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# Multi-leader multi-follower coordination with cohesion, dispersion, and containment control via proximity graphs

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**Abstract** This paper studies the problem of multi-leader multi-follower coordination with proximity-based network topologies. The particular interest is to drive all the followers towards the convex hull formed by the moving leaders while producing cohesion behavior and keeping group dispersion. First, in the case of stationary leaders, we design a gradient-based continuous control algorithm. We show that with this continuous algorithm the control objective can be achieved, and the tracking error bound can be controlled by tuning some control parameters. We apply the continuous control algorithm to the moving leaders case and show that the tracking error bound is related to the velocities of the leaders. However, in this case, the algorithm has one restriction that the velocities of the leaders should depend on neighboring followers' velocities, which might not be desirable in some scenarios. Therefore, we propose a nonsmooth algorithm for moving leaders which works under the mild assumption of boundedness of leaders' velocities. Finally, we present numerical examples to show the validity of the proposed algorithms.

**Keywords** cooperative control, multi-agent system, containment control, cohesion, dispersion, collision-free movement

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## 1 Introduction

One theme in cooperative control is to assemble and coordinate individual physical devices into a coherent whole to perform a common task [1–4]. In particular, in the control community, containment control has become one interesting research direction because cooperative control usually poses geometrical constraints during agents' movements. The objective of containment control is to drive a group of agents to a particular area specified by another group. For example when a collection of autonomous robots are to secure and then remove hazardous materials, the robots should not venture into populated areas or in other ways contaminate their surroundings [5]. Potential applications of containment control include combat and reconnaissance systems [6], hazardous material handling [7], and distributed mobile

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sensor networks [8], to name just a few. To enable these applications, some control capabilities need to be developed, including group cohesion, collision avoidance, and connectivity maintenance.

Ref. [5] studies the containment control problem from a leader-follower point of view, where the theory of partial difference equations is exploited and hybrid control schemes are proposed based on the stop-go rule. Ref. [9] extends the results of [5] to the case of switching directed network topologies by using a Lyapunov-based approach. For double-integrator dynamics, the distributed containment control problem under the constraints that the velocities and the accelerations of the agents are not available was studied in [10], where the control algorithms use only position measurements. Meanwhile, the finite-time containment control problem for double-integrator systems is discussed [11], where external disturbances are tackled by the homogeneous control technique. Distributed containment control for nonlinear Lagrangian systems is investigated in [12] for directed graphs and in [13] for flocking behavior. It is worth pointing out that containment control with a leader-follower architecture is related to the leader-follower model in the study of consensus problems, where algorithms are designed to drive all followers to track the trajectory of a leader agent [14, 15]. Meanwhile, the surrounding control problem, which can be seen as the inverse problem of containment control, is also studied for both the fixed leaders case [16] and time-varying leaders case [17].

In this paper, we aim at designing control algorithms for a team of agents which behave either as leaders or followers. Assuming that the followers obey the single-integrator dynamics and the communication graph depends upon proximity relations, the control scheme is based on the sum of potential functions in order to achieve: (1) containment control (The followers asymptotically converge to the convex hull formed by the leaders' positions, up to a scaling factor, which can be tuned by some control parameters); (2) persistent communication with the leaders (If a follower is initially within the transmitting range of a leader, it will stay in the range afterwards); (3) group cohesion (The inter-follower distances are upper bounded by a prescribed constant); (4) group dispersion and hence collision avoidance (The inter-agent distances are lower bounded by a prescribed constant). For all the control algorithms, we provide quantitative bound analysis for the containment control error, which expands our preliminary results reported in [18].

In contrast with the containment control work in [5, 9–12], the proposed algorithms have the following advantages: collision avoidance, which has become an increasingly important point; connectivity maintenance, therefore no assumptions on the connectivity being imposed during the movements of agents; group cohesion and dispersion, which lead to better system performance, are tunable by some control parameters. To guarantee collision-free movements, a potential function is constructed which incorporates only the position information of the agents. This is different from the one constructed in [19] which relies primarily on the velocity information. In contrast with the work in [20–22] on connectivity maintenance, we derive an explicit bound on the potential force. This greatly facilitates the design and analysis of the resulting system. There are three main differences which distinguish the current paper from the work in [23–27]. First, we analyze in this paper the spatial configuration of the multi-agent system, i.e., we derive the containment control error bound. This is the primary concern of the current paper and a challenge in stability analysis due to the collision avoidance term and connectivity maintenance term in the control inputs. Refs. [24, 25] did not analyze the spatial configuration of the resulting system. For example, Ref. [24] shows that the system converges to a local extremum of the potential functions. However, it does not establish the relationship between the local extremum and the spatial configuration. Although the spatial configuration of flocking behaviors is analyzed in [23], the results rely on two conjectures which are not proved. Second, in the current paper, we derive a bound on the control inputs, that is, $\|u_i(t)\| \leqslant \sigma_s$ for some derived constant $\sigma_s$ and $t \geqslant 0$. This greatly facilitates the design of the algorithms because for some pre-specified constant $o \in \mathbb{R}^+$ one can obtain the parameters of the system by solving the inequality $\sigma_s \leqslant o$. Using the derived parameters to initialize the system, the inequality $\|u_i(t)\| \leqslant o$ will hold for all $t \geqslant 0$. Note that the boundedness of the control inputs is not considered in [23, 24, 26, 27], while Ref. [25] only shows that $\|u_i(t)\| < \infty$ for all $t \geqslant 0$. Third, there are also some delicate but nontrivial technical differences between the current paper and the work in [23–25]. In [23], the coupling term between the leaders and the followers uses absolute information (see Eq. (32) of [23]), while the current paper uses only relative information. The potentials of the secondary objectives are required to be twice differentiable and radially unbounded in [25], while the potentials for containment control in the current paper is neither twice differentiable nor radially unbounded. A more distinguished difference is the second control algorithm of the current paper which uses the signum function that is not presented in the literature discussed above.

The rest of this paper is organized as follows. In Section 2, we define the notation and describe the problem to be studied. We solve the problem in Section 3 for the stationary leaders case and the moving leaders case. For the moving leaders case, we introduce a nonsmooth control algorithm in Section 4 such that the control objective is achieved as long as the leaders' velocities are bounded. We present two simulation examples in Section 5 to validate the theoretical results. Finally, Section 6 concludes this paper.

## 2 Problem description

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ the set of $n$-dimensional real vectors, and $\mathbb{R}^{m\times n}$ the set of $m\times n$ real matrices. Let $I_n\in\mathbb{R}^{n\times n}$ be the $n$-dimensional identity matrix, $\mathbf{0}_n\in\mathbb{R}^n$ the vector with all zeroes, and $\mathbf{1}_n\in\mathbb{R}^n$ the vector with all ones. The subscripts of $I_n$, $\mathbf{0}_n$, and $\mathbf{1}_n$ might be dropped if no confusion arises in the context. Let $\otimes$ denote the Kronecker product. Throughout this paper, we use the Euclidean norm, i.e., $\|x\|\triangleq\sqrt{x^{\mathrm{T}}x}$. The convex hull of a finite number of points is the set of all convex combinations of these points. For $x\in\mathbb{R}^2$ and $S\subseteq\mathbb{R}^2$, define $\|x-S\|\triangleq\inf_{y\in S}\|x-y\|$.

Consider a system of $n$ agents. We divide the agents into two groups: $\mathcal{V}_{\mathrm{L}}$ and $\mathcal{V}_{\mathrm{F}}$, where $\mathcal{V}_{\mathrm{L}}$ denotes the nonempty set of leader agents and $\mathcal{V}_{\mathrm{F}}$ denotes the nonempty set of follower agents. The motion of the leaders is described by

$$\dot{x}_i(t)=v_i(t),\quad i\in\mathcal{V}_{\mathrm{L}},\tag{1}$$

where $x_i(t)\in\mathbb{R}^2$ is the position and $v_i(t)\in\mathbb{R}^2$ is the velocity of leader $i$. The motion of the followers is governed by

$$\dot{x}_i(t)=u_i(t),\quad i\in\mathcal{V}_{\mathrm{F}},\tag{2}$$

where $x_i(t)\in\mathbb{R}^2$ is the position and $u_i(t)\in\mathbb{R}^2$ is the control input for follower $i$.

We assume that all the leaders and followers are equipped with transceivers. Let $r_{\mathrm{L}}$ be the transmitting radius of the leaders. We denote the transmitting range of leader $i$ by $S_i(t)\triangleq\{y\in\mathbb{R}^2|\,\|y-x_i(t)\|<r_{\mathrm{L}}\}$, $i\in\mathcal{V}_{\mathrm{L}}$. Let $r_{\mathrm{F}}\leqslant r_{\mathrm{L}}$ be the transmitting radius of the followers. We denote the transmitting range of follower $i$ by $S_i(t)\triangleq\{y\in\mathbb{R}^2|\,\|y-x_i(t)\|<r_{\mathrm{F}}\}$, $i\in\mathcal{V}_{\mathrm{F}}$. If agent $j$ is within the transmitting range of agent $i$ at time $t$, we write $x_j(t)\in S_i(t)$. Define the neighboring set $\mathcal{N}_i(t)\triangleq\{j\in\mathcal{V}_{\mathrm{L}}\cup\mathcal{V}_{\mathrm{F}}|\,j\neq i,x_i(t)\in S_j(t)\}$. We use a graph $\mathcal{G}(t)\triangleq(\mathcal{V},\mathcal{E}(t))$ to describe the information flows among the agents, where $\mathcal{V}\triangleq\mathcal{V}_{\mathrm{L}}\cup\mathcal{V}_{\mathrm{F}}$ and $\mathcal{E}(t)\triangleq\{(i,j)|\,x_j(t)\in S_i(t)\}$. The subgraph $\mathcal{G}_{\mathrm{F}}\triangleq(\mathcal{V}_{\mathrm{F}},\mathcal{E}_{\mathrm{F}}(t))$ depicts the information flows among the followers with $\mathcal{E}_{\mathrm{F}}(t)\triangleq\{(i,j)|\,x_j(t)\in S_i(t),i,j\in\mathcal{V}_{\mathrm{F}}\}$.

