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Stochastic evolution equations of jump type with random coefficients: existence, uniqueness and optimal control

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On a given filtrated probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\leqslant t\leqslant T}, P)$, we study the following stochastic partial differential equations (SPDEs) of evolutionary type with jump:

$$\begin{cases} \mathrm{d}X(s) = & [A(s)X(s) + b(s, X(s))]\mathrm{d}s \\ & + [B(s)X(s) + g(s, X(s))]\mathrm{d}W(s) \\ & + \int_E \sigma(s, e, X(s-))\tilde{\mu}(\mathrm{d}e, \mathrm{d}s), \\ X(0) = & x, \quad s \in [0, T], \end{cases} \tag{1}$$

in a Gelfand triple $V \subset H = H^* \subset V^*$. Here $\tilde{\mu}$ is a Poisson random martingale measure on a fixed nonempty Borel measurable subset E of $\mathbb{R}^1$ and W is a one-dimensional Brownian motion, $A : \Omega \times [0, T] \longrightarrow \mathscr{L}(V, V^*)$, $B : \Omega \times [0, T] \longrightarrow \mathscr{L}(V, H)$, $b : \Omega \times [0, T] \times H \longrightarrow H$, $g : \Omega \times [0, T] \times H \longrightarrow H$ and $\sigma : \Omega \times [0, T] \times E \times H \longrightarrow H$ are given random mappings. Here $\mathscr{L}(V, V^*)$ denotes the set of all bounded linear operators of V into V^* and $\mathscr{L}(V, H)$ denotes the set of all bounded linear operators of H into V. An adapted solution of (1) is a V-valued, $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$-adapted process $X(\cdot)$ which satisfies (1) under some appropriate sense. Such a model as (1) represents a large classes of stochastic partial differential equations, for instance, the nonlinear filtering equation and other stochastic parabolic PDEs, but it is by no means the largest one. Partial differential equations are too diverse to be covered by a single model, like ordinary equations. One of the purposes of our work is to establish the existence and uniqueness of solutions to the stochastic evolution equation (1). Most recently, thanks to comprehensive practical applications, many attentions have been paid to SPDEs driven by jump processes, (for example, [1–7] and the references therein). It is worth mentioning that Röckner and Zhang [3] obtained the uniqueness and existence results for stochastic evolution equations of type (1.1) by a successive approximations, in which case the operator B does not exist. Another purpose of this paper is to establish the maximum principle and verification theorem for the optimal control problem where the state process is driven by a controlled stochastic evolution equation (1). In 2005, Øksendal et al. [8] studied the optimal control problem of quasilinear semielliptic SPDEs driven by Poisson random measure and gave sufficient maximum principle results, not necessary ones. As an application, at last, we will present a linear quadratic optimal control problem of a controlled SPDE with jumps which our theoretical results can solve.

Assumption 1. The operator processes A and B are weakly predictable; i.e., $\langle A(\cdot)x, y\rangle$ and $(B(\cdot)x, y)_H$ are both predictable process for every

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$x, y \in V$, and satisfy the coercive condition, i.e., there exist some constants $C, \alpha > 0$ and λ such that for any $x \in V$ and each $(s, \omega) \in [0, T] \times \Omega$,

$$\lambda||x||_H - \alpha||x||_V \geqslant \langle A(s)x, x\rangle + ||Bx||_H \quad (2)$$

and

$$\sup_{(\omega,s)\in\Omega\times[0,T]} \|A(s,\omega)\|_{\mathscr{L}(V,V^*)} + \sup_{(\omega,s)\in\Omega\times[0,T]} \|B(s,\omega)\|_{\mathscr{L}(V,H)} \leqslant C.$$

