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Stochastic evolution equations of jump type with random coefficients: existence, uniqueness and optimal control

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Appendix A Proof of Theorem 1

In this section, we first introduce the notations which will be used in this paper. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space equipped with a one-dimensional standard Brownian motion $\{W(t), 0 \leqslant t \leqslant T\}$ and a stationary Poisson point process $\{\eta_t\}_{t\geqslant 0}$ defined on a fixed nonempty Borel measurable subset E of $\mathbb{R}^1$. Denote by $\mathbb{E}[\cdot]$ the expectation under the probability $\mathbb{P}$. We denote by $\mu(de, dt)$ the counting measure induced by $\{\eta_t\}_{t\geqslant 0}$ and by $\nu(de)$ the corresponding characteristic measure. Then the compensatory random martingale measure is denoted by $\tilde{\mu}(de, dt) := \mu(de, dt) - \nu(de)dt$ which is assumed to be independent of the Brownian motion $\{W(t), 0 \leqslant t \leqslant T\}$. Furthermore, we assume that $\nu(E) < \infty$. Let $\{\mathscr{F}_t\}_{0\leqslant t\leqslant T}$ be the P-augmentation of the natural filtration generated by $\{W_t\}_{t\geqslant 0}$ and $\{\eta_t\}_{t\geqslant 0}$. By $\mathscr{P}$ we denote the predictable σ field on $\Omega \times [0, T]$ and by $\mathscr{B}(\Lambda)$ the Borel σ-algebra of any topological space Λ. Let X be a separable Hilbert space with norm $\|\cdot\|_X$. Denote by $M^{\nu,2}(E; X)$ the set of all X-valued measurable functions $r = \{r(e), e \in E\}$ defined on the measure space $(E, \mathscr{B}(E); v)$ such that $\|r\|_{M^{\nu,2}(E;X)} \triangleq \sqrt{\int_E \|r(e)\|_X^2 v(de)} < \infty$, by $M^{\nu,2}_{\mathscr{F}}([0, T] \times E; X)$ the set of all $\mathscr{P} \times \mathscr{B}(E)$-measurable X-valued processes $r = \{r(t, \omega, e),\ (t, \omega, e) \in [0, T] \times \Omega \times E\}$ such that $\|r\|_{M^{\nu,2}_{\mathscr{F}}([0,T]\times E;X)} \triangleq \sqrt{\mathbb{E}\Big[\int_0^T \int_E \|r(t, e)\|_X^2 \nu(de)dt\Big]} < \infty$, by $M^2_{\mathscr{F}}(0, T; X)$ the set of all $\mathscr{F}_t$-adapted X-valued processes $f = \{f(t, \omega),\ (t, \omega) \in [0, T] \times \Omega\}$ such that $\|f\|_{M^2_{\mathscr{F}}(0,T;X)} \triangleq \sqrt{\mathbb{E}\Big[\int_0^T \|f(t)\|_X^2 dt\Big]} < \infty$, by $S^2_{\mathscr{F}}(0, T; X)$ the set of all $\mathscr{F}_t$-adapted X-valued càdlàg processes $f = \{f(t, \omega),\ (t, \omega) \in [0, T] \times \Omega\}$ such that $\|f\|_{S^2_{\mathscr{F}}(0,T;X)} \triangleq \sqrt{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} \|f(t)\|_X^2\Big]} < +\infty$, by $L^2(\Omega, \mathscr{F}, \mathbb{P}; X)$ the set of all X-valued random variables ξ on $(\Omega, \mathscr{F}, \mathbb{P})$ such that $\|\xi\|_{L^2(\Omega,\mathscr{F},\mathbb{P}; X)} \triangleq \sqrt{\mathbb{E}[\|\xi\|_X^2]} < \infty$. Throughout this paper, we let C and K be two generic positive constants, which may be different from line to line. To prove this Theorem 1, we now begi show the following result on the continuous dependence of the solution to the SEE (1).

Theorem 1. Let $X(\cdot)$ be a solution to the SEE (1) with the initial value $X(0) = x$ and the coefficients (A, B, b, g, σ) which satisfy Assumptions 1 and 2. Then the following estimate holds:

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} \|X(t)\|_H^2\Big] + \mathbb{E}\Big[\int_0^T \|X(t)\|_V^2 dt\Big] \\ &\leqslant K\Big\{||x||_H^2 + \mathbb{E}\Big[\int_0^T \|b(t, 0)\|_H^2 dt\Big] + \mathbb{E}\Big[\int_0^T \|g(t, 0)\|_H^2 dt\Big] + \mathbb{E}\Big[\int_0^T \int_E \|\sigma(t, e, 0)\|_H^2 \nu(de)dt\Big]\Big\}. \end{aligned} \tag{A1}$$

Furthermore, suppose that $\bar{X}(\cdot)$ is a solution to the SEE (1) with the initial value $\bar{X}(0) = \bar{x} \in H$ and the coefficients $(A, B, \bar{b}, \bar{g}, \bar{\sigma})$ satisfying Assumptions 1 and 2, then we have

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} \|X(t) - \bar{X}(t)\|_H^2\Big] + \mathbb{E}\Big[\int_0^T \|X(t) - \bar{X}(t)\|_V^2 dt\Big]$$