Let $\mathrm{co}(\mathcal{V}_{\mathrm{L}})$ denote the convex hull formed by the leaders. The main purpose of the paper is to design control inputs for the followers such that:

(1) All the followers are driven towards the convex hull of the leaders, i.e.,

$$\limsup_{t\to\infty}\|x_i(t)-\mathrm{co}(\mathcal{V}_{\mathrm{L}})\|\leqslant c,\quad i\in\mathcal{V}_{\mathrm{F}},\tag{3}$$

where $c\geqslant 0$ is a control parameter and $\limsup f(t)$ is the limit superior of the function $f(t)$.

(2) If follower $i$ is initially within the transmitting range of leader $j$, then it will be within the transmitting range of that leader for all $t\geqslant 0$, that is,

$$\|x_i(t)-x_j(t)\|<r_{\mathrm{L}},\quad i\in\mathcal{V}_{\mathrm{F}},\quad j\in\mathcal{V}_{\mathrm{L}},\tag{4}$$

for all $(j,i)\in\mathcal{E}(0)$ and $t\geqslant 0$.

(3) The followers move cohesively as a swarm while preserving connectivity, that is,

$$\|x_i(t)-x_j(t)\|<d_1,\quad 0<d_1<r_{\mathrm{F}},\tag{5}$$

for all $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_\mathrm{F}$, $j \in \mathcal{V}_\mathrm{F}$, and $t \geqslant 0$, where $d_1$ is a control parameter.

(4) Group dispersion is maintained, i.e.,

$$\|x_i(t) - x_j(t)\| > d_2, \quad 0 < d_2 < d_1, \tag{6}$$

for all $i \in \mathcal{V}_\mathrm{F}$, $j \in \mathcal{V}_\mathrm{L} \cup \mathcal{V}_\mathrm{F}$, $i \neq j$, and $t \geqslant 0$, where $d_2$ is a control parameter. If Eq. (6) holds, then collision avoidance is guaranteed.

## 3 Algorithm design and analysis

### 3.1 Algorithm design

To drive all followers towards the convex hull of the leaders, define

$$V_{i1} \triangleq \frac{1}{2}\left(\sum_{k\in(\mathcal{N}_i\cap\mathcal{V}_\mathrm{L})} b_{ik}\|x_i - x_k\|^2 + \sum_{k\in(\mathcal{V}_\mathrm{L}\backslash\mathcal{N}_i)} b_{ik} r_\mathrm{L}^2\right), \quad i \in \mathcal{V}_\mathrm{F},$$

where $b_{ik} > 0$ are positive constants. It is worth noticing that $-\frac{\partial V_{i1}}{\partial x_i} = -\sum_{k\in(\mathcal{N}_i\cap\mathcal{V}_\mathrm{L})} b_{ik}(x_i - x_k)$ is a consensus type protocol that is widely used in the literature. The purpose of introducing the second term in $V_{i1}$ is to make the function continuous when leaders leave or enter the "neighborhood" of follower $i$.

To keep group cohesion and to preserve connectivity between the followers, define

$$V_{i2} \triangleq \sum_{j\in(\mathcal{N}_i(0)\cap\mathcal{V}_\mathrm{F})} a_{ij} s_{ij}(d_{ij}), \quad i \in \mathcal{V}_\mathrm{F},$$

where

$$s_{ij}(d_{ij}) = \frac{1}{\frac{1}{2}d_1^2 - d_{ij}},$$

with $d_{ij} \triangleq \frac{1}{2}\|x_i - x_j\|^2$ and $a_{ij} = a_{ji} > 0$ for all $i \in \mathcal{V}_\mathrm{F}$ and $j \in \mathcal{V}_\mathrm{F}$. We can verify that $s_{ij} > 0$ for $d_{ij} < \frac{1}{2}d_1^2$ and $s_{ij}(\frac{1}{2}d_1^2) = \infty$. It can be shown that $\frac{\partial s_{ij}}{\partial d_{ij}} = (\frac{1}{2}d_1^2 - d_{ij})^{-2} > 0$ when $d_{ij} < \frac{1}{2}d_1^2$. Therefore, the decrease of $d_{ij}$ will lead to the decrease of $s_{ij}$. That is, $s_{ij}$ is an attractive function. Similar functions have been used in the literature [20].

In addition, to preserve the connectivity between the leaders and the followers, i.e., $(\mathcal{N}_i(0) \cap \mathcal{V}_\mathrm{L}) \subseteq (\mathcal{N}_i(t) \cap \mathcal{V}_\mathrm{L})$ for all $i \in \mathcal{V}_\mathrm{F}$ and $t \geqslant 0$, define

$$V_{i3} \triangleq \sum_{k\in(\mathcal{N}_i(0)\cap\mathcal{V}_\mathrm{L})} a_{ik} q_{ik}(d_{ik}), \quad i \in \mathcal{V}_\mathrm{F},$$

where

$$q_{ik}(d_{ik}) = \frac{1}{\frac{1}{2}r_\mathrm{L}^2 - d_{ik}},$$

with $d_{ik} = \frac{1}{2}\|x_i - x_k\|^2$ and $a_{ik} > 0$ for $i \in \mathcal{V}_\mathrm{F}$ and $k \in \mathcal{V}_\mathrm{L}$. The function $q_{ik}$ is of a similar form to that of $s_{ij}$, except that $d_1$ is replaced with $r_\mathrm{L}$.

To achieve group dispersion, we introduce the following potential function:

$$V_{i4} \triangleq \sum_{j\neq i} \frac{c_{ij}}{r_{ij}(d_{ij})}, \quad i \in \mathcal{V}_\mathrm{L} \cup \mathcal{V}_\mathrm{F},$$

where $c_{ij} = c_{ji} > 0$, and $r_{ij}$ is given by

$$r_{ij}(d_{ij}) = \begin{cases} a_1\left(d_{ij} - \frac{1}{2}d_2^2\right) + a_2\left(d_{ij} - \frac{1}{2}d_2^2\right)^2, & d_{ij} \leqslant \frac{1}{2}r_\mathrm{F}^2, \\ a_1\left(\frac{1}{2}r_\mathrm{F}^2 - \frac{1}{2}d_2^2\right) + a_2\left(\frac{1}{2}r_\mathrm{F}^2 - \frac{1}{2}d_2^2\right)^2, & d_{ij} > \frac{1}{2}r_\mathrm{F}^2, \end{cases} \tag{7}$$

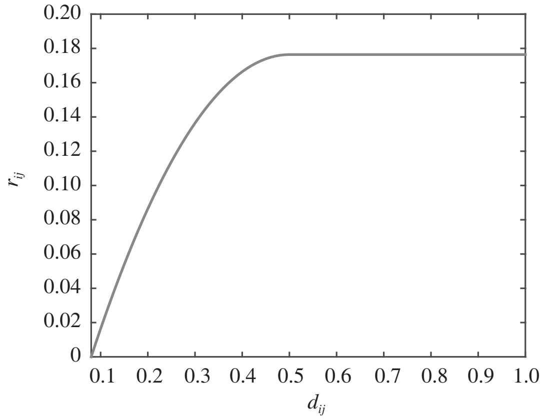


**Figure 1** An illustration of the function $r_{ij}$ with the parameters: $r_{\rm F} = 1$, $d_2 = 0.4$, $a_1 = 0.84$, $a_2 = -1$.

for $i, j \in \mathcal{V}_{\rm L} \cup \mathcal{V}_{\rm F}$ and $i \neq j$. Here $a_1 = a_2(d_2^2 - r_{\rm F}^2)$ and $a_2 < 0$. Figure 1 illustrates the function $r_{ij}$.

Note that $V_{i4}$ is a repulsive force, which should be spatially distributed. Such function $r_{ij}$ satisfies several key properties, as stated in the following lemma.

**Lemma 1.** The following properties hold for the function $r_{ij}$.

(1) $r_{ij}(d_{ij}) > 0$ for $\|x_i - x_j\| > d_2$,

(2) $r_{ij}(d_{ij})$ is continuously differentiable with respect to $d_{ij}$,

(3) $r_{ij}(d_{ij}) = 0$ if $\|x_i - x_j\| = d_2$,

(4) and $r_{ij}(d_{ij}) = r_{ij}(\frac{1}{2}r_{\rm F}^2)$ for all $\|x_i - x_j\| > r_{\rm F}$.

*Proof.* (1) Note that $\|x_i - x_j\| > d_2$ is equivalent to $d_{ij} > \frac{1}{2}d_2^2$. Therefore, one can consider the two cases: $\frac{1}{2}d_2^2 < d_{ij} \leqslant \frac{1}{2}r_{\rm F}^2$ and $d_{ij} > \frac{1}{2}r_{\rm F}^2$. If $\frac{1}{2}d_2^2 < d_{ij} \leqslant \frac{1}{2}r_{\rm F}^2$, then

$$\begin{aligned} r_{ij} &= a_1\left(d_{ij} - \frac{1}{2}d_2^2\right) + a_2\left(d_{ij} - \frac{1}{2}d_2^2\right)^2 \\ &= a_2\left(d_2^2 - r_{\rm F}^2\right)\left(d_{ij} - \frac{1}{2}d_2^2\right) + a_2\left(d_{ij} - \frac{1}{2}d_2^2\right)^2 \\ &= a_2\left(d_{ij} - \frac{1}{2}d_2^2\right)\left(d_2^2 - r_{\rm F}^2 + d_{ij} - \frac{1}{2}d_2^2\right). \end{aligned}$$

It follows from $a_2 < 0$ and $d_{ij} - \frac{1}{2}d_2^2 > 0$ that

$$\begin{aligned} r_{ij} &\geqslant a_2\left(d_{ij} - \frac{1}{2}d_2^2\right)\left(\frac{1}{2}d_2^2 - r_{\rm F}^2 + \frac{1}{2}r_{\rm F}^2\right) \\ &= a_2\left(d_{ij} - \frac{1}{2}d_2^2\right)\frac{1}{2}\left(d_2^2 - r_{\rm F}^2\right) \\ &> 0. \end{aligned}$$

If $d_{ij} > \frac{1}{2}r_{\rm F}^2$, then

$$\begin{aligned} r_{ij} &= a_1\left(\frac{1}{2}r_{\rm F}^2 - \frac{1}{2}d_2^2\right) + a_2\left(\frac{1}{2}r_{\rm F}^2 - \frac{1}{2}d_2^2\right)^2 \\ &= a_2\left(d_2^2 - r_{\rm F}^2\right)\left(\frac{1}{2}r_{\rm F}^2 - \frac{1}{2}d_2^2\right) + a_2\left(\frac{1}{2}r_{\rm F}^2 - \frac{1}{2}d_2^2\right)^2 \\ &= a_2\left(\frac{1}{2}r_{\rm F}^2 - \frac{1}{2}d_2^2\right)\left(d_2^2 - r_{\rm F}^2 + \frac{1}{2}r_{\rm F}^2 - \frac{1}{2}d_2^2\right) \\ &= a_2\left(\frac{1}{2}r_{\rm F}^2 - \frac{1}{2}d_2^2\right)\left(\frac{1}{2}d_2^2 - \frac{1}{2}r_{\rm F}^2\right) \\ &> 0. \end{aligned}$$

