The main results are presented in Section 3, where the maximum principle is developed and the necessary and sufficient solvability condition is explored. A numerical example is provided in Section 4 to illustrate the main results obtained in this paper. Finally, the paper is concluded in Section 5.

2 Problem formulation

The following linear controlled discrete-time system is considered:

$$\begin{cases} X_{k+1}^\tau = (A_{k,\tau} + w_k \bar{A}_{k,\tau}) X_k^\tau + (B_{k,\tau} + w_k \bar{B}_{k,\tau}) u_k + f_{k,\tau} + w_k \bar{f}_{k,\tau}, \\ X_\tau = x, \end{cases} \tag{1}$$

where $X_k^\tau \in \mathcal{R}^n$ is the system state, $u_k \in \mathcal{R}^m$ is the control, $\{w_k\}_{k=0}^N$ is the scalar-valued Gaussian white noise with zero mean and covariance 1, $A_{k,\tau}, \bar{A}_{k,\tau} \in \mathcal{R}^{n\times n}$, $B_{k,\tau}, \bar{B}_{k,\tau} \in \mathcal{R}^{n\times m}$, and the non-homogeneous terms $f_{k,\tau}, \bar{f}_{k,\tau} \in \mathcal{R}^n$ are given deterministic functions for any $k \in \mathbb{N}_\tau$ depending on the initial time τ. The initial state $X_\tau = x$ and (τ, x) is called the initial pair.

For an arbitrary initial pair (τ, x), the cost function associated with (1) is given as

$$\begin{aligned} J(\tau, x; u) = &\sum_{k=\tau}^{N} E_{\tau-1}[(X_k^\tau)' Q_{k,\tau} X_k^\tau + u_k' R_{k,\tau} u_k] + E_{\tau-1}[(X_{N+1}^\tau)' P_{N+1,\tau} X_{N+1}^\tau] \\ &+ (E_{\tau-1} X_{N+1}^\tau)' \Gamma_{N+1,\tau} E_{\tau-1} X_{N+1}^\tau + 2X_\tau' M_{N+1,\tau} E_{\tau-1} X_{N+1}^\tau + 2\Phi_{N+1,\tau}' E_{\tau-1} X_{N+1}^\tau, \end{aligned} \tag{2}$$

where $E_{\tau-1}$ denotes $E[\cdot|\mathcal{F}_{\tau-1}]$, $\{\mathcal{F}_k\}$ is the natural filtration generated by $\{x_0, w_0, \dots, w_k\}$ and $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$. The weighting matrices $Q_{k,\tau} \in \mathcal{R}^{n\times n}, R_{k,\tau} \in \mathcal{R}^{m\times m}, P_{N+1,\tau} \in \mathcal{R}^{n\times n}, \Gamma_{N+1,\tau} \in \mathcal{R}^{n\times n}$ and are symmetric, and $M_{N+1,\tau} \in \mathcal{R}^{n\times n}$, $\Phi_{N+1,\tau} \in \mathcal{R}^n$. All the coefficient matrices are deterministic and bounded. The initial time τ in the matrices and the state is to indicate that the matrices and state may change with τ.

Firstly, we define the admissible control set for the time-inconsistent control problem as below:

$$\mathcal{U}_\tau = \{u_k \in \mathcal{R}^m,\ \tau \leqslant k \leqslant N | u_k \text{ is } \mathcal{F}_{k-1}\text{-measurable},\ E(u_k' u_k) < \infty\}. \tag{3}$$

Any $u_k \in \mathcal{U}_\tau$ is called admissible control.

Remark 1. The time-inconsistency investigated in this paper arises from three aspects: 1) With the passage of time τ, the coefficient matrices and weighting matrices in (1) and (2) are changing accordingly; 2) The coefficient matrices of the first three terms in (2) depend on initial time τ; 3) The last three terms in (2) also result in the time-inconsistency. To be specific, $(E_{\tau-1} X_{N+1}^\tau)' \Gamma_{N+1,\tau} E_{\tau-1} X_{N+1}^\tau$ is from the mean-variance portfolio selection problem which can be viewed as the risk [11, 30]. The term $2X_\tau' M_{N+1,\tau} E_{\tau-1} X_{N+1}^\tau + 2\Phi_{N+1,\tau}' E_{\tau-1} X_{N+1}^\tau$ is motivated by the utility function in mathematical finance, which involves the initial state [7, 31].

In this paper, the following LQ control problem is mainly investigated:

Problem 1. For system (1) associated with cost function (2) and initial pair (τ, x), to seek $u_k^* \in \mathcal{U}_\tau$, $\tau \leqslant k \leqslant N$, such that cost function (2) is minimized, i.e.,

$$J(\tau, x; u_k^*) = \inf_{u_k \in \mathcal{U}_\tau} J(\tau, x; u_k), \quad \tau \leqslant k \leqslant N. \tag{4}$$

As analyzed in Section 1, the pre-committed control methods will not be adopted to solve Problem 1 in this paper, which will be dealt with in a dynamic manner instead. To be explicit, we aim to find the open-loop equilibrium control for Problem 1, which is optimal in an infinitesimal sense.

Firstly, by following from [17], we give the definition of open-loop equilibrium control as below.

Definition 1. For the initial pair (τ, x), the control $u_k^{\tau,x,*} \in \mathcal{U}_\tau$, and $X_k^{\tau,x,*}$ is the state corresponding to $u_k^{\tau,x,*}$, then $u_k^{\tau,x,*}$ is called the equilibrium control if $X_\tau^{\tau,x,*} = x$, and for any $u_k \in \mathcal{U}_\tau$, there holds

$$J(k, X_k^{\tau,x,*}; u^{\tau,x,*}|_{\mathbb{N}_k}) \leqslant J(k, X_k^{\tau,x,*}; (u_k, u^{\tau,x,*}|_{\mathbb{N}_{k+1}})), \tag{5}$$

where $u^{\tau,x,*}|_{\mathbb{N}_k}$ means the restrictions of control $u^{\tau,x,*}$ on $\mathbb{N}_k = \{k, \dots, N\}$.

Remark 2. It is noted from Definition 1 that the equilibrium control is local optimal. Specifically, from (5) we see that $u^{\tau,x,*}|_{\mathbb{N}_k}$ differs with $(u_k, u^{\tau,x,*}|_{\mathbb{N}_{k+1}})$ only at time k. On the other hand, the perturbation of equilibrium control u_k will not change the control $u_{k+1}, \dots, u_N$. In this sense, the equilibrium control given in Definition 1 is defined within the class of open-loop controls.

To guarantee the solvability of Problem 1, we make the following standard assumption on weighting matrices of (2).

Assumption 1. For $k \in \mathbb{N}_\tau$ and $l \in \mathbb{N}_k$, the weighting matrices in (2) satisfy: $Q_{l,k} \geqslant 0$, $R_{l,k} \geqslant 0$ and $P_{N+1,l} \geqslant 0$.

3 Main results

3.1 Maximum principle

To solve Problem 1, the necessary conditions (maximum principle) for the existence of the open-loop equilibrium control will be derived first. Clearly, from (3) we know that the control set in this paper is a convex and closed subset of $\mathcal{R}^m$, then the method adopted in this paper is the convex variational method, see [32,33]. For the case of the control set being non-convex, the corresponding results can also be derived by using spike variation, which will not be discussed in this paper, see [34].