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$$\leqslant K\bigg\{\|x-\bar{x}\|_H^2+\mathbb{E}\bigg[\int_0^T\|b(t,\bar{X}(t))-\bar{b}(t,\bar{X}(t))\|_H^2dt\bigg]$$
$$+\mathbb{E}\bigg[\int_0^T\|g(t,\bar{X}(t))-\bar{g}(t,\bar{X}(t))\|_H^2dt\bigg]+\mathbb{E}\bigg[\int_0^T\int_E\|\sigma(t,e,\bar{X}(t))-\bar{\sigma}(t,e,\bar{X}(t))\|_H^2\nu(de)dt\bigg]\bigg\}. \tag{A2}$$

Proof. The estimate (A1) can be directly obtained by the estimate (A2) by taking the initial value $\bar{X}(0)=0$ and the coefficients $(A,B,\bar{b},\bar{g},\bar{\sigma})=(A,B,0,0,0)$ which imply that $\bar{X}(\cdot)\equiv 0$. Therefore, it suffices to prove the estimate (A2). For the sake of simplicity, in the following discussion, we will use the following shorthand notation:

$$\begin{aligned}&\hat{X}(t)\triangleq X(t)-\bar{X}(t),\quad \hat{x}\triangleq x-\bar{x},\\&\Lambda\triangleq\|x-\bar{x}\|_H^2+\mathbb{E}\bigg[\int_0^T\|b(t,\bar{X}(t))-\bar{b}(t,\bar{X}(t))\|_H^2dt\bigg]+\mathbb{E}\bigg[\int_0^T\|g(t,\bar{X}(t))-\bar{g}(t,\bar{X}(t))\|_H^2dt\bigg]\\&\qquad+\mathbb{E}\bigg[\int_0^T\int_E\|\sigma(t,\bar{X}(t),e)-\bar{\sigma}(t,\bar{X}(t),e)\|_H^2\nu(de)dt\bigg],\end{aligned}\tag{A3}$$

and for $\phi=b,g,\sigma$

$$\tilde{\phi}(t)\triangleq\phi(t,X(t))-\bar{\phi}(t,\bar{X}(t)),\hat{\phi}(t)\triangleq\phi(t,\bar{X}(t))-\bar{\phi}(t,\bar{X}(t)),\Delta\phi(t)\triangleq\phi(t,X(t))-\phi(t,\bar{X}(t)),\quad t\in[0,T],\tag{A4}$$

where when $\phi=\sigma$, the terms $X(t)$ and $\bar{X}(t)$ will be replaced by $X(t-)$ and $\bar{X}(t-)$, respectively.

Applying Itô formula in Lemma **??** to $||\hat{X}(t)||_H^2$ and using Assumptions 1 and 2 and the elementary inequalities $|a+b|^2\leqslant 2a^2+2b^2$ and $2ab\leqslant a^2+b^2$, $\forall a,b>0$, we get that

$$\begin{aligned}||\hat{X}(t)||_H^2=\;&||\hat{x}||_H^2+2\int_0^t\langle A(s)\hat{X}(s),\hat{X}(s)\rangle ds+2\int_0^t(\hat{X}(s),\tilde{b}(s))_Hds+\int_0^t||B(s)\hat{X}(s)+\tilde{g}(s)||_H^2ds\\&+\int_0^t\int_E||\tilde{\sigma}(s,e)||_H^2\nu(de)ds+2\int_0^t(\hat{X}(s),B(s)\hat{X}(s)+\tilde{g}(s))_HdW(s)\\&+\int_0^t\int_E\Big[||\tilde{\sigma}(s,e)||_H^2+2(\hat{X}(s),\tilde{\sigma}(s,e))\Big]\tilde{\mu}(de,ds)\\=\;&||\hat{x}||_H^2+2\int_0^t\Big[\langle A(s)\hat{X}(s),\hat{X}(s)\rangle+||B(s)\hat{X}(s)||_H^2\Big]ds+\int_0^t||\Delta g(s)+\hat{g}(s)||_H^2\\&+2\int_0^t(B(s)\hat{X}(s),\Delta g(s)+\hat{g}(s))_Hds+2\int_0^t(\hat{X}(s),\Delta b(s)+\hat{b}(s))_Hds\\&+\int_0^t\int_E||\Delta\sigma(s,e)+\hat{\sigma}(s,e)||_H^2\nu(de)ds+2\int_0^t(\hat{X}(s),B(s)\hat{X}(s)+\tilde{g}(s))_HdW(s)\\&+\int_0^t\int_E\Big[||\tilde{\sigma}(s,e)||_H^2+2(\hat{X}(s),\tilde{\sigma}(s,e))\Big]\tilde{\mu}(de,ds)\\\leqslant\;&K\Lambda-2\alpha\mathbb{E}\bigg[\int_0^t\|\hat{x}(s)\|_V^2ds\bigg]+K\mathbb{E}\bigg[\int_0^t\|\hat{X}(s)\|_H^2dt\bigg]\\&+2\int_0^t(\hat{X}(s),B(s)\hat{X}(s)+\tilde{g}(s))_HdW(s)+\int_0^t\int_E\Big[||\tilde{\sigma}(s,e)||_H^2+2(\hat{X}(s),\tilde{\sigma}(s,e))\Big]\tilde{\mu}(de,ds).\end{aligned}\tag{A5}$$

Taking expectation on both sides of the above inequality, we get that

$$\mathbb{E}[||\hat{X}(t)||_H^2]+2\alpha\mathbb{E}\bigg[\int_0^t||\hat{X}(s)||_V^2ds\bigg]\leqslant K\Lambda+K\mathbb{E}\bigg[\int_0^t\|\hat{X}(s)\|_H^2dt\bigg].\tag{A6}$$