(2) The partial derivative of $r_{ij}$ with respect to $d_{ij}$ is

$$\frac{\partial r_{ij}}{\partial d_{ij}} = \begin{cases} a_1 + 2a_2\left(d_{ij} - \frac{1}{2}d_2^2\right), & d_{ij} \leqslant \frac{1}{2}r_{\rm F}^2, \\ 0, & d_{ij} > \frac{1}{2}r_{\rm F}^2. \end{cases}$$

To make $\frac{\partial r_{ij}}{\partial d_{ij}}$ continuously differentiable, it suffices to show

$$\frac{\partial r_{ij}}{\partial d_{ij}}\Big|_{\frac{1}{2}r_{\rm F}^2} = 0,$$

which leads to

$$a_1 + 2a_2\left(\frac{1}{2}r_{\rm F}^2 - \frac{1}{2}d_2^2\right) = 0,$$

which is equivalent to

$$a_1 = a_2\left(d_2^2 - r_{\rm F}^2\right).$$

(3) If $\|x_i - x_j\| = d_2$, then $d_{ij} = \frac{1}{2}d_2^2$. It follows from (7) that $r_{ij} = 0$.

(4) It is straightforward to verify that $r_{ij}(d_{ij}) = r_{ij}(\frac{1}{2}r_{\rm F}^2)$ for all $\|x_i - x_j\| > r_{\rm F}$.

Note that if $\|x_i - x_j\| = d_2$ for some $i, j \in \mathcal{V}_{\rm L} \cup \mathcal{V}_{\rm F}$, then $r_{ij} = 0$, which yields $V_{i4} = \infty$.

Define

$$V \triangleq \sum_{i\in\mathcal{V}_{\rm F}} V_{i1} + \frac{1}{2}\sum_{i\in\mathcal{V}_{\rm F}} V_{i2} + \sum_{i\in\mathcal{V}_{\rm F}} V_{i3} + \frac{1}{2}\sum_{i=1}^{n} V_{i4}. \tag{8}$$

The control law is given by

$$u_i = -\left(\frac{\partial V}{\partial x_i}\right)^{\rm T}, \quad i \in \mathcal{V}_{\rm F}. \tag{9}$$

Eqs. (8) and (9) yield

$$\begin{aligned} u_i = &- \sum_{k\in\mathcal{N}_i\cap\mathcal{V}_{\rm L}} b_{ik}(x_i - x_k) - \sum_{j\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm F}} a_{ij}\frac{\partial s_{ij}}{\partial d_{ij}}(x_i - x_j) \\ &- \sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} a_{ik}\frac{\partial q_{ik}}{\partial d_{ik}}(x_i - x_k) + \sum_{j\neq i} c_{ij}r_{ij}^{-2}\frac{\partial r_{ij}}{\partial d_{ij}}(x_i - x_j), \end{aligned}$$

for all $i \in \mathcal{V}_{\rm F}$. Since

$$\frac{\partial r_{ij}}{\partial d_{ij}} = 0, \quad \forall \|x_i - x_j\| \geqslant r_{\rm F},$$

one has

$$\begin{aligned} u_i = &- \sum_{k\in\mathcal{N}_i\cap\mathcal{V}_{\rm L}} b_{ik}(x_i - x_k) - \sum_{j\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm F}} a_{ij}\frac{\partial s_{ij}}{\partial d_{ij}}(x_i - x_j) \\ &- \sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} a_{ik}\frac{\partial q_{ik}}{\partial d_{ik}}(x_i - x_k) + \sum_{j\in\mathcal{N}_i} c_{ij}r_{ij}^{-2}\frac{\partial r_{ij}}{\partial d_{ij}}(x_i - x_j). \end{aligned} \tag{10}$$

Therefore, the algorithm defined by (9) is distributed.

**Remark 1.** It is possible to extend the control input design to the double-integrator agent model, i.e.,

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i$$

where $x_i$, $v_i$, and $u_i$ denote, respectively, the position, velocity, and control input of agent $i$. In this scenario, the consensus term of agents' velocities should be added to the control input (10). If the velocities of the leaders are constant, then a linear velocity consensus term, like $\sum_{j\in\mathcal{N}_i}(v_j - v_i)$, can be constructed. If the leaders' velocities are time-varying, then a nonlinear consensus term on velocities might be needed.

### 3.2 Analysis for multiple stationary leaders

In this subsection, we assume that the leaders are stationary, that is,

$$v_k(t) \equiv 0, \quad t \geqslant 0, \quad k \in \mathcal{V}_{\mathrm{L}}. \tag{11}$$

Using (9) for (2) and (11) for (1), we obtain a system with a leader-follower architecture.

**Lemma 2.** Define

$$e_i(t) \triangleq \sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}\left(x_i(t)-x_k(t)\right), \quad i\in\mathcal{V}_{\mathrm{F}}. \tag{12}$$

If $\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}} \neq \emptyset$ and $\|e_i(t)\| = 0$, then $x_i(t) \in \mathrm{co}(\mathcal{V}_{\mathrm{L}})$.

*Proof.* The proof is straightforward, and is hence omitted.

For notational convenience, define the distance between follower $i$ and the convex hull spanned by the leaders $\mathrm{co}(\mathcal{V}_{\mathrm{L}})$ as

$$\alpha_i(t) \triangleq \|x_i(t) - \mathrm{co}(\mathcal{V}_{\mathrm{L}})\|, \quad i\in\mathcal{V}_{\mathrm{F}}.$$

Let

$$\alpha(t) \triangleq \sum_{i\in\mathcal{V}_{\mathrm{F}}} \alpha_i^2(t), \tag{13}$$

which measures the distance from $\mathcal{V}_{\mathrm{F}}$ to $\mathrm{co}(\mathcal{V}_{\mathrm{L}})$. If $\alpha(t)=0$, then all the followers are in the convex hull spanned by the leaders. Define

$$b_{\min}(t) \triangleq \min_{i\in\mathcal{V}_{\mathrm{F}}}\left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}\right),$$

and

$$b_{\max}(t) \triangleq \max_{i\in\mathcal{V}_{\mathrm{F}}}\left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}\right).$$

One has the following result.

**Lemma 3.** Define $e(t) \triangleq [e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \ldots, e_{|\mathcal{V}_{\mathrm{F}}|}^{\mathrm{T}}(t)]^{\mathrm{T}}$ with $e_i(t)$ defined by (12). If $b_{\min}(t) > 0$, then $\alpha(t) \leqslant \frac{\|e(t)\|^2}{b_{\min}^2(t)}$.

*Proof.* Let

$$z_i(t) \triangleq \frac{\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}x_k(t)}{\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}}.$$

Then $e_i(t)$ can be rewritten as

$$e_i(t) = \left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}\right)\left(x_i(t) - z_i(t)\right),$$

which yields

$$\|e_i(t)\| = \left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}\right)\|x_i(t) - z_i(t)\|.$$

By Lemma 2, we know that $z_i(t)$ is in $\mathrm{co}(\mathcal{V}_{\mathrm{L}})$, which leads to

$$\|e_i(t)\| \geqslant \left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}\right)\alpha_i(t).$$

It follows that

$$\alpha_i(t) \leqslant \frac{\|e_i(t)\|}{\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}},$$

which further yields

$$\alpha(t) \leqslant \sum_{i\in\mathcal{V}_{\mathrm{F}}} \frac{\|e_i(t)\|^2}{\left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}\right)^2} \leqslant \sum_{i\in\mathcal{V}_{\mathrm{F}}} \frac{\|e_i(t)\|^2}{b_{\min}^2(t)} \leqslant \frac{\|e(t)\|^2}{b_{\min}^2(t)}.$$

The following lemma shows that the control input given by (9) is bounded.

**Lemma 4.** For any $i \in \mathcal{V}_{\mathrm{F}}$, the control input $u_i(t)$ given by (9) is bounded.

*Proof.* For $i \in \mathcal{V}_{\mathrm{F}}$ and $j \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{F}}$, let $\frac{1}{2}a_{ij}s_{ij}(s_{ij}^*) = V(0)$, where $s_{ij}^* \in (0, \frac{1}{2}d_1^2)$, and $V(0)$ is the value of the function $V$, defined by (8), at time 0. Since the function is monotonic, $s_{ij}^*$ is unique, and

$$\frac{1}{2}a_{ij}s_{ij}[d_{ij}(t)] \leqslant V(t) \leqslant V(0) = \frac{1}{2}a_{ij}s_{ij}(s_{ij}^*).$$

It follows that $d_{ij}(t) \leqslant s_{ij}^*$ for all $t \geqslant 0$. Define $s^* = \max_{i,j} s_{ij}^*$. One has $d_{ij}(t) \leqslant s^*$ for all $t \geqslant 0$, $i \in \mathcal{V}_{\mathrm{F}}$ and $j \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{F}}$, which leads to

$$\frac{1}{2}d_1^2 - d_{ij}(t) \geqslant \frac{1}{2}d_1^2 - s^*, \tag{14}$$

for all $t \geqslant 0$, $i \in \mathcal{V}_{\mathrm{F}}$ and $j \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{F}}$.

Similarly, for all $t \geqslant 0$, $i \in \mathcal{V}_{\mathrm{F}}$ and $k \in \mathcal{V}_{\mathrm{L}}$, one can show that

$$\frac{1}{2}r_{\mathrm{L}}^2 - d_{ik}(t) \geqslant \frac{1}{2}r_{\mathrm{L}}^2 - q^*, \tag{15}$$

for some $q^* \in (0, \frac{1}{2}r_{\mathrm{L}}^2)$.