Theorem 1. For initial pair (τ, x), the equilibrium control $u_k^{\tau,x,*}$ satisfies the stationary relationship:

$$0 = R_{k,k} u_k^{\tau,x,*} + E[(B_{k,k} + w_k \bar{B}_{k,k})' \lambda_k^{k,\tau,x} | \mathcal{F}_{k-1}], \quad k \in \mathbb{N}_\tau, \tag{6}$$

where the costate $\lambda_k^{k,\tau,x}$ can be calculated from the adjoint equation as below:

$$\lambda_{l-1}^{k,\tau,x} = Q_{l,k} X_l^{k,\tau,x} + E[(A_{l,k} + w_l \bar{A}_{l,k})' \lambda_l^{k,\tau,x} | \mathcal{F}_{l-1}], \quad l \in \mathbb{N}_k, \tag{7}$$

with final condition

$$\lambda_N^{k,\tau,x} = P_{N+1,k} X_{N+1}^{k,\tau,x} + \Gamma_{N+1,k} E_{k-1}(X_{N+1}^{k,\tau,x}) + M_{N+1,k} X_k^{\tau,x,*} + \Phi_{N+1,k}, \tag{8}$$

and $P_{N+1,k}$, $\Gamma_{N+1,k}$, $M_{N+1,k}$ and $\Phi_{N+1,k}$ are given in (2).

Proof. Since the control set $\mathcal{U}_\tau$ defined in (3) is convex, if we choose u_l, $\delta u_l \in \mathcal{U}_\tau$ for $l \in \mathbb{N}_\tau$, then $u_l^\varepsilon = u_l + \varepsilon \delta u_l \in \mathcal{U}_\tau$ for any $\varepsilon \in (0,1)$ can be developed. Accordingly, $J(k, X_k^{\tau,x,*}; u_l|_{\mathbb{N}_k})$ and $X_l^{k,\tau,x}$ are the cost function and state trajectory, respectively, with control u_l, and $J^\varepsilon(k, X_k^{\tau,x,*}; u_l^\varepsilon|_{\mathbb{N}_k})$ and $X_l^{\varepsilon,k,\tau,x}$ represent the cost function associated with u_k^ε.

It is noted that the variation of the initial $X_k^{\tau,x,*}$ is zero, i.e., $\delta X_k^{\tau,x,*} = 0$. Thus, it holds from (1) that

$$\begin{aligned}
\delta X_{l+1}^{k,\tau,x} &= X_{l+1}^{\varepsilon,k,\tau,x} - X_{l+1}^{k,\tau,x} = (A_{l,k} + w_l \bar{A}_{l,k}) \delta X_l^{k,\tau,x} + (B_{l,k} + w_l \bar{B}_{l,k}) \varepsilon \delta u_l \\
&= F_X(l,k) \delta X_k^{\tau,x,*} + \sum_{j=k}^{l} F_X(l, j+1)(B_{j,k} + w_j \bar{B}_{j,k}) \varepsilon \delta u_j \\
&= \sum_{j=k}^{l} F_X(l, j+1)(B_{j,k} + w_j \bar{B}_{j,k}) \varepsilon \delta u_j,
\end{aligned} \tag{9}$$

where $F_X(l,j)=(A_{l,k}+w_l\bar{A}_{l,k})\cdots(A_{j,k}+w_j\bar{A}_{j,k})$, and $F_X(l,l+1)=I$.

Since the coefficient matrices in (1) are deterministic and $E(w_j^2)$ is finite for any $k\leqslant j\leqslant N$, Then $\delta J=J^{\varepsilon}(k,X_k^{\tau,x,*};u_l^{\varepsilon}|_{\mathbb{N}_k})-J(k,X_k^{\tau,x,*};u_l|_{\mathbb{N}_k})$ can be calculated from cost function (2) as

$$\begin{aligned}\delta J&=2E\Bigg\{[P_{N+1,k}X_{N+1}^{k,\tau,x}+\Gamma_{N+1,k}E_{k-1}X_{N+1}^{k,\tau,x}+M_{N+1,k}X_k^{\tau,x,*}+\Phi_{N+1,k}]'\delta X_{N+1}^{k,\tau,x}\\&\quad+\sum_{l=k}^{N}\Big[(X_l^{k,\tau,x})'Q_{l,k}\delta X_l^{k,\tau,x}+u_l'R_{l,k}\varepsilon\delta u_l\Big]\Big|\mathcal{F}_{k-1}\Bigg\}+o(\varepsilon)\\&=2E\Bigg\{(\lambda_N^{k,\tau,x})'\delta X_{N+1}^{k,\tau,x}+\sum_{l=k}^{N}\Big[(X_l^{k,\tau,x})'Q_{l,k}\delta X_l^{k,\tau,x}+u_l'R_{l,k}\varepsilon\delta u_l\Big]\Big|\mathcal{F}_{k-1}\Bigg\}+o(\varepsilon),\end{aligned}\qquad(10)$$

where $o(\varepsilon)$ means that an infinitesimal of the higher order with ε and (8) has been used.

Now substituting (9) in (10), we have

$$\begin{aligned}\delta J&=2E\Bigg\{(\lambda_N^{k,\tau,x})'\Bigg[\sum_{j=k}^{N}F_X(N,j+1)(B_{j,k}+w_j\bar{B}_{j,k})\varepsilon\delta u_j\Bigg]\\&\quad+\sum_{l=k}^{N}(X_l^{k,\tau,x})'Q_{l,k}\Bigg[\sum_{j=k}^{l-1}F_X(l-1,j+1)(B_{j,k}+w_j\bar{B}_{j,k})\varepsilon\delta u_j\Bigg]+\sum_{l=k}^{N}u_l'R_{l,k}\varepsilon\delta u_l|\mathcal{F}_{k-1}\Bigg\}+o(\varepsilon)\\&=2E\Bigg\{\big[(\lambda_N^{k,\tau,x})'(B_{N,k}+w_N\bar{B}_{N,k})+u_N'R_{N,k}\big]\varepsilon\delta u_N\\&\quad+\sum_{j=k}^{N-1}\Bigg[u_j'R_{j,k}+\sum_{l=j+1}^{N}(X_l^{k,\tau,x})'Q_{l,k}F_X(l-1,j+1)(B_{j,k}+w_j\bar{B}_{j,k})\Bigg]\varepsilon\delta u_j\\&\quad+\sum_{j=k}^{N-1}\big[(\lambda_N^{k,\tau,x})'F_X(N,j+1)(B_{j,k}+w_j\bar{B}_{j,k})\big]\varepsilon\delta u_j\big|\mathcal{F}_{k-1}\Bigg\}+o(\varepsilon)\\&=2E\{\mathcal{G}(N+1,N)\varepsilon\delta u_N|\mathcal{F}_{k-1}\}+2E\Bigg\{\sum_{j=k}^{N-1}\mathcal{G}(j+1,N)\varepsilon\delta u_j|\mathcal{F}_{k-1}\Bigg\}+o(\varepsilon),\end{aligned}\qquad(11)$$

where the following fact has been used

$$\begin{aligned}&\sum_{l=k}^{N}(X_l^{k,\tau,x})'Q_{l,k}\Bigg[\sum_{j=k}^{l-1}F_X(l-1,j+1)(B_{j,k}+w_j\bar{B}_{j,k})\varepsilon\delta u_j\Bigg]\\&\quad=\sum_{j=k}^{N-1}\Bigg[\sum_{l=j+1}^{N}(X_l^{k,\tau,x})'Q_{l,k}F_X(l-1,j+1)(B_{j,k}+w_j\bar{B}_{j,k})\Bigg]\varepsilon\delta u_j,\end{aligned}\qquad(12)$$

and $\mathcal{G}(N+1,N)$ and $\mathcal{G}(j+1,N)$ are respectively given as

$$\mathcal{G}(N+1,N)=(\lambda_N^{k,\tau,x})'(B_{N,k}+w_N\bar{B}_{N,k})+u_N'R_{N,k},\qquad(13)$$

$$\begin{aligned}\mathcal{G}(j+1,N)&=(\lambda_N^{k,\tau,x})'F_X(N,j+1)(B_{j,k}+w_j\bar{B}_{j,k})\\&\quad+u_j'R_{j,k}+\sum_{l=j+1}^{N}(X_l^{k,\tau,x})'Q_{l,k}F_X(l-1,j+1)(B_{j,k}+w_j\bar{B}_{j,k}).\end{aligned}\qquad(14)$$