Then by virtue of Grönwall's inequality to $\mathbb{E}[||X(t)||_H^2]$, we obtain

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}[||\hat{X}(t)||_H^2]+\mathbb{E}\bigg[\int_0^T||\hat{X}(s)||_V^2ds\bigg]\leqslant K\Lambda.\tag{A7}$$

Furthermore, applying Burkholder-Davis-Gundy inequality to (A5) and using the elementary inequality $2ab\leqslant\frac{1}{\varepsilon}a^2+\varepsilon b^2$, $\forall a,b>0$, $\varepsilon>0$ and the estimate (A6), we get that

$$\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T}\|\hat{X}(t)\|_H^2\bigg]\leqslant\frac{1}{2}\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T}\|\hat{X}(t)\|_H^2\bigg]+K\Lambda,\tag{A8}$$

which implies that

$$\mathbb{E}\bigg[\sup_{0\leqslant t\leqslant T}\|\hat{X}(t)\|_H^2\bigg]\leqslant K\Lambda.\tag{A9}$$

Combining (A7) and (A9), we get the desired result. The proof is complete.

In the following, we give the existence and uniqueness result for the solution of the SEE (1) for a simple case where the coefficients (b,g,σ) is independent of the variable x.

Lemma 1. Given three stochastic processes b, g and σ such that $b \in M_{\mathscr{F}}^2(0,T;H), g(\cdot) \in M_{\mathscr{F}}^2(0,T;H)$ and $\sigma(\cdot) \in M_{\mathscr{F}}^{\nu,2}([0,T]\times E;H)$. Suppose that the operators A and B satisfy Assumption 1. Then there exists a unique solution to the following SEE:

$$\begin{cases} dX(t) = [A(t)X(t) + b(t)]dt + [B(t)X(t) + g(t)]dW(t) + \int_E \sigma(t,e)\tilde{\mu}(de,dt), \\ X(0) = x, \quad t \in [0,T]. \end{cases} \tag{A10}$$

Proof. The proof can be obtained by Galerkin approximations in the same way as the proof of Theorem 3.2 in [1] with minor change.

Proof of Theorem 1. The uniqueness of the solution to the SEE (1) can be got by the a priori estimate (A2) directly. For $\rho \in [0,1]$ and any three given stochastic processes $b_0 \in M_{\mathscr{F}}^2(0,T;H)$, $g_0 \in M_{\mathscr{F}}^2(0,T;H)$, and $\sigma_0 \in M_{\mathscr{F}^{\nu,2}([0,T]\times E;H)}$, we introduce a family of parameterized SEEs as follows:

$$\begin{aligned} X(t) = x &+ \int_0^t A(s)X(s)ds + \int_0^t \Big[\rho b(s,X(s))] + b_0(s)\Big]ds + \int_0^t \Big[B(s)X(s) + \rho g(s,X(s)) + g_0(s)\Big]dW(s) \\ &+ \int_0^t \int_E \Big[\rho\sigma(s,e,X(s)) + \sigma_0(s,e)\Big]\tilde{\mu}(de,ds). \end{aligned} \tag{A11}$$

It is easy to see that when we take the parameter $\rho = 1$ and $b_0 \equiv 0, g_0 \equiv 0, \sigma_0 \equiv 0$, the SEE (A11) is reduced to the original SEE (1). Obviously, the coefficients of the SEE (A11) satisfy Assumptions 1 and 2 with (A,B,b,g,σ) replaced by $(A, B, \rho b + b_0, \rho g + g_0, \rho\sigma + \sigma_0)$. Suppose for any $b_0 \in M_{\mathscr{F}}^2(0,T;H)$, $g_0 \in M_{\mathscr{F}}^2(0,T;H)$, $\sigma_0 \in M_{\mathscr{F}^{\nu,2}([0,T]\times E;H)}$, and some parameter $\rho = \rho_0$, there exists a unique solution $X(\cdot) \in M_{\mathscr{F}}^2(0,T;V)$ to the SEE (A11). For any parameter ρ, the SEE (A11) can be rewritten as

$$\begin{aligned} X(t) = x + \int_0^t A(s)X(s)ds &+ \int_0^t \Big[\rho_0 b(s,X(s)) + b_0(s) + (\rho-\rho_0)b(s,X(s))\Big]ds \\ &+ \int_0^t \Big[B(s)X(s) + \rho_0 g(s,X(s)) + g_0(s) + (\rho-\rho_0)g(s,X(s))\Big]dW(s) \\ &+ \int_0^t \int_E \Big[\rho_0\sigma(s,e,X(s)) + \sigma_0(t,e) + (\rho-\rho_0)\sigma(s,e,X(s))\Big]d\tilde{\mu}(de,ds). \end{aligned} \tag{A12}$$

Therefore, by the above assumption, for any $x(\cdot) \in M_{\mathscr{F}}^2(0,T;V)$, the following SEE

$$\begin{aligned} X(t) = x + \int_0^t A(s)X(s)ds &+ \int_0^t \Big[\rho_0 b(s,X(s)) + b_0(s) + (\rho-\rho_0)b(s,x(s))\Big]ds \\ &+ \int_0^t \Big[B(s)X(s) + \rho_0 g(s,X(s)) + g_0(s) + (\rho-\rho_0)g(s,x(s))\Big]dW(s) \\ &+ \int_0^t \int_E \Big[\rho_0\sigma(s,e,X(s)) + \sigma_0(s,e) + (\rho-\rho_0)\sigma(s,e,x(s))\Big]d\tilde{\mu}(de,ds) \end{aligned} \tag{A13}$$

admits a unique solution $X(\cdot) \in M_{\mathscr{F}}^2(0,T;V)$. Now define a mapping from $M_{\mathscr{F}}^2(0,T;V)$ onto itself denoted by