For $i \in \mathcal{V}_{\mathrm{F}}$ and $j \in \mathcal{N}_i(t)$, let $\frac{1}{2}\frac{c_{ij}}{r_{ij}(r_{ij}^*)} = V(0)$, with $r_{ij}^* \in (\frac{1}{2}d_2^2, \frac{1}{2}r_{\mathrm{F}}^2)$. It follows that

$$\frac{1}{2}\frac{c_{ij}}{r_{ij}[d_{ij}(t)]} \leqslant V(t) \leqslant V(0) = \frac{1}{2}\frac{c_{ij}}{r_{ij}(r_{ij}^*)},$$

which yields that $d_{ij}(t) \geqslant r_{ij}^*$ for all $t \geqslant 0$. Define $r^* \triangleq \min_{i,j} r_{ij}^*$. We have

$$d_{ij}(t) \geqslant r^*, \tag{16}$$

for all $t \geqslant 0$, $i \in \mathcal{V}_{\mathrm{F}}$ and $j \in \mathcal{N}_i(t)$. In addition, for all $t \geqslant 0$, $i \in \mathcal{V}_{\mathrm{F}}$ and $j \in (\mathcal{V}_{\mathrm{L}} \cup \mathcal{V}_{\mathrm{F}})\backslash\mathcal{N}_i(t)$, $d_{ij}(t) \geqslant \frac{1}{2}r_{\mathrm{F}}^2 > r^*$.

In view of (14)–(16), it follows that

$$\begin{aligned}
\|u_i(t)\| \leqslant& \sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}\|x_i(t)-x_k(t)\| + \sum_{j\in\mathcal{N}_i(0)\cap\mathcal{V}_{\mathrm{F}}} a_{ij}\left(\frac{1}{2}d_1^2 - d_{ij}(t)\right)^{-2}\|x_i(t)-x_j(t)\| \\
&+ \sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\mathrm{L}}} a_{ik}\left(\frac{1}{2}r_{\mathrm{L}}^2 - d_{ik}(t)\right)^{-2}\|x_i(t)-x_k(t)\| \\
&+ \sum_{j\in\mathcal{N}_i(t)} c_{ij}\left(r_{ij}\left(d_{ij}(t)\right)\right)^{-2}\left|a_1 + 2a_2\left(d_{ij}(t) - \frac{1}{2}d_2^2\right)\right|\|x_i(t)-x_j(t)\| \\
\leqslant& |\mathcal{V}_{\mathrm{L}}|\left(\max_{i,k} b_{ik}\right)r_{\mathrm{L}} + |\mathcal{V}_{\mathrm{F}}|\left(\max_{i,j} a_{ij}\right)\left(\frac{1}{2}d_1^2 - s^*\right)^{-2}\sqrt{2s^*} \\
&+|\mathcal{V}_{\mathrm{L}}|\left(\max_{i,k} a_{ik}\right)\left(\frac{1}{2}r_{\mathrm{L}}^2 - q^*\right)^{-2}\sqrt{2q^*} + n\left(\max_{i,j} c_{ij}\right)\left(r_{ij}(r^*)\right)^{-2}2|a_2|\left(\frac{1}{2}r_{\mathrm{F}}^2 - r^*\right)r_{\mathrm{L}} \\
\triangleq& \sigma_s.
\end{aligned} \tag{17}$$

Eq. (17) indicates that the bound on the control input can be tuned by choosing appropriate values of $b_{ik}$, $a_{ij}$, $a_{ik}$, and $c_{ij}$.

For notational convenience, define

$$g_i(t) \triangleq -\sum_{j\in\mathcal{N}_i(0)\cap\mathcal{V}_\mathrm{F}} a_{ij}\frac{\partial s_{ij}}{\partial d_{ij}}(x_i(t)-x_j(t)) - \sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_\mathrm{L}} a_{ik}\frac{\partial q_{ik}}{\partial d_{ik}}(x_i(t)-x_k(t)) + \sum_{j\neq i} c_{ij}\left(r_{ij}\left(d_{ij}(t)\right)\right)^{-2}\frac{\partial r_{ij}}{\partial d_{ij}}(x_i(t)-x_j(t)), \tag{18}$$

and let $g \triangleq [g_1^\mathrm{T}, g_2^\mathrm{T}, \ldots, g_{|\mathcal{V}_\mathrm{F}|}^\mathrm{T}]^\mathrm{T}$. By using similar arguments to those in the proof of Lemma 4, we can show that $\|g_i\| \leqslant \sigma_g$, where

$$\sigma_g \triangleq |\mathcal{V}_\mathrm{F}|\max_{i,j} a_{ij}\left(\frac{1}{2}d_1^2 - s^*\right)^{-2}\sqrt{2s^*} + |\mathcal{V}_\mathrm{L}|\max_{i,k} a_{ik}\left(\frac{1}{2}r_\mathrm{L}^2 - q^*\right)^{-2}\sqrt{2q^*} + n\max_{i,j} c_{ij}\left(r_{ij}(r^*)\right)^{-2} 2|a_2|\left(\frac{1}{2}r_\mathrm{F}^2 - r^*\right) r_\mathrm{L}.$$

Therefore, we have

$$\|g\| = \sqrt{\|g_1\|^2 + \|g_2\|^2 + \cdots + \|g_{|\mathcal{V}_\mathrm{F}|}\|^2} \leqslant \sqrt{|\mathcal{V}_\mathrm{F}|}\sigma_g. \tag{19}$$

We first consider the case where each follower has at least one leader as its neighbor at time 0.

**Theorem 1.** Use (9) for (2) with stationary leaders. If initially $\|x_i(0) - x_j(0)\| < d_1$ for all $(i,j) \in \mathcal{E}(0)$, $i,j \in \mathcal{V}_\mathrm{F}$, $\|x_i(0) - x_j(0)\| > d_2$ for all $i,j \in \mathcal{V}_\mathrm{L}\cup\mathcal{V}_\mathrm{F}$, $i \neq j$, and $\mathcal{N}_i(0)\cap\mathcal{V}_\mathrm{L} \neq \emptyset$ for all $i \in \mathcal{V}_\mathrm{F}$, then

(1) for all $t \geqslant 0$, $\|x_i(t) - x_j(t)\| < r_\mathrm{L}$ for all $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_\mathrm{L}$, $j \in \mathcal{V}_\mathrm{F}$, $\|x_i(t) - x_j(t)\| < d_1$ for all $(i,j) \in \mathcal{E}(0)$, $i,j \in \mathcal{V}_\mathrm{F}$, and $\|x_i(t) - x_j(t)\| > d_2$ for all $i,j \in \mathcal{V}_\mathrm{L}\cup\mathcal{V}_\mathrm{F}$, $i \neq j$;

(2) $\mathcal{N}_i(0) \subseteq \mathcal{N}_i(t)$ for all $t \geqslant 0$ and $i \in \mathcal{V}_\mathrm{F}$;

(3) $\limsup_{t\to\infty}\alpha(t) \leqslant |\mathcal{V}_\mathrm{L}|\left(\max_{i,k} b_{ik}\right) b_{\min}^{-3}(0)|\mathcal{V}_\mathrm{F}|\sigma_g^2$.

*Proof.* (1) Use $V$, defined in (8), as a Lyapunov function candidate. Because $\|x_i(0) - x_j(0)\| < r_\mathrm{L}$ for all $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_\mathrm{L}$, $j \in \mathcal{V}_\mathrm{F}$, $\|x_i(0) - x_j(0)\| < d_1$ for all $(i,j) \in \mathcal{E}(0)$, $i,j \in \mathcal{V}_\mathrm{F}$, $\|x_i(0) - x_j(0)\| > d_2$ for all $i,j \in \mathcal{V}_\mathrm{L}\cup\mathcal{V}_\mathrm{F}$, $i \neq j$, it follows from (8) and Lemma 1 that $V(0) < \infty$. Since

$$\dot{V} = \sum_{i\in\mathcal{V}_\mathrm{F}}\frac{\partial V}{\partial x_i}\dot{x}_i = -\sum_{i\in\mathcal{V}_\mathrm{F}}\frac{\partial V}{\partial x_i}\left(\frac{\partial V}{\partial x_i}\right)^\mathrm{T} \leqslant 0, \tag{20}$$

it follows that for all $t \geqslant 0$,

$$V(t) \leqslant V(0) < \infty. \tag{21}$$

Suppose that at time $t_1$, $\|x_i(t_1) - x_j(t_1)\| \leqslant d_2$ for some $i,j \in \mathcal{V}_\mathrm{L}\cup\mathcal{V}_\mathrm{F}$. Because agents are moving continuously, there exists a time $t_2 \leqslant t_1$ such that $\|x_i(t_2) - x_j(t_2)\| = d_2$, which indicates that $d_{ij}(t_2) = \frac{1}{2}d_2^2$. Therefore, one has

$$\lim_{t\to t_2} V(t) = \infty,$$

which contradicts (21). Similarly, we can prove that $\|x_i(t) - x_j(t)\| < r_\mathrm{L}$ for all $t \geqslant 0$, $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_\mathrm{L}$, $j \in \mathcal{V}_\mathrm{F}$, and $\|x_i(t) - x_j(t)\| < d_1$ for all $t \geqslant 0$, $(i,j) \in \mathcal{E}(0)$, $i,j \in \mathcal{V}_\mathrm{F}$. The proof of the first part is thus completed.

(2) Let $j \in \mathcal{N}_i(0)$ be a neighbor of follower $i$ at time 0. If $j \in \mathcal{V}_\mathrm{F}$, then step (1) of this theorem yields $\|x_i(t) - x_j(t)\| < d_1 < r_\mathrm{F}$ for all $t \geqslant 0$. If $j \in \mathcal{V}_\mathrm{L}$, one has $\|x_i(t) - x_j(t)\| < r_\mathrm{L}$ for all $t \geqslant 0$. It thus follows that $j \in \mathcal{N}_i(t)$ for all $t \geqslant 0$, which indicates that $\mathcal{N}_i(0) \subseteq \mathcal{N}_i(t)$ for all $t \geqslant 0$.