Assumption 2. The mappings b and g are both $\mathscr{P} \times \mathscr{B}(H)/\mathscr{B}(H)$-measurable with $b(\cdot, 0), g(\cdot, 0) \in M^2_{\mathscr{F}}(0, T; H)$; the mapping σ is $\mathscr{P} \times \mathscr{B}(E) \times \mathscr{B}(H)/\mathscr{B}(H)$-measurable with $\sigma(\cdot, \cdot, 0) \in M^{\nu,2}_{\mathscr{F}}([0, T] \times E; H)$. And there is a constant C such that for a.s.$(\omega, s) \in \Omega \times [0, T]$ and all $x, y \in V$,

$$\begin{aligned}&||b(s, x) - b(s, y)||_H + ||g(s, x) - g(s, y)||_H \\ &\quad + ||\sigma(s, \cdot, x) - \sigma(s, \cdot, y)||_{M^{\nu,2}(E;H)} \\ &\leqslant C||x - y||_H.\end{aligned} \quad (3)$$

Theorem 1 (Existence and uniqueness theorem of SEE with jumps)**.** Let Assumptions 1 and 2 be satisfied by any given coefficients (A, B, b, g, σ) of the SEE (1). Then for any initial value $X(0) = x$, the SEE (1) admits a unique solution $X(\cdot) \in S^2_{\mathscr{F}}(0, T; H) \bigcap M^2_{\mathscr{F}}(0, T; V)$.

On a real-valued Hilbert space U, consider a nonempty convex closed subset $\mathscr{U}$ which is our control domain. A predictable stochastic process $u(\cdot) \triangleq \{u(t), 0 \leqslant t \leqslant T\}$ is referred to as an admissible control process if $u(\cdot) \in \mathcal{M}^2(0, T; U)$ and $u(t) \in \mathscr{U}$, a.e., $t \in [0, T]$, $\mathbb{P}$-a.s.. The set of all admissible control processes is denoted by $\mathcal{A}$.

In the Gelfand triple (V, H, V^*), for any admissible control $u(\cdot) \in \mathcal{A}$, we consider the following controlled SEE with jumps:

$$\begin{cases} \mathrm{d}X(s) = [A(s)X(s) + b(t, X(s), u(s))]\mathrm{d}s \\ \qquad\qquad +[B(s)X(s) + g(t, X(s), u(s))]\mathrm{d}W(s) \\ \qquad\qquad +\displaystyle\int_E \sigma(s, e, X(s-), u(s))\tilde{\mu}(\mathrm{d}e, \mathrm{d}s), \\ X(0) = x, \quad s \in [0, T], \end{cases} \quad (4)$$

with the cost functional

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T l(s, x(s), u(s))\mathrm{d}t + \Phi(x(T))\right], \quad (5)$$

where the coefficients $A : [0, T] \times \Omega \longrightarrow \mathscr{L}(V, V^*)$, $B : [0, T] \times \Omega \longrightarrow \mathscr{L}(V, H)$, $b, g : [0, T] \times \Omega \times H \times \mathscr{U} \to H$, $\sigma : [0, T] \times \Omega \times E \times H \times \mathscr{U} \longrightarrow H$, $l : [0, T] \times \Omega \times H \times \mathscr{U} \to \mathbb{R}$ and $\Phi : \Omega \times H \to \mathbb{R}$ are given random mappings satisfying the following basic assumptions.

Assumption 3. (i) The operator-valued stochastic processes A and B satisfy Assumption 1.

(ii) $b(\cdot, 0, 0), g(\cdot, 0, 0) \in M^2_{\mathscr{F}}(0, T; H), \sigma(\cdot, \cdot, 0, 0) \in M^{\nu,2}_{\mathscr{F}}([0, T] \times E; H)$. Moreover, for almost all $(\omega, s, e) \in \Omega \times [0, T] \times E$, b, g and σ have continuous bounded Gâteaux derivatives $b_x, g_x, \sigma_x, b_u, g_u$ and σ_u.

(iii) For every $(\omega, s) \in \Omega \times [0, T]$, l has continuous Gâteaux derivatives l_x and l_u, and Φ has continuous Gâteaux derivative Φ_x. Moreover, for every $(\omega, s) \in \Omega \times [0, T]$, $|l(s, x, u)| \leqslant K(1 + \|u\|^2_U + \|x\|^2_H)$, $\|l_x(s, x, u)\|_H + \|l_u(s, x, u)\|_U \leqslant K(1 + \|u\|_U + \|x\|_H)$ and $|\Phi(x)| \leqslant K(1 + \|x\|^2_H), \|\Phi_x(x)\|_H \leqslant K(1 + \|x\|_H), (x, u) \in H \times \mathscr{U}$, where K is some positive constant.