$$X(\cdot) = \Gamma(x(\cdot)).$$

Then for any $x_i(\cdot) \in M_{\mathscr{F}}^2(0,T;V)$, $i = 1,2$, from the Lipschitz continuity of b, g, σ and a priori estimate (A2), it follows that

$$\begin{aligned} ||\Gamma(x_1(\cdot)) - \Gamma(x_2(\cdot))||^2_{M_{\mathscr{F}}^2(0,T;V)} &= ||X_1(\cdot) - X_2(\cdot)||^2_{M_{\mathscr{F}}^2(0,T;V)} \\ &\leqslant K|\rho-\rho_0|^2 \cdot ||x_1(\cdot) - x_2(\cdot)||^2_{M_{\mathscr{F}}^2(0,T;V)}. \end{aligned}$$

Here K is a positive constant independent of ρ. If $|\rho - \rho_0| < \frac{1}{2\sqrt{K}}$, the mapping Γ is strictly contractive in $M_{\mathscr{F}}^2(0,T;V)$. Hence it implies that the SEE (A11) with the coefficients $(A, B, \rho b + b_0, \rho g + g_0, \rho\sigma + \sigma_0)$ admits a unique solution $X(\cdot) \in M_{\mathcal{F}}^2(0,T;V)$. From Lemma 1, the uniqueness and existence of the solution to the SEE (A11) is true for $\rho = 0$. Then starting from $\rho = 0$, one can reach $\rho = 1$ in finite steps and this finishes the proof of solvability of the SEE (**??**). Moreover, from Lemma **??** and the estimate (**??**), we obtain $X(\cdot) \in \mathcal{S}_{\mathscr{F}}^2(0,T;H)$. This completes the proof.

Appendix B Proof of Theorems 3 and 4

This section is devoted to giving the proof of our main results. We first gives the well-posedness of the state equation as well as some useful estimates.

Lemma 2. Let Assumption 3 be satisfied. Then for any admissible control $u(\cdot)$, the state equation (4) has a unique solution $X^u(\cdot) \in M_{\mathscr{F}}^2(0,T;V) \bigcap S_{\mathscr{F}}^2(0,T;H)$. Moreover, the following estimate holds

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T} \|X^u(t)\|_H^2\Big] + \mathbb{E}\Big[\int_0^T \|X^u(t)\|_V^2 dt\Big] \leqslant K\Big\{1 + ||x||_H^2 + \mathbb{E}\Big[\int_0^T \|u(t)\|_U^2 dt\Big]\Big\} \tag{B1}$$

and

$$|J(u(\cdot))| < \infty. \tag{B2}$$

Furthermore, let $X^v(\cdot)$ be the state process corresponding to another admissible control $v(\cdot)$, then

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\|X^u(t)-X^v(t)\|_H^2\Big]+\mathbb{E}\Big[\int_0^T\|X^u(t)-X^v(t)\|_V^2dt\Big]\leqslant K\mathbb{E}\Big[\int_0^T\|u(t)-v(t)\|_U^2dt\Big]. \tag{B3}$$

Proof. Under Assumption 3, by Theorem 1, we can get directly the existence and uniqueness of the solution of the state equation (4). And the estimates (B1) and (B3) can be obtained by the estimates (A1) and (A2), respectively. Furthermore, from Assumption 3 and the estimate (B1), it follows that

$$|J(u(\cdot))|\leqslant K\bigg\{\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}||X(t)||_H^2\Big]+\mathbb{E}\Big[\int_0^T||u(t)||_U^2dt\Big]+1\bigg\}\leqslant K\bigg\{1+||x||_H^2+\mathbb{E}\Big[\int_0^T||u(t)||_U^2dt\Big]\bigg\}<\infty. \tag{B4}$$

The proof is complete.

Therefore, by Lemma 2, we claim that the cost functional (5) is well-defined. The admissible control $\bar{u}(\cdot)$ satisfying (6) is called an optimal control process of Problem 2. Correspondingly, the state process $\bar{X}(\cdot)$ associated with $\bar{u}(\cdot)$ is called an optimal state process. Then $(\bar{u}(\cdot);\bar{X}(\cdot))$ is called an optimal pair of Problem 2.

For any admissible pair $(\bar{u}(\cdot);\bar{X}(\cdot))$, the corresponding adjoint processes is defined as a triple $(\bar{p}(\cdot),\bar{q}(\cdot),\bar{r}(\cdot,\cdot))$ of stochastic processes, which is a solution to the following backward stochastic evolution equation (BSEE for short) with jump, called the adjoint equation,

$$\begin{cases} d\bar{p}(t)= -\Big[A^*(t)\bar{p}(t)+b_x^*(t,\bar{X}(t),\bar{u}(t))\bar{p}(t)+B^*(t)\bar{q}(t)+g_x^*(t,\bar{X}(t),\bar{u}(t))\bar{q}(t)\\ \qquad\qquad +\int_E\sigma_x^*(t,e,\bar{X}(t),\bar{u}(t))\bar{r}(t,e)\nu(de)dt+l_x(t,\bar{X}(t),\bar{u}(t))\Big]dt\\ \qquad\qquad +\bar{q}(t)dW(t)+\int_E\bar{r}(t,e)\tilde{\mu}(de,dt), \qquad 0\leqslant t\leqslant T,\\ \bar{p}(T)= \Phi_x(\bar{X}(T)). \end{cases} \tag{B5}$$

Here A^* denotes the adjoint operator of the operator A. Similarly, we can define the corresponding adjoint operator for other coefficients.