(3) Because $\mathcal{N}_i(0)\cap\mathcal{V}_\mathrm{L} \neq \emptyset$ for all $i \in \mathcal{V}_\mathrm{F}$ and $\mathcal{N}_i(0) \subseteq \mathcal{N}_i(t)$ for $t \geqslant 0$, one has $b_{\min}(t) > 0$. Define

$$E(t) \triangleq \frac{1}{2}\sum_{i\in\mathcal{V}_\mathrm{F}} e_i^\mathrm{T}(t)e_i(t), \tag{22}$$

with $e_i(t)$ defined in (12). The derivative of $E(t)$ is given by

$$\dot{E}(t) = \sum_{i\in\mathcal{V}_{\rm F}} e_i^{\rm T}(t)\dot{e}_i(t). \tag{23}$$

Because $\dot{x}_k(t) \equiv 0$ for all $t \geqslant 0$ and $k \in \mathcal{V}_{\rm L}$, one has

$$\dot{e}_i(t) = \sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}} b_{ik}\dot{x}_i(t) = \left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}} b_{ik}\right)(g_i(t) - e_i(t)), \tag{24}$$

which yields

$$\begin{aligned}\dot{E}(t) &= \sum_{i\in\mathcal{V}_{\rm F}} -\left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}} b_{ik}\right) e_i^{\rm T}(t)e_i(t) + \sum_{i\in\mathcal{V}_{\rm F}}\left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}} b_{ik}\right) e_i^{\rm T}(t)g_i(t)\\ &\leqslant \sum_{i\in\mathcal{V}_{\rm F}} -\left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}} b_{ik}\right) e_i^{\rm T}(t)e_i(t) + \sum_{i\in\mathcal{V}_{\rm F}}\frac{1}{2}\left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}} b_{ik}\right)\left(e_i^{\rm T}(t), g_i^{\rm T}(t)\right)\begin{pmatrix} e_i(t)\\ g_i(t)\end{pmatrix}\\ &= \frac{1}{2}\sum_{i\in\mathcal{V}_{\rm F}}\left(-\left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}} b_{ik}\right) e_i^{\rm T}(t)e_i(t) + \left(\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}} b_{ik}\right) g_i^{\rm T}(t)g_i(t)\right),\end{aligned}$$

where $g_i(t)$ is defined in (18). Define

$$B(t) \triangleq {\rm diag}\left\{\sum_{k\in\mathcal{N}_1(t)\cap\mathcal{V}_{\rm L}} b_{1k}, \sum_{k\in\mathcal{N}_2(t)\cap\mathcal{V}_{\rm L}} b_{2k}, \ldots, \sum_{k\in\mathcal{N}_{|\mathcal{V}_{\rm F}|}(t)\cap\mathcal{V}_{\rm L}} b_{|\mathcal{V}_{\rm F}|k}\right\}, \tag{25}$$

it follows from (25) that

$$\begin{aligned}\dot{E}(t) &\leqslant \frac{1}{2}\left(-e^{\rm T}(t)(B(t)\otimes I_2)e(t) + g^{\rm T}(t)(B(t)\otimes I_2)g(t)\right)\\ &\leqslant \frac{1}{2}(-b_{\min}(t)\|e(t)\|^2 + b_{\max}(t)\|g(t)\|^2).\end{aligned} \tag{26}$$

If $\|e(t)\|^2 > \frac{b_{\max}(t)}{b_{\min}(t)}\|g(t)\|^2$, Eq. (26) yields $\dot{E}(t) < 0$, which further leads to

$$\begin{aligned}\limsup_{t\to\infty}\|e(t)\| &\leqslant \limsup_{t\to\infty}\sqrt{\frac{b_{\max}(t)}{b_{\min}(t)}}\|g(t)\|\\ &\leqslant \left(\limsup_{t\to\infty}\sqrt{\frac{b_{\max}(t)}{b_{\min}(t)}}\right)\left(\limsup_{t\to\infty}\|g(t)\|\right)\\ &\leqslant \sqrt{\frac{\limsup_{t\to\infty} b_{\max}(t)}{\liminf_{t\to\infty} b_{\min}(t)}}\left(\limsup_{t\to\infty}\|g(t)\|\right).\end{aligned} \tag{27}$$

For $i \in \mathcal{V}_{\rm F}$ and $g(t) \geqslant 0$, one has

$$\sum_{k\in\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}} b_{ik} \geqslant \sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} b_{ik},$$

because the connectivity between the leaders and the followers is preserved. It follows that

$$b_{\min}(t) \geqslant b_{\min}(0),$$

which indicates

$$\liminf_{t\to\infty} b_{\min}(t) = b_{\min}(0). \tag{28}$$

It thus follows that

$$\limsup_{t\to\infty}\|e(t)\| \leqslant \sqrt{\frac{\limsup_{t\to\infty} b_{\max}(t)}{b_{\min}(0)}}\left(\limsup_{t\to\infty}\|g(t)\|\right)$$
$$\leqslant \sqrt{\frac{|\mathcal{V}_{\rm L}|\left(\max_{i,k} b_{ik}\right)}{b_{\min}(0)}}\left(\limsup_{t\to\infty}\|g(t)\|\right).$$

By Lemma 3,

$$\limsup_{t\to\infty}\alpha(t) \leqslant \limsup_{t\to\infty} b_{\min}^{-2}(0)\|e(t)\|^2 \leqslant |\mathcal{V}_{\rm L}|\left(\max_{i,k} b_{ik}\right) b_{\min}^{-3}(0)\left(\limsup_{t\to\infty}\|g(t)\|^2\right).$$

In view of (19), it is straightforward to obtain that

$$\limsup_{t\to\infty}\alpha(t) \leqslant |\mathcal{V}_{\rm L}|\left(\max_{i,k} b_{ik}\right) b_{\min}^{-3}(0)|\mathcal{V}_{\rm F}|\sigma_g^2. \tag{29}$$

**Remark 2.** From (27) and (29), one knows that to control the bound on $\limsup_{t\to\infty}\|e\|$,

(1) one can control the parameters $b_{ik}$ such that $\sqrt{\frac{b_{\max}}{b_{\min}}}$ is as small as possible. However, since $b_{\max} \geqslant b_{\min}$, the best result that can be obtained is $\sqrt{\frac{b_{\max}}{b_{\min}}} = 1$, which means that for all $i,j \in \mathcal{V}_{\rm F}$, $(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} b_{ik}) = (\sum_{k\in\mathcal{N}_j(0)\cap\mathcal{V}_{\rm L}} b_{jk})$, or

(2) one can control the parameters $a_{ij}$, $c_{ij}$, $b_{ik}$, and $a_{ik}$ to make $\sigma_g$ smaller. In addition, notice that $\lim_{a_{ij},c_{ij},b_{ik},a_{ik}\to 0}\sigma_g = 0$. Therefore the constant $c$ in (3) can be specified a priori.

**Remark 3.** Theorem 1 says that the tracking error under the proposed containment control algorithm is ultimately upper bounded, where the bound is given by $|\mathcal{V}_{\rm L}|\left(\max_{i,k} b_{ik}\right) b_{\min}^{-3}(0)|\mathcal{V}_{\rm F}|\sigma_g^2$. Although the result is only "bounded", it offers the required flexibility of tuning the tracking error to facilitate practical applications. For example, one can increase the value of $b_{ik}$ (the coupling between followers and leaders). In this case, $b_{\min}^{-3}(0)$ decreases, while $\max_{i,k} b_{ik}$ increases. Since the higher-order term $b_{\min}^{-3}(0)$ dominates the tracking error bound, the tracking error will decrease. Therefore, in order to make the tracking error arbitrarily small, one can choose a sufficiently large value of $b_{ik}$.

In Theorem 1, it is required that each follower has at least one leader as its neighbor at time 0. This condition can be relaxed by assuming the connectivity of $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0),\mathcal{V}_{\rm F})$, the induced subgraph of $\mathcal{G}(0)$, which describes the information flows between the followers at time 0.

**Theorem 2.** Use (9) for (2) with stationary leaders. Assume that $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0),\mathcal{V}_{\rm F})$ is connected, and $K \triangleq \{i \in \mathcal{V}_{\rm F} | \mathcal{N}_i(0)\cap\mathcal{V}_{\rm L} \neq \emptyset\} \neq \emptyset$[1]. If $\|x_i(0)-x_j(0)\| < d_1$ for all $(i,j)\in\mathcal{E}(0)$, $i,j\in\mathcal{V}_{\rm F}$, and $\|x_i(0)-x_j(0)\| > d_2$ for all $i,j\in\mathcal{V}_{\rm L}\cup\mathcal{V}_{\rm F}$, $i\neq j$, then

(1) for all $t\geqslant 0$, $\|x_i(t)-x_j(t)\| < r_{\rm L}$ for all $(i,j)\in\mathcal{E}(0)$, $i\in\mathcal{V}_{\rm L}$, $j\in\mathcal{V}_{\rm F}$, $\|x_i(t)-x_j(t)\| < d_1$ for all $(i,j)\in\mathcal{E}(0)$, $i,j\in\mathcal{V}_{\rm F}$, and $\|x_i(t)-x_j(t)\| > d_2$ for all $i,j\in\mathcal{V}_{\rm L}\cup\mathcal{V}_{\rm F}$, $i\neq j$;

(2) $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(t),\mathcal{V}_{\rm F})$ is connected for all $t\geqslant 0$;

(3) for any $j\in\mathcal{V}_{\rm F}$, $\limsup_{t\to\infty}\alpha_j < (|\mathcal{V}_{\rm F}|-1)d_1+\sigma_s$.

*Proof.* (1) The proof of this part is similar to that in Theorem 1, and is hence omitted.

(2) For any edge $(i,j)\in\mathcal{E}_{\rm F}(0)$, where $\mathcal{E}_{\rm F}(0)$ is the edge set of $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0),\mathcal{V}_{\rm F})$, one has $\|x_i(0)-x_j(0)\| < d_1 < r_{\rm F}$. From the first part of this theorem, one knows that $\|x_i(t)-x_j(t)\| < d_1 < r_{\rm F}$ for all $(i,j)\in\mathcal{E}_{\rm F}(0)$ and $t\geqslant 0$, which yields $(i,j)\in\mathcal{E}_{\rm F}(t)$ for all $t\geqslant 0$. Therefore, one can obtain $\mathcal{E}_{\rm F}(0)\subseteq\mathcal{E}_{\rm F}(t)$. Since $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0),\mathcal{V}_{\rm F})$ is connected, one knows that $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(t),\mathcal{V}_{\rm F})$ is connected for all $t\geqslant 0$.

(3) Let

$$i \triangleq \arg\min_{k\in K}\left(\limsup_{t\to\infty}\|g_k\|\right). \tag{30}$$

1) Equivalently, the graph for the followers is connected, and at least one follower is within the transmitting range of a leader at time 0.