It is easy to check that under Assumption 3, for any given admissible control process $u(\cdot)$, the state equation (4) has a unique solution, denoted by $X^u(\cdot)$ or $X(\cdot)$, if its dependence on $u(\cdot)$ is clear from the context. In the following, the solution $X^u(\cdot)$ is referred to as the state process associated with the control process $u(\cdot)$, and $(u(\cdot); X(\cdot))$ is referred to as an admissible pair.

Now we begin to present our optimal control problem.

Problem 1. Choose an admissible control process $\bar{u}(\cdot) \in \mathcal{A}$ satisfies

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{A}} J(u(\cdot)). \quad (6)$$

The admissible control process $\bar{u}(\cdot)$ satisfying the above (6) and the corresponding state process $\bar{X}(\cdot)$ are said to be an optimal control process and an optimal state process of Problem 1, respectively. Then $(\bar{u}(\cdot); \bar{X}(\cdot))$ is referred to as an optimal pair of Problem 1. For any admissible pair $(\bar{u}(\cdot); \bar{X}(\cdot))$, the corresponding adjoint processes are defined as a triple $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{r}(\cdot, \cdot))$ of stochastic processes, which is a solution to the following backward stochastic evolution equation (BSEE for short) with jump, called the adjoint equation,

$$\begin{cases} \mathrm{d}\bar{p}(s) = -\left[A^*(s)\bar{p}(s) + B(s)^*\bar{q}(s) + \bar{\mathcal{H}}_x(s)\right]\mathrm{d}s \\ \qquad\qquad +\bar{q}(s)\mathrm{d}W(s) + \displaystyle\int_E \bar{r}(s, e)\tilde{\mu}(\mathrm{d}e, \mathrm{d}s), \\ \bar{p}(S) = \Phi_x(\bar{X}(T)), \end{cases} \quad (7)$$

where we define the Hamiltonian $\mathcal{H}(s, x, u, p, q, r(\cdot)) := (b, p)_H + (g, q)_H + \int_E (\sigma, r)_H\, \nu(\mathrm{d}e) + l$, and denote

$$\bar{\mathcal{H}}(s) \triangleq \mathcal{H}(t, \bar{X}(s), \bar{u}(s), \bar{p}(s), \bar{q}(s), \bar{r}(s, \cdot)). \quad (8)$$

Now we are in position to state the maximum principle and the verification theorem for Problem 1, respectively.

Theorem 2 (Maximum principle)**.** Let Assumption 3 be satisfied. Let $(\bar{u}(\cdot);\bar{X}(\cdot))$ be an optimal pair of Problem 1 associated with the adjoint processes $(\bar{p}(\cdot),\bar{q}(\cdot),\bar{r}(\cdot,\cdot))$. Then for all $v\in\mathscr{U}$, a.e., $(\omega,s)\in\Omega\times[0,T]$, the following minimum value condition holds: $\big(\mathcal{H}_u(s,\bar{X}(s-),\bar{u}(s),\bar{p}(s-),\bar{q}(s),\bar{r}(s,\cdot)),v-\bar{u}(s)\big)_U\geqslant 0$.

Theorem 3 (Verification theorem)**.** Assume that Assumption 3 holds. Suppose that $(\bar{u}(\cdot);\bar{X}(\cdot))$ is a given admissible pair of Problem 1 associated with the adjoint processes $(\bar{p}(\cdot),\bar{q}(\cdot),\bar{r}(\cdot,\cdot))$. Assume that $\mathcal{H}(s,x,u,\bar{p}(s),\bar{q}(s),\bar{r}(s,\cdot))$ is a convex function with respect to (x,u), and $\Phi(x)$ is a convex function with respect to x. Moreover assume that the following minimum value condition is satisfied for almost all $(\omega,s)\in\Omega\times[0,T]$:

$$\begin{aligned}&\mathcal{H}(s,\bar{X}(s-),\bar{u}(s),\bar{p}(s-),\bar{q}(s),\bar{r}(s,\cdot))\\&\quad=\min_{u\in\mathscr{U}}\mathcal{H}(t,\bar{X}(s-),u,\bar{p}(s-),\bar{q}(s),\bar{r}(s,\cdot)).\end{aligned}$$

Then $(\bar{u}(\cdot);\bar{X}(\cdot))$ is an optimal pair of Problem 1.