Under Assumption 3, we have the following basic result for the adjoint process.

Lemma 3. Let Assumptions 3 be satisfied. Then for any admissible pair $(\bar{u}(\cdot);\bar{X}(\cdot))$, there exists a unique adjoint process $(\bar{p}(\cdot),\bar{q}(\cdot),\bar{r}(\cdot,\cdot))\in M_{\mathscr{F}}^2(0,T;H)\bigcap S_{\mathscr{F}}^2(0,T;V)\times M_{\mathscr{F}}^2(0,T;H)\times M_{\mathscr{F}}^{\nu,2}([0,T]\times E;H)$. Moreover, the following estimate holds:

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\|\bar{p}(t)\|_H^2\Big]+\mathbb{E}\Big[\int_0^T\|\bar{p}(t)\|_V^2dt\Big]+\mathbb{E}\Big[\int_0^T\|\bar{q}(t)\|_H^2dt\Big]+\mathbb{E}\Big[\int_0^T\int_E\|r(t,e)\|_H^2\nu(de)dt\Big]\\ &\leqslant K\bigg\{\mathbb{E}\Big[\int_0^T\|l_x(t,\bar{X}(t),\bar{u}(t))\|_H^2dt\Big]+\mathbb{E}[||\Phi_x(\bar{X}(T))||_H^2]\bigg\}.\end{aligned} \tag{B6}$$

Proof. From the property of adjoint operator, the adjoint operator A^* of A and the adjoint operator B^* of B also satisfies (i) in Assumption 1. Therefore, the desired result can be obtained by the existence and uniqueness theorem of solution of BSEE with jumps established in [2].

Define the Hamiltonian $\mathcal{H}:[0,T]\times\Omega\times H\times\mathscr{U}\times H\times H\times M^{\nu,2}(E;H)\to\mathbb{R}$ by

$$\mathcal{H}(t,x,u,p,q,r(\cdot)):=(b(t,x,u),p)_H+(g(t,x,u),q)_H+\int_E(\sigma(t,e,x,u),r(t,e))_H\,\nu(de)+l(t,x,u). \tag{B7}$$

Using Hamiltonian $\mathcal{H}$, the adjoint equation (B5) can be written in the following form:

$$\begin{cases} d\bar{p}(t)= -\Big[A^*(t)\bar{p}(t)+B(t)^*\bar{q}(t)+\bar{\mathcal{H}}_x(t)\Big]dt+\bar{q}(t)dW(t)+\int_E\bar{r}(t,e)\tilde{\mu}(de,dt), \qquad 0\leqslant t\leqslant T,\\ \bar{p}(T)= \Phi_x(\bar{X}(T)), \end{cases} \tag{B8}$$

where we denote

$$\bar{\mathcal{H}}(t)\triangleq\mathcal{H}(t,\bar{x}(t),\bar{u}(t),\bar{p}(t),\bar{q}(t),\bar{r}(t,\cdot)). \tag{B9}$$

Let$(\bar{u}(\cdot);\bar{X}(\cdot))$ be an optimal pair of Problem 2. Define a convex perturbation of $\bar{u}(\cdot)$ as follows:

$$u^\varepsilon(\cdot)\triangleq\bar{u}(\cdot)+\varepsilon(v(\cdot)-\bar{u}(\cdot)), \qquad 0\leqslant\varepsilon\leqslant1,$$

where $v(\cdot)$ is an arbitrarily admissible control. Since the control domain $\mathscr{U}$ is convex, $u^\varepsilon(\cdot)$ is also an element of $\mathcal{A}$. We denote by $X^\varepsilon(\cdot)$ the state process corresponding to the control $u^\varepsilon(\cdot)$. Now we introduce the following first order variation equation:

$$\begin{cases} dY(t)=[A(t)Y(t)+b_x(t,\bar{X}(t),\bar{u}(t))Y(t)+b_u(t,\bar{X}(t),\bar{u}(t))(v(t)-\bar{u}(t))]dt\\ \qquad\quad +[B(t)Y(t)+g_x(t,\bar{X}(t),\bar{u}(t))Y(t)+g_u(t,\bar{X}(t),\bar{u}(t))(v(t)-\bar{u}(t))]dW(t)\\ \qquad\quad +\int_E\Big[\sigma_x(t,e,\bar{X}(t-),\bar{u}(t))Y(t-)+\sigma_u(t,e,\bar{X}(t-),\bar{u}(t))(v(t)-\bar{u}(t))\Big]\tilde{\mu}(de,dt),\\ Y(0)=0. \end{cases} \tag{B10}$$

Under Assumption 3, by Theorem 1, we see that the variation equation (B10) has a unique solution $Y(\cdot)\in M_{\mathscr{F}}^2(0,T;V)\bigcap S_{\mathscr{F}}^2(0,T;H)$.