Furthermore, it is noted that (11) can be rewritten as

$$\delta J=2E_{k-1}\left\{E\left[\mathcal{G}(N+1,N)\mid\mathcal{F}_{N-1}\right]\varepsilon\delta u_N\right\}+2E_{k-1}\Bigg\{\sum_{j=k}^{N-1}E\left[\mathcal{G}(j+1,N)\mid\mathcal{F}_{j-1}\right]\varepsilon\delta u_j\Bigg\}+o(\varepsilon),\qquad(15)$$

where

$$E_{k-1}\{\{\mathcal{G}(N+1,N)-E[\mathcal{G}(N+1,N)|\mathcal{F}_{N-1}]\}\varepsilon\delta u_N\}=0,$$
$$E_{k-1}\left\{\sum_{j=k}^{N-1}\{\mathcal{G}(j+1,N)-E[\mathcal{G}(j+1,N)|\mathcal{F}_{j-1}]\}\varepsilon\delta u_j\right\}=0$$

has been inserted into (15).

Since δu_k is arbitrary, and the equilibrium control $u_k^{\tau,x,*}$ is optimal only at time k, from (15) we know that the following relationship should be satisfied

$$E\{\mathcal{G}(k+1,N)\mid\mathcal{F}_{k-1}\}=0. \tag{16}$$

Next, we will show that (6) and (7) are the necessary condition (16).

In fact, following from (7) and (8), we can obtain

$$\begin{aligned}\lambda_k^{k,\tau,x}&=Q_{k+1,k}X_k^{\tau,x,*}+E[(A_{k+1,k}+w_{k+1}\bar{A}_{k+1,k})'\lambda_{k+1}^{k,\tau,x}|\mathcal{F}_{k-1}]\\&=Q_{k+1,k}X_k^{\tau,x,*}+E\Big\{(A_{k+1,k}+w_{k+1}\bar{A}_{k+1,k})'[Q_{k+2,k}X_{k+1}^{k,\tau,x}\\&\quad+E[(A_{k+2,k}+w_{k+2}\bar{A}_{k+2,k})'\lambda_{k+2}^{k,\tau,x}|\mathcal{F}_k]]|\mathcal{F}_{k-1}\Big\}\\&=E\{Q_{k+1,k}X_k^{\tau,x,*}+(A_{k+1,k}+w_{k+1}\bar{A}_{k+1,k})'[Q_{k+2,k}X_{k+1}^{k,\tau,x}+(A_{k+2,k}+w_{k+2}\bar{A}_{k+2,k})'\lambda_{k+2}^{k,\tau,x}]|\mathcal{F}_{k-1}\}\\&=E\left\{\sum_{j=k+1}^{N}F_X'(j-1,k+1)Q_{j,k}X_j^{k,\tau,x}+F_X'(N,k+1)\lambda_N^{k,\tau,x}\middle|\mathcal{F}_{k-1}\right\}.\end{aligned} \tag{17}$$

Substituting (17) into (6), one has

$$\begin{aligned}0=R_{k,k}u_k^{\tau,x,*}+E\Bigg\{&(B_{k,k}+w_k\bar{B}_{k,k})'\sum_{j=k+1}^{N}F_X'(j-1,k+1)Q_{j,k}X_j^{k,\tau,x}\\&+(B_{k,k}+w_k\bar{B}_{k,k})'F_X'(N,k+1)(\lambda_N^{k,\tau,x})'|\mathcal{F}_{k-1}\Bigg\},\end{aligned} \tag{18}$$

which is indeed the relationship (16). The proof is complete.

Remark 3. It is noted that (7) is the costate equation (backward) with final condition (8), associated with the system (forward) below:

$$\begin{cases}X_{l+1}^{k,\tau,x}=(A_{l,k}+w_l\bar{A}_{l,k})X_l^{k,\tau,x}+(B_{l,k}+w_l\bar{B}_{l,k})u_k^{\tau,x,*}+f_{l,k}+w_l\bar{f}_{l,k},\\X_k^{k,\tau,x}=X_k^{\tau,x,*},\quad l\in\mathbb{N}_k,\end{cases} \tag{19}$$

which forms a flow of the FBSDE system. The equilibrium control $u_k^{\tau,x,*}$ can be calculated by decoupling the FBSDE system with the stationary condition (6).

Remark 4. The proposed maximum principle in Theorem 1 and the solution to the FBSDE serve as the key tools for the time-inconsistent equilibrium control problem investigated in this paper, as well as other stochastic control problems [32, 33]. It is stressed that the derivation methods in Theorem 1 differ from those in previous work on time-inconsistent control problems [22, 24].

3.2 Solution to Problem 1

Time-inconsistent control problems usually arise in financial applications, like the mean-variance portfolio selection problem. In these cases, the wealth of the investor is scalar-valued. Therefore, we mainly investigate the $n=1$ case in this section, i.e., the state of system (1) is one-dimensional. By using the maximum principle and the solution to the FBSDE developed in Theorem 1, we will explore the necessary and sufficient solvability conditions of Problem 1.

The main results of this section can be stated as follows.

Theorem 2. Under Assumption 1, Problem 1 is uniquely solved if and only if $\Upsilon_k,\ k \in \mathbb{N}_\tau$ as given below are all positive definite.

In this case, for any $\tau \leqslant k \leqslant N$, the equilibrium control can be given as

$$u_k^{\tau,x,*} = -\Upsilon_k^{-1}\Delta_k X_k^{\tau,x,*} - \Upsilon_k^{-1}\Pi_k, \tag{20}$$

where $\Upsilon_k, \Delta_k, \Pi_k$ follow the relationship as below:

$$\Upsilon_k = R_{k,k} + (P_{k+1,k} + \Gamma_{k+1,k})B'_{k,k}B_{k,k} + P_{k+1,k}\bar{B}'_{k,k}\bar{B}_{k,k}, \tag{21}$$
$$\Delta_k = (P_{k+1,k} + \Gamma_{k+1,k})B'_{k,k}A_{k,k} + P_{k+1,k}\bar{B}'_{k,k}\bar{A}_{k,k} + B'_{k,k}M_{k+1,k}, \tag{22}$$
$$\Pi_k = (P_{k+1,k} + \Gamma_{k+1,k})B'_{k,k}f_{k,k} + P_{k+1,k}\bar{B}'_{k,k}\bar{f}_{k,k} + B'_{k,k}\Phi_{k+1,k}, \tag{23}$$

in which $P_{k+1,k}, \Gamma_{k+1,k}$ can be calculated from the following nonsymmetric Riccati equations for $l \in \mathbb{N}_{k+1}$,

$$P_{l,k} = Q_{l,k} + (A_{l,k}^2 + \bar{A}_{l,k}^2)P_{l+1,k} - (A_{l,k}P_{l+1,k}B_{l,k} + \bar{A}_{l,k}P_{l+1,k}\bar{B}_{l,k})\Upsilon_l^{-1}\Delta_l, \tag{24}$$
$$\Gamma_{l,k} = A_{l,k}^2\Gamma_{l+1,k} - A_{l,k}\Gamma_{l+1,k}B_{l,k}\Upsilon_l^{-1}\Delta_l, \tag{25}$$

with final condition $P_{N+1,k}$ and $\Gamma_{N+1,k}$ given by cost function (2).