As an application, we consider a controlled Cauchy problem:

$$\begin{cases}\mathrm{d}y(s,z)=\big\{\partial_{z^i}[a^{ij}(s,z)\partial_{z^j}y(s,z)]+b^i(s,z)\partial_{z^i}y(s,z)\\ +c(s,z)y(s,z)+u(s,z)\big\}\mathrm{d}t+\{\partial_{z^i}[\eta^i(s,z)y(s,z)]\\ +\rho(s,z)y(s,z)+u(s,z)\}\mathrm{d}W(s)\\ +\displaystyle\int_E[\Gamma(s,e,z)y(s,z)+u(s,z)]\tilde{\mu}(\mathrm{d}e,\mathrm{d}s),\\ y(0,z)=\ \xi(z)\in\mathbb{R}^d,\quad (z,s)\in\mathbb{R}^d\times[0,T].\end{cases}\tag{9}$$

We define $V=H^1$, $H=H^0$, $V^*=H^{-1}$, where H^1 and H^0 are the classical Sobolev spaces. Then (V,H,V^*) is a Gelfand triple. We assume that control domain $\mathscr{U}=U=H$. The admissible control set $\mathcal{A}$ becomes $M^2_{\mathscr{F}}(0,T;U)$. For any admissible control process $u(\cdot,\cdot)$ and the corresponding solution $y(\cdot,\cdot)$ of the state equation (9), the purpose of the optimal control problem is to minimize the following cost functional:

$$\begin{aligned}J(u(\cdot))=\mathbb{E}\bigg[&\int_{\mathbb{R}^d}y^2(T,z)\mathrm{d}z+\iint_{[0,T]\times\mathbb{R}^d}y^2(s,z)\mathrm{d}s\mathrm{d}z\\&+\iint_{[0,T]\times\mathbb{R}^d}u^2(s,z)\mathrm{d}s\mathrm{d}z\bigg].\end{aligned}$$

To order to apply our maximum principle and verification theorem, for the coefficients a, b, c, η, ρ, Γ, we need the following basic assumptions.

Assumption 4. The functions a, b, c, η, and ρ are $\mathscr{P}\times\mathscr{B}(\mathbb{R}^d)$-measurable with values in the set of real symmetric $d\times d$ matrices, $\mathbb{R}^d$, $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{R}$, respectively, and are bounded by K. The function Γ is $\mathscr{P}\times\mathscr{B}(E)\times\mathscr{B}(\mathbb{R}^d)$-measurable with value $\mathbb{R}$ and is bounded by K. $\xi\in L^2(\mathbb{R}^d)$. The super-parabolic condition is satisfied, i.e., $\kappa I+\eta(s,z)(\eta(s,z))^*\leqslant 2a(s,\omega,z)\leqslant KI,\forall(s,\omega,z)\in[0,T]\times\Omega\times\mathbb{R}^d$, where $K\in(1,\infty)$ and $\kappa\in(0,1)$ are some fixed constants and I denotes the $(d\times d)$-identity matrix.

Let $(\bar{u}(\cdot);\bar{X}(\cdot))$ be an optimal pair. Under Assumption 4, by applying maximum principle, the optimal control $\bar{u}(\cdot)$ has the following adjoint representation:

$$\bar{u}(s)=-\frac{1}{2}\bigg[\bar{p}(s-)+\bar{q}(s)+\int_E\bar{r}(s,e)\nu(\mathrm{d}e)\bigg],$$

where $(\bar{p}(\cdot),\bar{q}(\cdot),\bar{r}(\cdot,\cdot))$ is the adjoint process corresponding to the optimal pair $(\bar{u}(\cdot);\bar{X}(\cdot))$.

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Supporting information Appendix A. The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

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