Lemma 4. Let Assumption 3 be satisfied. Then we have the following estimates:

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\|X^\varepsilon(t)-\bar X(t)\|_H^2\Big]+\mathbb{E}\Big[\int_0^T\|X^\varepsilon(t)-\bar X(t)\|_V^2dt\Big]=O(\varepsilon^2), \tag{B11}$$

$$\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\|X^\varepsilon(t)-\bar X(t)-\varepsilon Y(t)\|_H^2\Big]+\mathbb{E}\Big[\int_0^T\|X^\varepsilon(t)-\bar X(t)-\varepsilon Y(t)\|_V^2dt\Big]=o(\varepsilon^2)\ . \tag{B12}$$

Proof. From the estimate (B3), we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\|X^\varepsilon(t)-\bar X(t)\|_H^2\Big]+\mathbb{E}\Big[\int_0^T\|X^\varepsilon(t)-\bar X(t)\|_V^2dt\Big]&\leqslant K\mathbb{E}\Big[\int_0^T\|u^\varepsilon(t)-\bar u(t)\|_U^2dt\Big]\\&=K\varepsilon^2\mathbb{E}\Big[\int_0^T\|v(t)-\bar u(t)\|_U^2dt\Big].\\&=O(\varepsilon^2).\end{aligned} \tag{B13}$$

Denote

$$\Xi^\varepsilon(t):=X^\varepsilon(t)-\bar X(t)-\varepsilon Y(t). \tag{B14}$$

From Taylor expanding, we have

$$\begin{cases}d\Xi^\varepsilon(t)=[A(t)\Xi^\varepsilon(t)+b_x(t,\bar X(t),\bar u(t))\Xi^\varepsilon(t)+\alpha^\varepsilon(t)]dt+[B(t)\Xi^\varepsilon(t)+g_x(t,\bar X(t),\bar u(t))\Xi^\varepsilon(t)+\beta^\varepsilon(t)]dW(t)\\ \qquad\qquad+\displaystyle\int_E\Big[\sigma_x(t,e,\bar X(t-),\bar u(t))\Xi^\varepsilon(t)+\gamma^\varepsilon(t,e)\Big]d\tilde\mu(de,dt),\\ X(0)=x,\quad t\in[0,T],\end{cases} \tag{B15}$$

where

$$\begin{cases}\alpha^\varepsilon(t)=\displaystyle\int_0^1\Big[\big(b_x(t,\bar X(t)+\lambda(X^\varepsilon(t)-\bar X(t)),\bar u(t)+\lambda(u^\varepsilon(t)-\bar u(t)))-b_x(t,\bar X(t),\bar u(t))\big)(X^\varepsilon(t)-\bar X(t))\\ \qquad+\big(b_u(t,\bar X(t)+\lambda(X^\varepsilon(t)-\bar X(t)),\bar u(t)+\lambda(u^\varepsilon(t)-\bar u(t)))-b_u(t,\bar X(t),\bar u(t))\big)(u^\varepsilon(t)-\bar u(t))\Big]d\lambda,\\ \beta^\varepsilon(t)=\displaystyle\int_0^1\Big[\big(g_x(t,\bar X(t)+\lambda(X^\varepsilon(t)-\bar X(t)),\bar u(t)+\lambda(u^\varepsilon(t)-\bar u(t)))-g_x(t,\bar X(t),\bar u(t))\big)(X^\varepsilon(t)-\bar X(t))\\ \qquad+\big(g_u(t,\bar X(t)+\lambda(X^\varepsilon(t)-\bar X(t)),\bar u(t)+\lambda(u^\varepsilon(t)-\bar u(t)))-g_u(t,\bar X(t),\bar u(t))\big)(u^\varepsilon(t)-\bar u(t))\Big]d\lambda,\\ \gamma^\varepsilon(t,e)=\displaystyle\int_0^1\Big[\big(\sigma_x(t,e,\bar X(t-)+\lambda(X^\varepsilon(t-)-\bar X(t-)),\bar u(t)+\lambda(u^\varepsilon(t)-\bar u(t))-\sigma_x(t,e,\bar X(t-),\bar u(t))\big)(X^\varepsilon(t-)-\bar X(t-))\\ \qquad+\big(\sigma_u(t,e,\bar X(t-)+\lambda(X^\varepsilon(t-)-\bar X(t-)),\bar u(t)+\lambda(u^\varepsilon(t)-\bar u(t)))-\sigma_x(t,e,\bar X(t-),\bar u(t))\big)(u^\varepsilon(t)-\bar u(t))\Big]d\lambda.\end{cases} \tag{B16}$$

From the estimates (A1), (B11) and Lebesgue dominated convergence theorem, we get that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\|\Xi(t)\|_H^2\Big]+\mathbb{E}\Big[\int_0^T\|\Xi(t)\|_V^2dt\Big]\\ \leqslant\ & K\Big\{\mathbb{E}\Big[\int_0^T||\alpha^\varepsilon(t)||_H^2dt\Big]+\mathbb{E}\Big[\int_0^T||\beta^\varepsilon(t)||_H^2dt\Big]+\mathbb{E}\Big[\int_0^T\int_E||\gamma^\varepsilon(t,e)||_H^2\nu(de)dt\Big]\Big\}\\ =\ & o(\varepsilon).\end{aligned} \tag{B17}$$

Lemma 5. Let Assumption 3 be satisfied. Let $(\bar u(\cdot);\bar X(\cdot))$ be an optimal pair of Problem 2 associated with the first order variation process $Y(\cdot)$ (see (B10)). Then,

$$\begin{aligned}J(u^\varepsilon(\cdot))-J(\bar u(\cdot))=&\varepsilon\mathbb{E}\Big[(\Phi_x(\bar X(T)),Y(T))_H\Big]+\varepsilon\mathbb{E}\Big[\int_0^T(l_x(t,\bar X(t),\bar u(t)),Y(t))_Hdt\Big]\\&+\varepsilon\mathbb{E}\Big[\int_0^T(l_u(t,\bar X(t),\bar u(t)),v(t)-u(t))_Udt\Big]+o(\varepsilon).\end{aligned} \tag{B18}$$

Proof.