Moreover, $M_{k+1,k},\ \Phi_{k+1,k}$ satisfy the difference equations for $l \in \mathbb{N}_{k+1}$ as below:

$$M_{l,k} = A_{l,k}M_{l+1,k}, \tag{26}$$
$$\begin{aligned}\Phi_{l,k} &= (P_{l+1,k} + \Gamma_{l+1,k})A_{l,k}f_{l,k} + \bar{A}_{l,k}P_{l+1,k}\bar{f}_{l,k} + A_{l,k}\Phi_{l+1,k}\\ &\quad - \Big[(P_{l+1,k} + \Gamma_{l+1,k})A_{l,k}B_{l,k} + \bar{A}_{l,k}P_{l+1,k}\bar{B}_{l,k}\Big]\Upsilon_l^{-1}\Pi_l,\end{aligned} \tag{27}$$

while $M_{N+1,k}$ and $\Phi_{N+1,k}$ are as in (2).

In this case, the system state $X_l^{k,\tau,x}$ and costate $\lambda_l^{k,\tau,x}$ have the following relationship (the solution to the FBSDE developed in Theorem 1):

$$\lambda_l^{k,\tau,x} = P_{l+1,k}X_{l+1}^{k,\tau,x} + \Gamma_{l+1,k}E_{k-1}X_{l+1}^{k,\tau,x} + M_{l+1,k}X_k^{\tau,x,*} + \Phi_{l+1,k}, \tag{28}$$

where $P_{l+1,k},\ \Gamma_{l+1,k},\ M_{l+1,k},\ \Phi_{l+1,k}$ can be calculated by (24)–(27), respectively.

Proof. "Sufficiency": Suppose $\Upsilon_k,\ k \in \mathbb{N}_\tau$ are strictly positive definite. We will show that Problem 1 is uniquely solvable.

In fact, for simplicity, we denote

$$V_{l,k}^{(1)} \triangleq E_{k-1}[P_{l,k}(X_l^{k,\tau,x})^2], \qquad V_{l,k}^{(2)} \triangleq \Gamma_{l,k}(E_{k-1}X_l^{k,\tau,x})^2,$$
$$V_{l,k}^{(3)} \triangleq E_{k-1}(X_k^{\tau,x,*}M_{l,k}X_l^{k,\tau,x}), \qquad V_{l,k}^{(4)} \triangleq \Phi_{l,k}E_{k-1}X_l^{k,\tau,x}, \tag{29}$$
$$V_{l,k} \triangleq V_{l,k}^{(1)} + V_{l,k}^{(2)} + 2V_{l,k}^{(3)} + 2V_{l,k}^{(4)}, \tag{30}$$

where $P_{l,k}$ and $\Gamma_{l,k}$ satisfy Riccati equations (24) and (25), and $M_{l,k}$ and $\Phi_{l,k}$ satisfy (26) and (27), respectively.

Through simple calculation, we can obtain

$$\begin{aligned}V_{l,k}^{(1)} - V_{l+1,k}^{(1)} &= E_{k-1}\Big[(P_{l,k} - P_{l+1,k}A_{l,k}^2 - P_{l+1,k}\bar{A}_{l,k}^2)(X_l^{k,\tau,x})^2\\ &\quad - 2X_l^{k,\tau,x}(A_{l,k}P_{l+1,k}B_{l,k} + \bar{A}_{l,k}P_{l+1,k}\bar{B}_{l,k})u_l\\ &\quad - 2X_l^{k,\tau,x}(A_{l,k}P_{l+1,k}f_{l,k} + \bar{A}_{l,k}P_{l+1,k}\bar{f}_{l,k}) - u'_l[P_{l+1,k}(B'_{l,k}B_{l,k} + \bar{B}'_{l,k}\bar{B}_{l,k})]u_l\\ &\quad - 2u'_l(B'_{l,k}P_{l+1,k}f_{l,k} + \bar{B}'_{l,k}P_{l+1,k}\bar{f}_{l,k}) - P_{l+1,k}(f_{l,k}^2 + \bar{f}_{l,k}^2)\Big],\end{aligned} \tag{31}$$
$$\begin{aligned}V_{l,k}^{(2)} - V_{l+1,k}^{(2)} &= (E_{k-1}X_l^{k,\tau,x})(\Gamma_{l,k} - A_{l,k}^2\Gamma_{l+1,k})E_{k-1}X_l^{k,\tau,x}\\ &\quad - E_{k-1}\Big(2u'_lB'_{l,k}\Gamma_{l+1,k}A_{l,k}X_l^{k,\tau,x} + u'_l\Gamma_{l+1,k}B'_{l,k}B_{l,k}u_l\end{aligned}$$

$$+ 2A_{l,k}\Gamma_{l+1,k}f_{l,k}X_l^{k,\tau,x} + 2u_l'B_{l,k}'\Gamma_{l+1,k}f_{l,k} + f_{l,k}^2\Gamma_{l+1,k}\Big), \tag{32}$$

$$V_{l,k}^{(3)} - V_{l+1,k}^{(3)} = E_{k-1}\Big[X_k^{\tau,x,*}(M_{l,k} - M_{l+1,k}A_{l,k})X_l^{k,\tau,x} - X_k^{\tau,x,*}M_{l+1,k}B_{l,k}u_l - X_k^{\tau,x,*}M_{l+1,k}f_{l,k}\Big], \tag{33}$$

$$V_{l,k}^{(4)} - V_{l+1,k}^{(4)} = E_{k-1}\Big\{(\Phi_{l,k} - \Phi_{l+1,k}A_{l,k})X_l^{k,\tau,x} - \Phi_{l+1,k}B_{l,k}u_l - \Phi_{l+1,k}f_{l,k}\Big\}. \tag{34}$$

Thus, combining (31)–(34) yields

$$\begin{aligned} V_{l,k} - V_{l+1,k} = E_{k-1}\Big\{&(P_{l,k} - A_{l,k}^2P_{l+1,k} - \bar{A}_{l,k}^2P_{l+1,k})(X_l^{k,\tau,x})^2 \\ &- 2u_l'(B_{l,k}'P_{l+1,k}A_{l,k} + \bar{B}_{l,k}'P_{l+1,k}\bar{A}_{l,k})X_l^{k,\tau,x} \\ &- u_l'[(P_{l+1,k} + \Gamma_{l+1,k})B_{l,k}'B_{l,k} + P_{l+1,k}\bar{B}_{l,k}'\bar{B}_{l,k}]u_l \\ &- 2X_l^{k,\tau,x}\Big[A_{l,k}P_{l+1,k}f_{l,k} + \bar{A}_{l,k}'P_{l+1,k}\bar{f}_{l,k} + A_{l,k}'\Gamma_{l+1,k}f_{l,k} \\ &+ (A_{l,k}M_{l+1,k} - M_{l,k})X_k^{\tau,x,*} + A_{l,k}\Phi_{l+1,k} - \Phi_{l,k}\Big] \\ &- 2u_l'\Big(B_{l,k}'\Gamma_{l+1,k}A_{l,k}E_{k-1}X_l^{k,\tau,x} + B_{l,k}'P_{l+1,k}f_{l,k} + \bar{B}_{l,k}'P_{l+1,k}\bar{f}_{l,k} + B_{l,k}'\Gamma_{l+1,k}f_{l,k} \\ &+ B_{l,k}'M_{l+1,k}X_k^{\tau,x,*} + B_{l,k}'\Phi_{l+1,k}\Big) + E_{k-1}(X_l^{k,\tau,x})(\Gamma_{l,k} - A_{l,k}\Gamma_{l+1,k}A_{l,k})E_{k-1}X_l^{k,\tau,x} \\ &- \Big[2X_k^{\tau,x,*}M_{l+1,k}f_{l,k} + f_{l,k}P_{l+1,k}f_{l,k} + \bar{f}_{l,k}P_{l+1,k}\bar{f}_{l,k} + f_{l,k}\Gamma_{l+1,k}f_{l,k} + 2\Phi_{l+1,k}f_{l,k}\Big]\Big\}. \end{aligned} \tag{35}$$