Define

$$E_i \triangleq \frac{1}{2} e_i^{\mathrm{T}} e_i, \tag{31}$$

which yields

$$\dot{E}_i = \left( \sum_{k \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{L}}} b_{ik} \right) e_i^{\mathrm{T}} (g_i - e_i).$$

Since $K \neq \emptyset$, one has $\sum_{k \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{L}}} b_{ik} > 0$. It follows that

$$\begin{aligned} \dot{E}_i &= -\left( \sum_{k \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{L}}} b_{ik} \right) e_i^{\mathrm{T}} e_i + \left( \sum_{k \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{L}}} b_{ik} \right) e_i^{\mathrm{T}} g_i \\ &\leqslant -\left( \sum_{k \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{L}}} b_{ik} \right) e_i^{\mathrm{T}} e_i + \frac{1}{2} \left( \sum_{k \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{L}}} b_{ik} \right) (e_i^{\mathrm{T}} g_i^{\mathrm{T}}) \begin{pmatrix} e_i \\ g_i \end{pmatrix} \\ &= -\frac{1}{2} \left( \sum_{k \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{L}}} b_{ik} \right) (\|e_i\|^2 - \|g_i\|^2). \end{aligned} \tag{32}$$

If $\|e_i\|^2 > \|g_i\|^2$, then $\dot{E}_i < 0$. It thus follows that

$$\limsup_{t \to \infty} \|e_i\| \leqslant \limsup_{t \to \infty} \|g_i\| = \min_{k \in K} \left\{ \limsup_{t \to \infty} \|g_k\| \right\}.$$

Since $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0), \mathcal{V}_{\mathrm{F}})$ is connected, for any $j \in \mathcal{V}_{\mathrm{F}}$ and $j \neq i$, there is a path from node $j$ to node $i$ in $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0), \mathcal{V}_{\mathrm{F}})$. Because $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0), \mathcal{V}_{\mathrm{F}}) \subseteq \mathcal{G}_{\mathcal{I}}(\mathcal{G}(t), \mathcal{V}_{\mathrm{F}})$ for all $t \geqslant 0$, this path is preserved. Suppose that the path is $j, j_1, \ldots, j_p, i$, where $p \leqslant |\mathcal{V}_{\mathrm{F}}| - 2$. It follows that

$$\begin{aligned} \|x_j - \mathrm{co}(\mathcal{V}_{\mathrm{L}})\| &\leqslant \|x_j - x_i\| + \|x_i - \mathrm{co}(\mathcal{V}_{\mathrm{L}})\| \\ &= \|(x_j - x_{j_1}) + (x_{j_1} - x_{j_2}) + \cdots + (x_{j_p} - x_i)\| + \|x_i - \mathrm{co}(\mathcal{V}_{\mathrm{L}})\| \\ &< (|\mathcal{V}_{\mathrm{F}}| - 1) d_1 + \|x_i - \mathrm{co}(\mathcal{V}_{\mathrm{L}})\|. \end{aligned} \tag{33}$$

Therefore, one has

$$\begin{aligned} \limsup_{t \to \infty} \alpha_j &\triangleq \limsup_{t \to \infty} \|x_j - \mathrm{co}(\mathcal{V}_{\mathrm{L}})\| \\ &< (|\mathcal{V}_{\mathrm{F}}| - 1) d_1 + \min_{k \in K} \left\{ \limsup_{t \to \infty} \|g_k\| \right\} \\ &\leqslant (|\mathcal{V}_{\mathrm{F}}| - 1) d_1 + \sigma_s. \end{aligned}$$

### 3.3 Analysis for multiple moving leaders

In this subsection, we consider moving leaders, i.e., $x_k(t)$, $k \in \mathcal{V}_{\mathrm{L}}$, is time varying. We assume that $\|v_k\|$, $k \in \mathcal{V}_{\mathrm{L}}$, is bounded. Define

$$h_i \triangleq \sum_{k \in \mathcal{N}_i(0) \cap \mathcal{V}_{\mathrm{L}}} b_{ik} v_k, \quad i \in V_F, \tag{34}$$

where $v_k$ is given in Eq. (1), and let

$$h \triangleq [h_1^{\mathrm{T}}, h_2^{\mathrm{T}}, \ldots, h_{|\mathcal{V}_{\mathrm{F}}|}^{\mathrm{T}}]^{\mathrm{T}}. \tag{35}$$

In addition, define

$$h_m \triangleq \sqrt{|\mathcal{V}_{\mathrm{F}}|} |\mathcal{V}_{\mathrm{L}}| \max_{i,k} b_{ik} \max_{k \in \mathcal{V}_{\mathrm{L}}} \sup \|v_k\|.$$

The first result in this subsection is stated below.

**Theorem 3.** Use (9) for (2) with moving leaders. If initially $\|x_i(0)-x_j(0)\|<d_1$ for all $(i,j)\in\mathcal{E}(0)$, $i,j\in\mathcal{V}_{\rm F}$, $\|x_i(0)-x_j(0)\|>d_2$ for all $i,j\in\mathcal{V}_{\rm L}\cup\mathcal{V}_{\rm F}$, $i\neq j$, $b_{\min}(0)>1$, and

$$\|v_k(t)\|\leqslant\frac{\sum_{i\in\mathcal{N}_k(t)\cap\mathcal{V}_{\rm F}}\frac{1}{|\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}|}\left(\frac{\partial V}{\partial x_i}\right)\left(\frac{\partial V}{\partial x_i}\right)^{\rm T}}{\|\frac{\partial V}{\partial x_k}\|},\quad t\geqslant 0,\quad k\in V_L, \tag{36}$$

then

(1) for all $t\geqslant 0$, $\|x_i(t)-x_j(t)\|<r_{\rm L}$ for all $(i,j)\in\mathcal{E}(0)$, $i\in\mathcal{V}_{\rm L}$, $j\in\mathcal{V}_{\rm F}$, $\|x_i(t)-x_j(t)\|<d_1$ for all $(i,j)\in\mathcal{E}(0)$, $i,j\in\mathcal{V}_{\rm F}$, and $\|x_i(t)-x_j(t)\|>d_2$ for all $i,j\in\mathcal{V}_{\rm L}\cup\mathcal{V}_{\rm F}$, $i\neq j$;

(2) $\mathcal{N}_i(0)\subseteq\mathcal{N}_i(t)$ for all $i\in\mathcal{V}_{\rm F}$ and $t\geqslant 0$;

(3) $\limsup_{t\to\infty}\alpha(t)\leqslant b_{\min}^{-2}(0)\frac{|\mathcal{V}_{\rm L}|(\max_{i,k}b_{ik})|\mathcal{V}_{\rm F}|\sigma_g^2+h_m^2}{b_{\min}(0)-1}$.

*Proof.* (1) and (2) From (36), one knows that

$$\left\|\frac{\partial V}{\partial x_k}\right\|\|\dot{x}_k(t)\|\leqslant\sum_{i\in\mathcal{N}_k(t)\cap\mathcal{V}_{\rm F}}\frac{1}{|\mathcal{N}_i(t)\cap\mathcal{V}_{\rm L}|}\frac{\partial V}{\partial x_i}\left(\frac{\partial V}{\partial x_i}\right)^{\rm T},\quad t\geqslant 0,\quad k\in V_L,$$

which leads to

$$\begin{aligned}\dot{V}&=\sum_{k\in\mathcal{V}_{\rm L}}\frac{\partial V}{\partial x_k}\dot{x}_k-\sum_{i\in\mathcal{V}_{\rm F}}\frac{\partial V}{\partial x_i}\left(\frac{\partial V}{\partial x_i}\right)^{\rm T}\\&\leqslant\sum_{k\in\mathcal{V}_{\rm L}}\sum_{i\in\mathcal{N}_k\cap\mathcal{V}_{\rm F}}\frac{1}{|\mathcal{N}_i\cap\mathcal{V}_{\rm L}|}\frac{\partial V}{\partial x_i}\left(\frac{\partial V}{\partial x_i}\right)^{\rm T}-\sum_{i\in\mathcal{V}_{\rm F}}\frac{\partial V}{\partial x_i}\left(\frac{\partial V}{\partial x_i}\right)^{\rm T}\\&\leqslant 0.\end{aligned}$$

By using the same arguments as those in the proof of Theorem 1, the first two parts of this theorem can be proved.

(3) Define $E$ as in (22). Because

$$\dot{e}_i=\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}}b_{ik}(\dot{x}_i-\dot{x}_k)=\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}}b_{ik}\,(g_i-e_i-v_k),$$

one has

$$\dot{E}=\sum_{i\in\mathcal{V}_{\rm F}}-\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}}b_{ik}\right)e_i^{\rm T}e_i+\sum_{i\in\mathcal{V}_{\rm F}}\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}}b_{ik}\right)e_i^{\rm T}g_i-\sum_{i\in\mathcal{V}_{\rm F}}e_i^{\rm T}\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}}b_{ik}v_k\right).$$

It thus follows that

$$\begin{aligned}\dot{E}&=\sum_{i\in\mathcal{V}_{\rm F}}\left(-\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}}b_{ik}\right)e_i^{\rm T}e_i+\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}}b_{ik}\right)e_i^{\rm T}g_i-e_i^{\rm T}h_i\right)\\&\leqslant\sum_{i\in\mathcal{V}_{\rm F}}-\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}}b_{ik}\right)e_i^{\rm T}e_i+\sum_{i\in\mathcal{V}_{\rm F}}\frac{1}{2}\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}}b_{ik}\right)(e_i^{\rm T}g_i^{\rm T})\begin{pmatrix}e_i\\g_i\end{pmatrix}\\&\quad+\sum_{i\in\mathcal{V}_{\rm F}}\frac{1}{2}(e_i^{\rm T}h_i^{\rm T})\begin{pmatrix}e_i\\h_i\end{pmatrix}\\&=\frac{1}{2}\left(-e^{\rm T}(B\otimes I_2-I_{2|\mathcal{V}_{\rm F}|})e+g^{\rm T}(B\otimes I_2)g+h^{\rm T}h\right)\\&\leqslant\frac{1}{2}\left(-(b_{\min}-1)\|e\|^2+b_{\max}\|g\|^2+\|h\|^2\right).\end{aligned}\tag{37}$$

From (37), one knows that if $b_{\min}>1$ and

$$\|e\|^2>\frac{b_{\max}\|g\|^2+\|h\|^2}{b_{\min}-1},$$

then $\dot{E} < 0$. Therefore,

$$\limsup_{t\to\infty} \|e\|^2 \leqslant \limsup_{t\to\infty} \left(\frac{b_{\max}\|g\|^2 + \|h\|^2}{b_{\min} - 1}\right).$$

It follows from Lemma 3 that

$$\limsup_{t\to\infty} \alpha \leqslant \limsup_{t\to\infty} \left(b_{\min}^{-2} \frac{b_{\max}\|g\|^2 + \|h\|^2}{b_{\min} - 1}\right). \tag{38}$$

We can show that

$$\|h_i\| \leqslant |\mathcal{V}_{\rm L}| \max_{i,k} b_{ik} \max_{k\in\mathcal{V}_{\rm L}} \sup \|v_k\|,$$

which leads to

$$\|h\| \leqslant h_m.$$

Therefore, from (38), one can conclude

$$\limsup_{t\to\infty} \alpha \leqslant b_{\min}^{-2} \frac{b_{\max}|\mathcal{V}_{\rm F}|\sigma_s^2 + h_m^2}{b_{\min} - 1}. \tag{39}$$

**Remark 4.** (1) The condition (36) implies that, to achieve containment control, the leaders cannot move arbitrarily fast. How fast a leader can move is determined by the velocities of its followers. If the leaders are cooperative (move according to the velocities of its neighboring followers) and the followers can transmit its velocities to their neighboring leaders, then Eq. (36) can be satisfied. Even if there exists communication failure at some time instants, the leaders can stop and wait until they receive the velocity information from neighboring followers.