From the definition of the cost functional , we have

$$J(u^\varepsilon(\cdot))-J(\bar u(\cdot))=\mathbb{E}\Big[\int_0^T\Big(l(t,X^\varepsilon(t),u^\varepsilon(t))-l(t,\bar X(t),\bar u(t)\Big)dt\Big]+\mathbb{E}\Big[\Phi(X^\varepsilon(T))-\Phi(\bar X(T))\Big]=:I_1+I_2, \tag{B19}$$

where

$$\begin{aligned}I_1=&\mathbb{E}\Big[\int_0^T\Big(l(t,X^\varepsilon(t),u^\varepsilon(t))-l(t,\bar X(t),\bar u(t)\Big)dt\Big],\\ I_2=&\mathbb{E}\Big[\Phi(X^\varepsilon(T))-\Phi(\bar X(T))\Big].\end{aligned} \tag{B20}$$

Let us concentrate on I_1. In terms of Taylor expanding, Lemma 4 and the control convergence theorem, we have

$$
\begin{aligned}
I_1 =&\mathbb{E}\Big[\int_0^T\int_0^1\big(l_x(t,\bar{X}(t)+\lambda(X^\varepsilon(t)-\bar{X}(t)),\bar{u}(t)+\lambda(u^\varepsilon(t)-\bar{u}(t)))-l_x(t,\bar{X}(t),\bar{u}(t))\big)(X^\varepsilon(t)-\bar{X}(t))d\lambda dt\Big]\\
&+E\Big[\int_0^T\int_0^1\big(l_u(t,\bar{X}(t)+\lambda(X^\varepsilon(t)-\bar{X}(t)),\bar{u}(t)+\lambda(u^\varepsilon(t)-\bar{u}(t)))-l_u(t,\bar{X}(t),\bar{u}(t))\big)(u^\varepsilon(t)-\bar{u}(t))d\lambda dt\Big]\\
&+\mathbb{E}\Big[\int_0^T l_x(t,\bar{X}(t),\bar{u}(t))\Xi^\varepsilon(t)dt\Big]+\varepsilon\mathbb{E}\Big[\int_0^T l_x(t,\bar{X}(t),\bar{u}(t))Y(t)dt\Big]\\
&+\varepsilon E\Big[\int_0^T l_u(t,\bar{X}(t),\bar{u}(t))(u(t)-\bar{u}(t))dt\Big]\\
=&\varepsilon\mathbb{E}\Big[\int_0^T l_x(t,\bar{X}(t),\bar{u}(t))Y(t)dt\Big]+\varepsilon E\Big[\int_0^T l_u(t,\bar{X}(t),\bar{u}(t))(u(t)-\bar{u}(t))dt\Big]\\
&+o(\varepsilon).
\end{aligned}\tag{B21}
$$

Similarly, we have

$$
I_2=\varepsilon\mathbb{E}\Big[\Phi_x(\bar{X}(T))Y(T)\Big]+o(\varepsilon).\tag{B22}
$$

Then putting (B21) and (B22) into (B19), we get (B18). The proof is complete.

Proof of Theorem 3. Recalling the adjoint equation (B8) and the first order variational equation (B10), and then applying Itô formula to $(\bar{p}(t),Y(t))_H$, we have

$$
\begin{aligned}
&\mathbb{E}[(\Phi_x(\bar{X}(T)),Y(T))_H]+\mathbb{E}\Big[\int_0^T(l_x(t,\bar{X}(t),\bar{u}(t)),Y(t))_H dt\Big]\\
=&\mathbb{E}\Big[\int_0^T\Big(v(t)-\bar{u}(t),b_u^*(t,\bar{X}(t),\bar{u}(t))\bar{p}(t)+g_u^*(t,\bar{X}(t),\bar{u}(t))\bar{q}(t)+\int_E\sigma_u^*(t,e,\bar{X}(t),\bar{u}(t))\bar{r}(t,e)\nu(de)\Big)_U dt\Big].
\end{aligned}\tag{B23}
$$

Since $\bar{u}(\cdot)$ is the optimal control, from (B18), the duality relation (B23) and the definition of the Hamiltonian $\mathcal{H}$ (see (B7)) , we have

$$
\begin{aligned}
0\leqslant&\lim_{\varepsilon\longrightarrow 0+}\frac{J(u^\varepsilon(\cdot))-J(\bar{u}(\cdot))}{\varepsilon}\\
=&\mathbb{E}[\big(\Phi_x(\bar{X}(T)),Y(T)\big)_H]+\mathbb{E}\Big[\int_0^T(l_x(t,\bar{X}(t),\bar{u}(t)),Y(t))_H dt\Big]+\mathbb{E}\Big[\int_0^T(l_u(t,\bar{X}(t),\bar{u}(t)),v(t)-u(t))_U dt\Big]\\
=&\mathbb{E}\Big[\int_0^T\Big(v(t)-\bar{u}(t),b_u^*(t,\bar{X}(t),\bar{u}(t))\bar{p}(t)+g_u^*(t,\bar{X}(t),\bar{u}(t))\bar{q}(t)+\int_E\sigma_u^*(t,e,\bar{X}(t),\bar{u}(t))\bar{r}(t,e)\nu(de)\Big)_U dt\Big]\\
&+\mathbb{E}\Big[\int_0^T(l_u(t,\bar{X}(t),\bar{u}(t)),v(t)-\bar{u}(t))_U dt\Big]\\
=&E\Big[\int_0^T\Big(v(t)-\bar{u}(t),\mathcal{H}_u(t,\bar{X}(t),\bar{u}(t),\bar{p}(t),\bar{q}(t),\bar{r}(t,\cdot))\Big)_U dt\Big].
\end{aligned}\tag{B24}
$$

This implies the minimum condition

$$
\big(\mathcal{H}_u(t,\bar{X}(t-),\bar{u}(t),\bar{p}(t-),\bar{q}(t),\bar{r}(t,\cdot)),v-\bar{u}(t)\big)_U\geqslant 0.
$$

holds since $v(\cdot)$ is any given admissible control. Proof of Theorem 1 is complete.