Especially, noticing (24)–(27), (35) indicates that

$$\begin{aligned} V_{k,k} - V_{k+1,k} = {}& u_k'R_{k,k}u_k + Q_{k,k}(X_k^{\tau,x,*})^2 - [u_k + \Upsilon_k^{-1}(\Delta_kX_k^{\tau,x,*} + \Pi_k)]'\Upsilon_k[u_k + \Upsilon_k^{-1}(\Delta_kX_k^{\tau,x,*} + \Pi_k)] \\ &+ \Big[P_{k,k} - Q_{k,k} - A_{k,k}^2P_{k+1,k} - \bar{A}_{k,k}^2P_{k+1,k} \\ &+ (A_{k,k}P_{k+1,k}B_{k,k} + \bar{A}_{k,k}P_{k+1,k}\bar{B}_{k,k})\Upsilon_k^{-1}\Delta_k\Big](X_k^{\tau,x,*})^2 + 2(M_{k,k} - A_kM_{k+1,k})(X_k^{\tau,x,*})^2 \\ &+ \Big(\Gamma_{k,k} - A_{k,k}^2\Gamma_{k+1,k} + A_{k,k}\Gamma_{k+1,k}B_{k,k}\Upsilon_k^{-1}\Delta_k\Big)(X_k^{\tau,x,*})^2 \\ &+ 2X_k^{\tau,x,*}\Big[\Phi_{k,k} - A_{k,k}\Phi_{k+1,k} - A_{k,k}P_{k+1,k}f_{k,k} - A_{k,k}\Gamma_{k+1,k}f_{k,k} - \bar{A}_{k,k}P_{k+1,k}\bar{f}_{k,k} \\ &+ (A_{k,k}\Gamma_{k+1,k}B_{k,k} + A_{k,k}P_{k+1,k}B_{k,k} + \bar{A}_{k,k}P_{k+1,k}\bar{B}_{k,k})\Upsilon_k^{-1}\Pi_k\Big] \\ &- \Big(2X_k^{\tau,x,*}M_{k+1,k}f_{k,k} + f_{k,k}P_{k+1,k}f_{k,k} + \bar{f}_{k,k}P_{k+1,k}\bar{f}_{k,k} \\ &+ f_{k,k}'\Gamma_{k+1,k}f_{k,k} + 2f_{k,k}\Phi_{k+1,k} - M_{k+1,k}B_{k,k}\Upsilon_k^{-1}\Delta_k(X_k^{\tau,x,*})^2 \\ &- 2X_k^{\tau,x,*}M_{k+1,k}B_{k,k}\Upsilon_k^{-1}\Pi_k - \Pi_k'\Upsilon_k^{-1}\Pi_k\Big) \\ = {}& u_k'R_{k,k}u_k + Q_{k,k}(X_k^{\tau,x,*})^2 - [u_k + \Upsilon_k^{-1}(\Delta_kX_k^{\tau,x,*} + \Pi_k)]'\Upsilon_k[u_k + \Upsilon_k^{-1}(\Delta_kX_k^{\tau,x,*} + \Pi_k)] \\ &- \Big[2X_k^{\tau,x,*}M_{k+1,k}f_{k,k} + f_{k,k}^2P_{k+1,k} + \bar{f}_{k,k}^2P_{k+1,k} + f_{k,k}^2\Gamma_{k+1,k} + 2f_{k,k}\Phi_{k+1,k} \\ &- (X_k^{\tau,x,*})^2M_{k+1,k}B_{k,k}\Upsilon_k^{-1}\Delta_k - 2X_k^{\tau,x,*}M_{k+1,k}B_{k,k}\Upsilon_k^{-1}\Pi_k - \Pi_k'\Upsilon_k^{-1}\Pi_k\Big]. \end{aligned} \tag{36}$$

Taking summation on both sides of (35) from l to N and using (36), we can obtain

$$\begin{aligned} V_{k,k} - V_{N+1,k} = {}& u_k'R_{k,k}u_k + Q_{k,k}(X_k^{\tau,x,*})^2 + E_{k-1}\left\{\sum_{l=k+1}^{N}[u_l^{k,\tau,x}R_{l,k}u_l + Q_{l,k}(X_l^{k,\tau,x})^2]\right\} \\ &- [u_k + \Upsilon_k^{-1}(\Delta_kX_k^{\tau,x,*} + \Pi_k)]'\Upsilon_k[u_k + \Upsilon_k^{-1}(\Delta_kX_k^{\tau,x,*} + \Pi_k)] + \Omega_k + \Psi_k, \end{aligned} \tag{37}$$

where $X_l^{k,\tau,x}$ is the regulated state with equilibrium control u_l as given in (20) for $l \in \mathbb{N}_{k+1}$, and Φ_k and Ψ_k are as given below:

$$\Omega_k = -\Big[2X_k^{\tau,x,*}M_{k+1,k}f_{k,k} + f_{k,k}^2P_{k+1,k} + \bar{f}_{k,k}^2P_{k+1,k} + f_{k,k}^2\Gamma_{k+1,k} + 2f_{k,k}\Phi_{k+1,k}$$

$$- (X_k^{\tau,x,*})^2 M_{k+1,k} B_{k,k} \Upsilon_k^{-1} \Delta_k - 2X_k^{\tau,x,*} M_{k+1,k} B_{k,k} \Upsilon_k^{-1} \Pi_k - \Pi_k' \Upsilon_k^{-1} \Pi_k \Big], \quad (38)$$

$$\begin{aligned}
\Psi_k = \sum_{l=k+1}^{N} & E_{k-1}\Big\{ (P_{l,k} - Q_{l,k} - A_{l,k}^2 P_{l+1,k} - \bar{A}_{l,k}^2 P_{l+1,k})(X_l^{k,\tau,x})^2 \\
& - 2u_l'(B_{l,k}' P_{l+1,k} A_{l,k} + \bar{B}_{l,k}' P_{l+1,k} \bar{A}_{l,k}) X_l^{k,\tau,x} \\
& - u_l'(R_{l,k} + B_{l,k}' P_{l+1,k} B_{l,k} + \bar{B}_{l,k}' P_{l+1,k} \bar{B}_{l,k} + B_{l,k}' \Gamma_{l+1,k} B_{l,k}) u_l \\
& - 2\Big[A_{l,k} P_{l+1,k} f_{l,k} + \bar{A}_{l,k} P_{l+1,k} \bar{f}_{l,k} + A_{l,k} \Gamma_{l+1,k} f_{l,k} \\
& + (A_{l,k} M_{l+1,k} - M_{l,k}) X_k^{\tau,x,*} + A_{l,k} \Phi_{l+1,k} - \Phi_{l,k} \Big] X_l^{k,\tau,x} \\
& - 2u_l'\Big(B_{l,k}' \Gamma_{l+1,k} A_{l,k} E_{k-1} X_l^{k,\tau,x} + B_{l,k}' P_{l+1,k} f_{l,k} + \bar{B}_{l,k}' P_{l+1,k} \bar{f}_{l,k} + B_{l,k}' \Gamma_{l+1,k} f_{l,k} \\
& + B_{l,k}' M_{l+1,k} X_k^{\tau,x,*} + B_{l,k}' \Phi_{l+1,k} \Big) + (\Gamma_{l,k} - A_{l,k} \Gamma_{l+1,k} A_{l,k})(E_{k-1} X_l^{k,\tau,x})^2 \\
& - (2X_k^{\tau,x,*} M_{l+1,k} f_{l,k} + f_{l,k} P_{l+1,k} f_{l,k} + \bar{f}_{l,k} P_{l+1,k} \bar{f}_{l,k} + f_{l,k} \Gamma_{l+1,k} f_{l,k} + 2\Phi_{l+1,k} f_{l,k}) \Big\}. \quad (39)
\end{aligned}$$