(2) Compared with Theorem 1, containment control for moving leaders is more difficult because it requires an additional condition that $b_{\min} > 1$. This means that the coupling strength between each follower and the leaders at time 0 should be larger than 1. At the same time, the bound on $\alpha$ is larger than that for the stationary leaders case as indicated by (39).

(3) For the system with moving leaders, the bound on $\alpha$ is also related to the velocities of the leaders. The larger $h_i$, $i \in \mathcal{V}_{\rm L}$, the larger the bound.

By arguments similar to those in the proof of Theorem 2, the condition $b_{\min} > 1$ can be relaxed by assuming the connectivity of $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0), \mathcal{V}_{\rm F})$, where $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0), \mathcal{V}_{\rm F})$ is the induced subgraph of $\mathcal{G}(0)$.

**Theorem 4.** Use (9) for the system (2) with moving leaders. Assume that $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0), \mathcal{V}_{\rm F})$ is connected and there exists a follower $i$, which satisfies $(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} b_{ik}) - 1 > 0$. If Eq. (36) holds, then

(1) for all $t \geqslant 0$, $\|x_i(t) - x_j(t)\| < r_{\rm L}$ for all $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_{\rm L}$, $j \in \mathcal{V}_{\rm F}$, $\|x_i(t) - x_j(t)\| < d_1$ for all $(i,j) \in \mathcal{E}(0)$, $i,j \in \mathcal{V}_{\rm F}$, and $\|x_i(t) - x_j(t)\| > d_2$ for all $i,j \in \mathcal{V}_{\rm L} \cup \mathcal{V}_{\rm F}$, $i \neq j$;

(2) $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(t), \mathcal{V}_{\rm F})$ is connected for all $t \geqslant 0$;

(3) for any $j \in \mathcal{V}_{\rm F}$,

$$\limsup_{t\to\infty} \alpha_j \leqslant \sqrt{\frac{\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} b_{ik}\right)\sigma_s^2 + h_m^2}{\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} b_{ik}\right) - 1}} + (|V_F| - 1)d_1.$$

*Proof.* The theorem can be proved by arguments similar to those in the proof of Theorem 2.

## 4 A nonsmooth control algorithm

In the last section, condition (36) is imposed for the moving leaders case. This condition implies that to guarantee containment control, the leaders should not move too fast. How fast the leaders can move depends on how fast the followers move. This is another form of the stop-and-go policy, which might not be desirable in some scenarios. In the following, we employ a nonsmooth control algorithm to relax the condition (36). The new control law is designed to be

$$u_i = -\left(\frac{\partial V}{\partial x_i}\right)^{\mathrm{T}} - \beta \mathrm{sgn}\left[\left(\frac{\partial V}{\partial x_i}\right)^{\mathrm{T}}\right], \quad i \in \mathcal{V}_{\mathrm{F}}, \tag{40}$$

for $i \in \mathcal{V}_{\mathrm{F}}$, where $\mathrm{sgn}(\cdot)$ is the signum function defined entrywise.

**Theorem 5.** Use (40) for the system (2) with moving leaders. If the conditions of Theorem 1 hold, $b_{\min} > 1$, and $\beta \geqslant \max_{k\in\mathcal{V}_{\mathrm{L}}} \sup \|v_k\|$, then

(1) for all $t \geqslant 0$, $\|x_i(t) - x_j(t)\| < r_{\mathrm{L}}$ for all $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_{\mathrm{L}}$, $j \in \mathcal{V}_{\mathrm{F}}$, $\|x_i(t) - x_j(t)\| < d_1$ for all $(i,j) \in \mathcal{E}(0)$, $i,j \in \mathcal{V}_{\mathrm{F}}$, and $\|x_i(t) - x_j(t)\| > d_2$ for all $i,j \in \mathcal{V}_{\mathrm{L}} \cup \mathcal{V}_{\mathrm{F}}$, $i \neq j$;

(2) $\mathcal{N}_i(0) \subseteq \mathcal{N}_i(t)$ for all $i \in \mathcal{V}_{\mathrm{F}}$ and $t \geqslant 0$;

(3) $\limsup_{t\to\infty} \alpha \leqslant b_{\min}^{-2} \frac{b_{\max}|\mathcal{V}_{\mathrm{F}}|(\sigma_s^2+\beta^2)+h_m^2}{b_{\min}-1}$.

*Proof.* (1) and (2) Let $V_{ij}$ be the sum of the coupling terms between $x_i$ and $x_j$ in $V$. It follows that

$$\sum_{i\in\mathcal{V}_{\mathrm{F}}} \frac{\partial V}{\partial x_i} = \sum_{i\in\mathcal{V}_{\mathrm{F}}} \sum_{j\in\mathcal{V}_{\mathrm{L}}\cup\mathcal{V}_{\mathrm{F}}} \frac{\partial V_{ij}}{\partial x_i} = \sum_{i\in\mathcal{V}_{\mathrm{F}}} \sum_{j\in\mathcal{V}_{\mathrm{F}}} \frac{\partial V_{ij}}{\partial x_i} + \sum_{i\in\mathcal{V}_{\mathrm{F}}} \sum_{j\in\mathcal{V}_{\mathrm{L}}} \frac{\partial V_{ij}}{\partial x_i}.$$

It is straightforward to verify that

$$\sum_{i\in\mathcal{V}_{\mathrm{F}}} \sum_{j\in\mathcal{V}_{\mathrm{F}}} \frac{\partial V_{ij}}{\partial x_i} = 0, \tag{41}$$

which yields

$$\sum_{i\in\mathcal{V}_{\mathrm{F}}} \frac{\partial V}{\partial x_i} = \sum_{i\in\mathcal{V}_{\mathrm{F}}} \sum_{j\in\mathcal{V}_{\mathrm{L}}} \frac{\partial V_{ij}}{\partial x_i} = -\sum_{i\in\mathcal{V}_{\mathrm{F}}} \sum_{j\in\mathcal{V}_{\mathrm{L}}} \frac{\partial V_{ij}}{\partial x_j} = -\sum_{j\in\mathcal{V}_{\mathrm{L}}} \frac{\partial V}{\partial x_j}. \tag{42}$$

Using (42), one has

$$\begin{aligned} \dot{V} &= \sum_{k\in\mathcal{V}_{\mathrm{L}}} \left(\frac{\partial V}{\partial x_k}\right) \dot{x}_k + \sum_{i\in\mathcal{V}_{\mathrm{F}}} \left(\frac{\partial V}{\partial x_i}\right) \dot{x}_i \\ &\leqslant \max_{k\in\mathcal{V}_{\mathrm{L}}} \sup \|v_k\| \left\|\sum_{i\in\mathcal{V}_{\mathrm{F}}} \frac{\partial V}{\partial x_i}\right\| - \sum_{i\in\mathcal{V}_{\mathrm{F}}} \left(\frac{\partial V}{\partial x_i}\right)\left(\frac{\partial V}{\partial x_i}\right)^{\mathrm{T}} - \beta \left\|\sum_{i\in\mathcal{V}_{\mathrm{F}}} \frac{\partial V}{\partial x_i}\right\|. \end{aligned} \tag{43}$$

If $\beta \geqslant \max_{k\in\mathcal{V}_{\mathrm{L}}} \sup \|v_k\|$, then $\dot{V} \leqslant 0$. The first two parts of this theorem are proved.

(3) For system (40), one has

$$\begin{aligned} \dot{e}_i &= \sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}(\dot{x}_i - \dot{x}_k) \\ &= \sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\mathrm{L}}} b_{ik}(g_i - e_i - \beta\mathrm{sgn}(e_i - g_i) - \dot{x}_k), \end{aligned}$$

which yields

$$\dot{E} \leqslant \frac{1}{2}(-(b_{\min}-1)\|e\|^2 + b_{\max}\|g - \beta\mathrm{sgn}(e-g)\|^2 + \|h\|^2).$$

Thus, one can conclude that

$$\begin{aligned} \limsup_{t\to\infty} \alpha &\leqslant \limsup_{t\to\infty} b_{\min}^{-2} \frac{b_{\max}\|g - \beta\mathrm{sgn}(e-g)\|^2 + \|h\|^2}{b_{\min}-1} \\ &\leqslant b_{\min}^{-2} \frac{b_{\max}|\mathcal{V}_{\mathrm{F}}|(\sigma_s^2 + 2\beta^2) + h_m^2}{b_{\min}-1}. \end{aligned}$$