Proof of Theorem 4. Let $(u(\cdot);X(\cdot))$ be an any given admissible pair. To simplify our notations, we define

$$
\begin{aligned}
&b(t)\triangleq b(t,X(t),u(t)),\bar{b}(t)\triangleq b(t,\bar{X}(t),\bar{u}(t)),\\
&g(t)\triangleq g(t,X(t),u(t)),\bar{g}(t)\triangleq g(t,\bar{X}(t),\bar{u}(t)),\\
&\sigma(t,e)\triangleq\sigma(t,e,X(t-),u(t)),\bar{\sigma}(t)\triangleq\sigma(t,e,\bar{X}(t-),\bar{u}(t)),\\
&\mathcal{H}(t)\triangleq\mathcal{H}(t,X(t),u(t),\bar{p}(t),\bar{q}(t),\bar{r}(t,\cdot)),\\
&\bar{\mathcal{H}}(t)\triangleq\mathcal{H}(t,\bar{X}(t),\bar{u}(t),\bar{p}(t),\bar{q}(t),\bar{r}(t,\cdot)).
\end{aligned}\tag{B25}
$$

From the definitions of the cost functional $J(u(\cdot))$ and the Hamiltonian $\mathcal{H}$, we can represent $J(u(\cdot))-J(\bar{u}(\cdot))$ as follows:

$$
\begin{aligned}
J(u(\cdot))-J(\bar{u}(\cdot))\ =\ &\mathbb{E}\Big[\int_0^T\Big(\mathcal{H}(t)-\bar{\mathcal{H}}(t)-(\bar{p}(t),b(t)-\bar{b}(t))_H-(\bar{q}(t),g(t)-\bar{g}(t))_H\\
&-\int_E(\bar{r}(t,e),\sigma(t,e)-\bar{\sigma}(t,e))_H\nu(de)\Big)dt\Big]+\mathbb{E}\Big[\Phi(X(T))-\Phi(\bar{X}(T))\Big].
\end{aligned}\tag{B26}
$$

Then recalling the adjoint equation (B5) and applying Itô's formula to $(\bar{p}(t),X(t)-\bar{X}(t))_H$, we get that

$$
\mathbb{E}\Big[\int_0^T\Big((\bar{p}(t),b(t)-\bar{b}(t))_H+(\bar{q}(t),g(t)-\bar{g}(t))_H+\int_E(\bar{r}(t,e),\sigma(t,e)-\bar{\sigma}(t,e))_H\nu(de)\Big)dt\Big]
$$

$$= \mathbb{E}\Big[\int_0^T (\bar{\mathcal{H}}_x(t), X(t) - \bar{X}(t))_H dt\Big] + \mathbb{E}\Big[(\Phi_x(\bar{X}(T)), X(T) - \bar{X}(T))_H\Big]. \tag{B27}$$

Then substituting (B27) into (B26) leads to

$$\begin{aligned} J(u(\cdot)) - J(\bar{u}(\cdot)) \;=\; & \mathbb{E}\Big[\int_0^T \Big(\mathcal{H}(t) - \bar{\mathcal{H}}(t) - (\bar{\mathcal{H}}_x(t), X(t) - \bar{X}(t))_H\Big)dt\Big] \\ & + \mathbb{E}[\Phi(X(T)) - \Phi(\bar{X}(T)) - (\Phi_x(\bar{X}(T)), X(T) - \bar{x}(T))_H]. \end{aligned} \tag{B28}$$

On the other hand, the convexity of $\mathcal{H}(t)$ and $\Phi(x)$ yields

$$\mathcal{H}(t) - \bar{\mathcal{H}}(t) \;\geqslant\; (\bar{\mathcal{H}}_x(t), X(t) - \bar{X}(t))_H + (\bar{\mathcal{H}}_u(t), u(t) - \bar{u}(t))_U, \tag{B29}$$

and

$$\Phi(X(T)) - \Phi(\bar{X}(T)) \geqslant (\Phi_x(\bar{X}(T)), x(T) - \bar{x}(T))_H. \tag{B30}$$

In addition, the optimality condition

$$\begin{aligned} & \mathcal{H}(t, \bar{X}(t-), \bar{u}(t), \bar{p}(t), \bar{q}(t), \bar{r}(t, \cdot)) \\ = & \min_{u \in \mathscr{U}} \mathcal{H}(t, \bar{X}(t-), u, \bar{p}(t), \bar{q}(t), \bar{r}(t, \cdot)). \end{aligned}$$

and the convex optimization principle (see Proposition 2.21 of [**?**]) yield that for almost all $(t, \omega) \in [0, T] \times \Omega$,

$$(\bar{\mathcal{H}}_u(t), u(t) - \bar{u}(t))_U \geqslant 0. \tag{B31}$$

Then putting (B29), (B30) and (B31) into (B28), we get that

$$J(u(\cdot)) - J(\bar{u}(\cdot)) \geqslant 0. \tag{B32}$$

Therefore, since $u(\cdot)$ is arbitrary, $\bar{u}(\cdot)$ is an optimal control process and $(\bar{u}(\cdot); \bar{X}(\cdot))$ is an optimal pair. The proof is complete.

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