From (36)–(39), the cost function (2) can be rewritten as:

$$\begin{aligned}
J(k, X_k^{\tau,x,*}; (u_k, u^{\tau,x,*}|_{\mathbb{N}_{k+1}})) = {} & P_{k,k}(X_k^{\tau,x,*})^2 + \Gamma_{k,k}(X_k^{\tau,x,*})^2 + 2M_{k,k}(X_k^{\tau,x,*})^2 + 2\Phi_{k,k} X_k^{\tau,x,*} - \Omega_k - \Psi_k \\
& + [u_k + \Upsilon_k^{-1}(\Delta_k X_k^{\tau,x,*} + \Pi_k)]' \Upsilon_k [u_k + \Upsilon_k^{-1}(\Delta_k X_k^{\tau,x,*} + \Pi_k)]. \quad (40)
\end{aligned}$$

It is stressed that u_l for $l \in \mathbb{N}_{k+1}$ in (37)–(40) is the equilibrium control given by (20), which is fixed, though unknown (see (19)). From Definition 1 of the equilibrium control and Remark 2, we know that u_k is the local optimal only at time k.

Moreover, since it is assumed that $\Upsilon_k > 0$, from (40) we can conclude that equilibrium control u_k satisfies (20) for $k \in \mathbb{N}_\tau$. The optimal cost function is as below:

$$J(k, X_k^{\tau,x,*}; u^{\tau,x,*}|_{\mathbb{N}_k}) = P_{k,k}(X_k^{\tau,x,*})^2 + \Gamma_{k,k}(X_k^{\tau,x,*})^2 + 2M_{k,k}(X_k^{\tau,x,*})^2 + 2\Phi_{k,k} X_k^{\tau,x,*} - \Omega_k - \Psi_k, \quad (41)$$

where Ω_k, Ψ_k satisfy (38) and (39), respectively.

"Necessity": For the initial pair (τ, x), suppose Problem 1 is uniquely solvable. We will show $\Upsilon_k > 0$ for $k \in \mathbb{N}_\tau$.

Firstly, we will show $\Upsilon_N > 0$. From Definition 1 and Remark 2 we know u_N is only optimal at time N, i.e., u_N minimizes $J(N, X_N^{\tau,x,*}; u_N)$. Moreover, we know $J(N, X_N^{\tau,x,*}; u_N)$ can be calculated as:

$$\begin{aligned}
J(N, X_N^{\tau,x,*}; u_N) = {} & E_{N-1}[Q_{N,N}(X_N^{\tau,x,*})^2 + u_N' R_{N,N} u_N + P_{N+1,N}(X_{N+1}^{N,\tau,x})^2] \\
& + \Gamma_{N+1,N}(E_{N-1} X_{N+1}^{N,\tau,x})^2 + 2(M_{N+1,N} X_N^{\tau,x,*} + \Phi_{N+1,N}) E_{N-1} X_{N+1}^{N,\tau,x} \\
= {} & \Gamma_{N+1,N} A_{N,N}^2 (X_N^{\tau,x,*})^2 + (Q_{N,N} + P_{N+1,N} A_{N,N}^2 + P_{N+1,N} \bar{A}_{N,N}^2)(X_N^{\tau,x,*})^2 \\
& + 2X_N^{\tau,x,*}[A_{N,N}(P_{N+1,N} + \Gamma_{N+1,N}) B_{N,N} + \bar{A}_{N,N} P_{N+1,N} \bar{B}_{N,N}] u_N \\
& + u_N'[R_{N,N} + B_{N,N}'(P_{N+1,N} + \Gamma_{N+1,N}) B_{N,N} + \bar{B}_{N,N}' P_{N+1,N} \bar{B}_{N,N}] u_N \\
& + 2X_N^{\tau,x,*}[A_{N,N}(P_{N+1,N} + \Gamma_{N+1,N}) f_{N,N} + \bar{A}_{N,N} P_{N+1,N} \bar{f}_{N,N}] \\
& + 2u_N'[B_{N,N}'(P_{N+1,N} + \Gamma_{N+1,N}) f_{N,N} + \bar{B}_{N,N}' P_{N+1,N} \bar{f}_{N,N}] \\
& + (P_{N+1,N} + \Gamma_{N+1,N}) f_{N,N}^2 + P_{N+1,N} \bar{f}_{N,N}^2 \\
& + 2(M_{N+1,N} X_N^{\tau,x,*} + \Phi_{N+1,N})(A_{N,N} X_N^{\tau,x,*} + B_{N,N} u_N + f_{N,N}). \quad (42)
\end{aligned}$$

Since Problem 1 is uniquely solvable, i.e., $J(N, X_N^{\tau,x,*}; u_N)$ is minimized with a unique equilibrium control $u_N^{\tau,x,*}$. In what follows, we set $X_N^{\tau,x,*} = 0$ and $\Phi_{N+1,N} = f_{N,N} = \bar{f}_{N,N} = 0$, so that (42) reduces to

$$J(N, X_N^{\tau,x,*}; u_N) = u_N' \Upsilon_N u_N, \quad (43)$$

where Υ_N obeys (21) for $k = N$.

It is noted that the equilibrium control $u_N^{\tau,x,*}$ is unique; thus for the case of $X_N^{\tau,x,*}=0$ and $\Phi_{N+1,N}=f_{N,N}=\bar{f}_{N,N}=0$, Problem 1 is also uniquely solvable.

In this case, we choose λ to be any fixed eigenvalue of Υ_N. We will show $\lambda\geqslant 0$. Otherwise, $\lambda<0$, we choose v_λ to be a unit eigenvector associated with λ satisfying $v_\lambda' v_\lambda=1$. Then $|\lambda|^{-1}\Upsilon v_\lambda=-v_\lambda$ holds.

Choose a fixed constant $\delta\in\mathcal{R}$, set $u_N=\frac{\delta}{|\lambda|^{1/2}}v_\lambda$; then from (43) we know $J(N,X_N^{\tau,x,*};u_N)=-\delta^2$. By letting $\delta\to\infty$, we have $J(N,X_N^{\tau,x,*};u_N)\to-\infty$, which is a contradiction with Assumption 1 and the restrictions for weighting matrices of (2). Hence, we have $\Upsilon_N\geqslant 0$. On the other hand, since the equilibrium control is unique, and $J(N,X_N^{\tau,x,*};u_N)\geqslant 0$. Thus $u_N^{\tau,x,*}=0$ is the unique equilibrium control of minimizing (43), and $\Upsilon_N>0$ can be derived.

In what follows, notice (8) and use the system dynamics (1), and letting $k=N$, it follows from (6) that

$$\begin{aligned}
0&=R_{N,N}u_N+E_{N-1}[(B_{N,N}+w_N\bar{B}_{N,N})'\lambda_{N,N}],\\
&=R_{N,N}u_N+E_{N-1}[(B_{N,N}+w_N\bar{B}_{N,N})'(P_{N+1,N}X_{N+1}^{N,\tau,x}+\Gamma_{N+1,N}E_{N-1}X_{N+1}^{N,\tau,x}+M_{N+1,N}X_N^{\tau,x,*}+\Phi_{N+1,N})]\\
&=R_{N,N}u_N+E_{N-1}[(B_{N,N}+w_N\bar{B}_{N,N})'P_{N+1,N}X_{N+1}^{N,\tau,x}]\\
&\quad+B_{N,N}'\Gamma_{N+1,N}E_{N-1}X_{N+1}^{N,\tau,x}+B_{N,N}'M_{N+1,N}X_N^{\tau,x,*}+B_{N,N}'\Phi_{N+1,N}\\
&=R_{N,N}u_N+E_{N-1}\{(B_{N,N}+w_N\bar{B}_{N,N})'P_{N+1,N}[(A_{N,N}+w_N\bar{A}_{N,N})X_N^{\tau,x,*}+(B_{N,N}+w_N\bar{B}_{N,N})u_N\\
&\quad+f_{N,N}+w_N\bar{f}_{N,N}]\}+B_{N,N}'\Gamma_{N+1,N}(A_{N,N}X_N^{\tau,x,*}+B_{N,N}u_N+f_{N,N})\\
&\quad+B_{N,N}'M_{N+1,N}X_N^{\tau,x,*}+B_{N,N}'\Phi_{N+1,N}\\
&=(R_{N,N}+B_{N,N}'P_{N+1,N}B_{N,N}+\bar{B}_{N,N}'P_{N+1,N}\bar{B}_{N,N})u_N+(B_{N,N}'P_{N+1,N}A_{N,N}+\bar{B}_{N,N}'P_{N+1,N}\bar{A}_{N,N})X_N^{\tau,x,*}\\
&\quad+B_{N,N}'\Gamma_{N+1,N}A_{N,N}X_N^{\tau,x,*}+B_{N,N}'\Gamma_{N+1,N}B_{N,N}u_N+B_{N,N}'\Gamma_{N+1,N}f_{N,N}+B_{N,N}'P_{N+1,N}f_{N,N}\\
&\quad+\bar{B}_{N,N}P_{N+1,N}\bar{f}_{N,N}+B_{N,N}'M_{N+1,N}X_N+B_{N,N}'\Phi_{N+1,N}\\
&=\Big(R_{N,N}+B_{N,N}'P_{N+1,N}B_{N,N}+\bar{B}_{N,N}'P_{N+1,N}\bar{B}_{N,N}+B_{N,N}'\Gamma_{N+1,N}B_{N,N}\Big)u_N\\
&\quad+\Big(B_{N,N}'P_{N+1,N}A_{N,N}+\bar{B}_{N,N}'P_{N+1,N}\bar{A}_{N,N}+B_{N,N}'\Gamma_{N+1,N}A_{N,N}+B_{N,N}'M_{N+1,N}\Big)X_N^{\tau,x,*}\\
&\quad+B_{N,N}'\Gamma_{N+1,N}f_{N,N}+B_{N,N}'P_{N+1,N}f_{N,N}+\bar{B}_{N,N}'P_{N+1,N}\bar{f}_{N,N}+B_{N,N}'\Phi_{N+1,N}\\
&=\Upsilon_Nu_N+\Delta_NX_N^{\tau,x,*}+\Pi_N,
\end{aligned}\tag{44}$$

where Υ_N, Δ_N and Π_N satisfy (21)–(23) for $k=N$.

Since Problem 1 is solvable and $\Upsilon_N>0$ has been proved as above, (44) indicates that the equilibrium control $u_N^{\tau,x,*}$ is

$$u_N^{\tau,x,*}=-\Upsilon_N^{-1}(\Delta_NX_N^{\tau,x,*}+\Pi_N),\tag{45}$$

i.e., the equilibrium control (20) has been verified for $k=N$.

Next, from (6) we know that to obtain the equilibrium control $u_{N-1}^{\tau,x,*}$ for time $N-1$, we should calculate $\lambda_{N-1}^{N-1,\tau,x}$ first. In fact, following from (7), we can obtain

$$\begin{aligned}
\lambda_{N-1}^{N-1,\tau,x}&=Q_{N,N-1}X_N^{N-1,\tau,x}+E_{N-1}[(A_{N,N-1}+w_N\bar{A}_{N,N-1})'\lambda_N^{N-1,\tau,x}]\\
&=Q_{N,N-1}X_N^{N-1,\tau,x}+E_{N-1}[(A_{N,N-1}+w_N\bar{A}_{N,N-1})'\\
&\quad\times(P_{N+1,N-1}X_{N+1}^{N-1,\tau,x}+\Gamma_{N+1,N-1}E_{N-2}X_{N+1}^{N-1,\tau,x}+M_{N+1,N-1}X_{N-1}^{\tau,x,*}+\Phi_{N+1,N-1})]\\
&=Q_{N,N-1}X_N^{N-1,\tau,x}+E_{N-1}\{(P_{N+1,N-1}A_{N,N-1}^2+P_{N+1,N-1}\bar{A}_{N,N-1}^2)X_N^{N-1,\tau,x}\\
&\quad+(A_{N,N-1}P_{N+1,N-1}B_{N,N-1}+\bar{A}_{N,N-1}P_{N+1,N-1}\bar{B}_{N,N-1})u_N^{\tau,x,*}\\
&\quad+A_{N,N-1}P_{N+1,N-1}f_{N,N-1}+\bar{A}_{N,N-1}P_{N+1,N-1}\bar{f}_{N,N-1}\\
&\quad+\Gamma_{N+1,N-1}A_{N,N-1}^2E_{N-2}X_N^{N-1,\tau,x}+A_{N,N-1}\Gamma_{N+1,N-1}B_{N,N-1}E_{N-2}u_N^{\tau,x,*}\\
&\quad+A_{N,N-1}\Gamma_{N+1,N-1}f_{N,N-1}+A_{N,N-1}M_{N+1,N-1}X_{N-1}^{\tau,x,*}+A_{N,N-1}\Phi_{N+1,N-1}\\
&=\Big[Q_{N,N-1}+P_{N+1,N-1}A_{N,N-1}^2+P_{N+1,N-1}\bar{A}_{N,N-1}^2
\end{aligned}$$

$$
\begin{aligned}
&- (A_{N,N-1}P_{N+1,N-1}B_{N,N-1} + \bar{A}_{N,N-1}P_{N+1,N-1}\bar{B}_{N,N-1})\Upsilon_N^{-1}\Delta_N\Big]X_N^{N-1,\tau,x}\\
&+ \Big(\Gamma_{N+1,N-1}A_{N,N-1}^2 - A_{N,N-1}\Gamma_{N+1,N-1}B_{N,N-1}\Upsilon_N^{-1}\Delta_N\Big)E_{N-2}X_N^{N-1,\tau,x}\\
&+ A_{N,N-1}M_{N+1,N-1}X_{N-1}^{\tau,x,*} + [A_{N,N-1}P_{N+1,N-1}f_{N,N-1}\\
&+ \bar{A}_{N,N-1}P_{N+1,N-1}\bar{f}_{N,N-1} + A_{N,N-1}\Gamma_{N+1,N-1}f_{N,N-1}\\
&+ A_{N,N+1}\Phi_{N+1,N-1} - A_{N,N-1}\Gamma_{N+1,N-1}B_{N,N-1}\Upsilon_N^{-1}\Pi_N\\
&- (A_{N,N-1}P_{N+1,N-1}B_{N,N-1} + \bar{A}_{N,N-1}P_{N+1,N-1}\bar{B}_{N,N-1})\Upsilon_N^{-1}\Pi_N]\\
= {}& P_{N,N-1}X_N^{N-1,\tau,x} + \Gamma_{N,N-1}E_{N-2}X_N^{N-1,\tau,x} + M_{N,N-1}X_{N-1}^{\tau,x,*} + \Phi_{N,N-1},
\end{aligned} \tag{46}
$$

where the equilibrium control $u_N^{\tau,x,*}$ in (45) has been inserted.

From (46) we know that (28) has been verified for $l = N-1, k = N-1$.

To use the induction method, take $\tau \leqslant k \leqslant N$, for any $j \geqslant k+1$, we assume:

- $\Upsilon_j > 0$;
- The equilibrium control u_j for minimizing $J(j, X_j^{\tau,x,*}; u_j, \ldots, u_N)$ satisfies (20);
- The costate $\lambda_j^{k,\tau,x}$ obeys the relationship (28).

In what follows, we shall show that the above statements are also true for $j = k$.

Firstly, we will show $\Upsilon_k > 0$. In fact, similar to the derivation of (47), by letting $X_k^{\tau,x,*} = 0$ and $f_{k,k} = \bar{f}_{k,k} = \Phi_{k+1,k} = 0$, we have

$$
J(k, X_k^{\tau,x,*}; u^{\tau,x,*}|_{\mathbb{N}_k}) = u_k'\Upsilon_k u_k - \Psi_k, \tag{47}
$$

where Ψ_k is given by (39) with $u_j^{\tau,x,*}$ satisfying (20) as assumed for $k+1 \leqslant j \leqslant N$.

Similar to the discussion for Υ_N of (42)–(43), it follows from (47) that if Problem 1 has a unique solution, then $\Upsilon_k > 0$.

Since $\Upsilon_k > 0$ has been shown, from (44)–(46) we know the equilibrium control u_k can be given as (20).

Finally, noting that $\lambda_j^{k,\tau,x}$ for $j \in \mathbb{N}_{k+1}$ satisfies (28), along the lines of (46), $\lambda_k^{k,\tau,x}$ can be derived from (7) as in (28) for $l = k$. This ends the necessity proof by using induction method.

Remark 5. The explicit necessary and sufficient condition for the solvability of the time-inconsistent control problem has been obtained for $n = 1$ case in Theorem 2, which has never been derived for time-inconsistent equilibrium control problems. The solvability conditions in previous work like [17, 22, 28] were based on the solvability of FBSDE, which cannot be easily verified.

The Riccati difference equations given in (24) and (25) are nonsymmetric, but they can be calculated backwardly. It is noted that the proposed algorithm in Theorem 2 is feasible.

The procedure for calculating the feedback gain matrices $\Upsilon_k, \Delta_k, \Pi_k$ for $k \in \mathbb{N}_\tau$ can be stated as:

1) From (21)–(23) for $k = N$, then $\Upsilon_N, \Delta_N, \Pi_N$ can be derived;

2) Plugging $\Upsilon_N, \Delta_N, \Pi_N$ into (21)–(23) and (24)–(27) for $l = N$ and $k = N-1$, we can obtain $\Upsilon_{N-1}, \Delta_{N-1}, \Pi_{N-1}$;

3) By repeating the above steps backward iteratively, $\Upsilon_k, \Delta_k, \Pi_k$ for $k \in \mathbb{N}_\tau$ can be developed.

4 Numerical example

This section provides an example to clarify the results obtained in Theorem 2 and verify the effectiveness of the procedures proposed in the above algorithm. It is noted that the presented example can be applied to solve mean-variance portfolio selection problems [11, 17].

We consider system (1) and associated cost function (2) with $\tau = 0, N = 2$ and the coefficients as below:

$$
\begin{aligned}
&A_{0,0} = 1.5,\ A_{1,0} = -2,\ A_{2,0} = 1.6,\ A_{1,1} = -3.2,\ A_{2,1} = 1.5,\ A_{2,2} = 2,\\
&B_{0,0} = 2.5,\ B_{1,0} = -4.5,\ B_{2,0} = 0.8,\ B_{1,1} = 0.9,\ B_{2,1} = 1.4,\ B_{2,2} = -3,\\
&\bar{A}_{0,0} = 0.8,\ \bar{A}_{1,0} = -1.2,\ \bar{A}_{2,0} = 2.6,\ \bar{A}_{1,1} = -2,\ \bar{A}_{2,1} = 0.5,\ \bar{A}_{2,2} = 3.2,
\end{aligned}
$$

$$\bar{B}_{0,0}=0.6,\ \bar{B}_{1,0}=8.2,\ \bar{B}_{2,0}=-4.7,\ \bar{B}_{1,1}=3.8,\ \bar{B}_{2,1}=-2.5,\ \bar{B}_{2,2}=3,$$
$$f_{0,0}=0.5,\ f_{1,0}=0,\ f_{2,0}=2.4,\ f_{1,1}=-1.5,\ f_{2,1}=2,\ f_{2,2}=0,$$
$$\bar{f}_{0,0}=-1.8,\ \bar{f}_{1,0}=5.5,\ \bar{f}_{2,0}=0.2,\ \bar{f}_{1,1}=1.2,\ \bar{f}_{2,1}=-0.5,\ \bar{f}_{2,2}=-1.5,$$
$$Q_{0,0}=0,\ Q_{1,0}=2,\ Q_{2,0}=3.5,\ Q_{1,1}=2.8,\ Q_{2,1}=0,\ Q_{2,2}=1,$$
$$R_{0,0}=1.2,\ R_{1,0}=0,\ R_{2,0}=0,\ R_{1,1}=0,\ R_{2,1}=1.5,\ R_{2,2}=3.6,$$
$$P_{3,0}=2.3,\ P_{3,1}=1.2,\ P_{3,2}=0,\ \Gamma_{3,0}=0.8,\ \Gamma_{3,1}=-0.8,\ \Gamma_{3,2}=2,$$
$$M_{3,0}=0.3,\ M_{3,1}=-2.2,\ M_{3,2}=0,\ \Phi_{3,0}=-5,\ \Phi_{3,1}=2,\ \Phi_{3,2}=3.$$

Firstly, by (21)–(23) for $k=N=2$, we have $\Upsilon_2=21.6>0,\ \Delta_2=-12,\ \Pi_2=-9$.

Next, by calculating (24)–(27) with the obtained $\Upsilon_2,\Delta_2,\Pi_2$, there holds

$$P_{2,1}=3.5667,\ M_{2,1}=-3.3,\ \Gamma_{2,1}=-2.7333,\ \Phi_{2,1}=3.6250,$$

then $\Upsilon_1=52.1782>0,\ \Delta_1=-32.4771,\ \Pi_1=18.4015$ can be obtained by solving (21)–(23) for $k=1$.

Finally, by plugging $\Upsilon_i,\Delta_i,\Pi_i,\ i=1,2$ obtained above, again using (24)–(27), it can be obtained:

$$P_{2,0}=9.2745,\ M_{2,0}=0.48,\ \Gamma_{2,0}=2.6854,\ \Phi_{2,0}=13.6127,$$
$$P_{1,0}=47.6042,\ M_{1,0}=-0.96,\ \Gamma_{1,0}=25.7848,\ \Phi_{1,0}=-42.1266.$$

Then naturally, we have

$$\Upsilon_0=477.0188>0,\ \Delta_0=295.6588,\ \Pi_0=-64.9928.$$

In conclusion, the equilibrium control can be calculated from (20) as

$$u_2^{0,x,*}=0.5556X_2^{0,x,*}+0.4167,\quad u_1^{0,x,*}=0.6224X_1^{0,x,*}-0.3527,\quad u_0^{0,x,*}=-0.6198X_0^{0,x,*}-0.1362,$$

where x is the initial state at initial time $\tau=0$.

5 Conclusion

The general time-inconsistent equilibrium control problem for discrete-time systems has been studied in this paper. We have developed the maximum principle by using variational methods, and thus, a flow of FBSDE is obtained. Based on the established maximum principle and the solution to the FBSDE, the open-loop equilibrium control of $n=1$ case has been derived and the explicit necessary and sufficient solvability condition has been developed for the first time. For future research, we would like to extend our work to solve the stabilization problems for time-inconsistent stochastic control.

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Conflict of interest The authors declare that they have no conflict of interest.

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