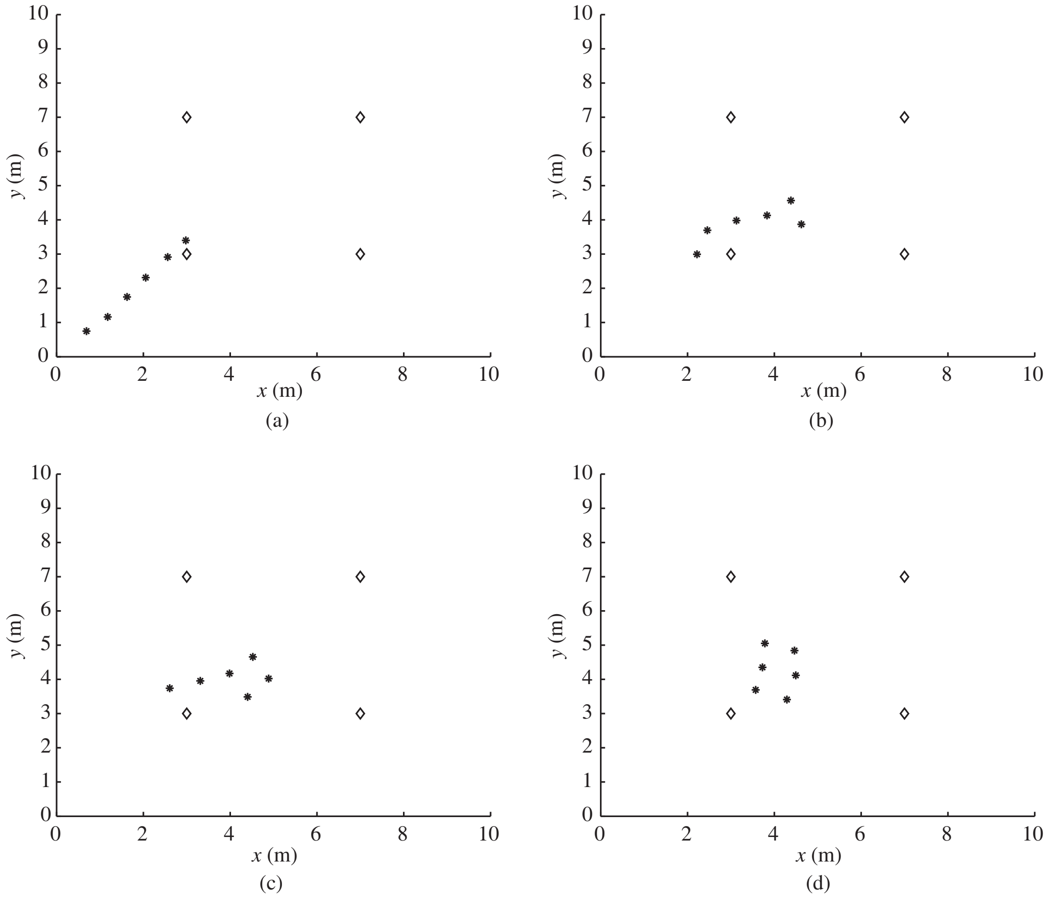

**Figure 2** Snapshots for the stationary leaders case. The leaders are denoted by "◇", while the followers are denoted by "*". The parameters are specified as follows: $r_{\rm L} = 5$, $r_{\rm F} = 1$, $d_1 = 0.8$, $d_2 = 0.4$, $a_2 = -1$, $a_1 = a_2(d_2^2 - r_{\rm F}^2)$, $b_{ik} = 1$ for all $i \in \mathcal{V}_{\rm F}$ and $k \in (\mathcal{N}_i \cap \mathcal{V}_{\rm L})$. (a) $t = 0$ s; (b) $t = 0.25$ s; (c) $t = 0.75$ s; (d) $t = 1$ s.

**Theorem 6.** Use (40) for the system (2) with moving leaders. Assume that $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(0), \mathcal{V}_{\rm F})$ is connected and there exists a follower $i$ satisfying $(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} b_{ik}) - 1 > 0$. If $\beta \geqslant \max_{k\in\mathcal{V}_{\rm L}} \sup \|v_k\|$, then

(1) for all $t \geqslant 0$, $\|x_i(t) - x_j(t)\| < r_{\rm L}$ for all $(i, j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_{\rm L}$, $j \in \mathcal{V}_{\rm F}$, $\|x_i(t) - x_j(t)\| < d_1$ for all $(i, j) \in \mathcal{E}(0)$, $i, j \in \mathcal{V}_{\rm F}$, and $\|x_i(t) - x_j(t)\| > d_2$ for all $i, j \in \mathcal{V}_{\rm L} \cup \mathcal{V}_{\rm F}$, $i \neq j$;

(2) $\mathcal{G}_{\mathcal{I}}(\mathcal{G}(t), \mathcal{V}_{\rm F})$ is connected for all $t \geqslant 0$;

(3) for any $j \in \mathcal{V}_{\rm F}$,

$$\limsup_{t\to\infty} \alpha_j \leqslant \sqrt{\frac{\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} b_{ik}\right)(\sigma_s^2 + 2\beta^2) + h_m^2}{\left(\sum_{k\in\mathcal{N}_i(0)\cap\mathcal{V}_{\rm L}} b_{ik}\right) - 1}} + (|V_{\rm F}| - 1)d_1.$$

*Proof.* The theorem can be proved by arguments similar to those in the proof of Theorem 2.

## 5 Simulation results

### 5.1 Example 1: stationary leaders

We first present a numerical example for stationary leaders. The example includes 4 leaders and 6 followers. Figure 2 shows the snapshots of the movements. Initially, the states of the agents are generated in such a way that $\|x_i(0) - x_j(0)\| < d_1$ for all $(i, j) \in \mathcal{E}(0)$, $i, j \in \mathcal{V}_{\rm F}$, and $\|x_i(0) - x_j(0)\| > d_2$ for all $i, j \in \mathcal{V}_{\rm L} \cup \mathcal{V}_{\rm F}$, $i \neq j$. As can be seen in Figure 2, all followers move toward the convex hull of the leaders

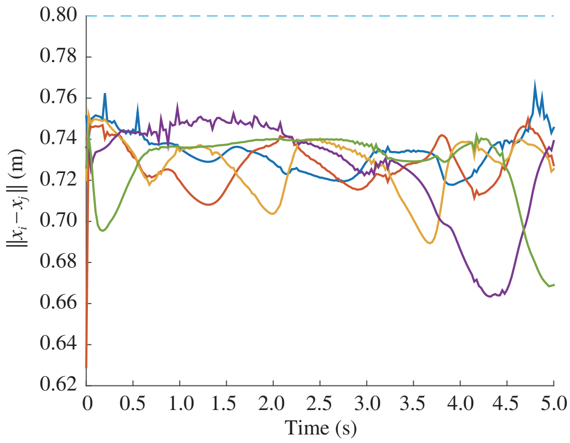


**Figure 3** (Color online) Stationary leaders: the trajectory of $\|x_i - x_j\|$ for $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_{\mathrm{F}}$ and $j \in \mathcal{V}_{\mathrm{F}}$ with the dashed line $\|x_i - x_j\| = d_1$.

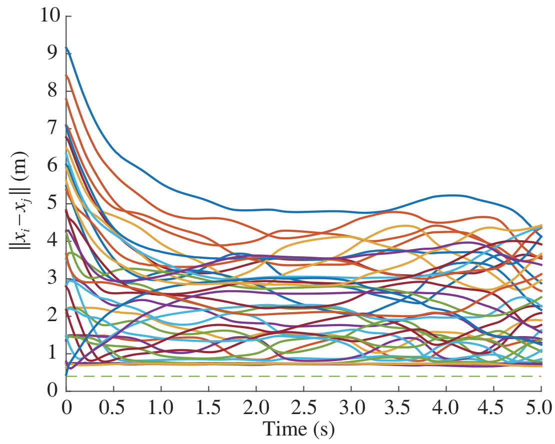


**Figure 4** (Color online) Stationary leaders: the trajectory of $\|x_i - x_j\|$ for all $i,j \in \mathcal{V}_{\mathrm{L}} \cup \mathcal{V}_{\mathrm{F}}$ and $i \neq j$ with the dashed line $\|x_i - x_j\| = d_2$.

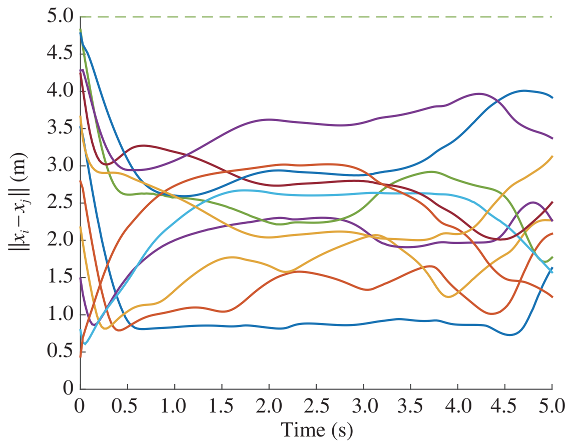


**Figure 5** (Color online) Stationary leaders: the trajectory of $\|x_i - x_j\|$ for $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_{\mathrm{L}}$, and $j \in \mathcal{V}_{\mathrm{F}}$ with the dashed line $\|x_i - x_j\| = r_{\mathrm{L}}$.

while avoiding collision. Figure 3 gives the trajectory of $\|x_i - x_j\|$ for $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_{\mathrm{F}}$ and $j \in \mathcal{V}_{\mathrm{F}}$. The dashed line is the curve of $\|x_i - x_j\| = d_1$. It is clear in Figure 3 that $\|x_i - x_j\| < d_1$ for all $t \geqslant 0$. Figure 4 shows the trajectory of $\|x_i - x_j\|$ for all $i,j \in \mathcal{V}_{\mathrm{L}} \cup \mathcal{V}_{\mathrm{F}}$ and $i \neq j$. The dashed line is the curve of $\|x_i - x_j\| = d_2$. It can be observed that $\|x_i - x_j\| > d_2$ for all $i,j \in \mathcal{V}_{\mathrm{L}} \cup \mathcal{V}_{\mathrm{F}}$, $i \neq j$ and $t \geqslant 0$. Figure 5 gives the trajectory of $\|x_i - x_j\|$ for $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_{\mathrm{L}}$ and $j \in \mathcal{V}_{\mathrm{F}}$. The dashed line is the curve of $\|x_i - x_j\| = r_{\mathrm{L}}$. It is clear in Figure 5 that $\|x_i - x_j\| < r_{\mathrm{L}}$ for $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_{\mathrm{L}}$, $j \in \mathcal{V}_{\mathrm{F}}$ and $t \geqslant 0$.

### 5.2 Example 2: moving leaders

The second example involves moving leaders. The parameters are set to the same values as those in the stationary leader case except that $r_{\mathrm{L}} = 8$. The reason for enlarging $r_{\mathrm{L}}$ is to ensure that every leader has a neighboring follower initially. The leaders move with the same speed. In particular, $v_k = [1,1]^{\mathrm{T}}$, $k \in \mathcal{V}_{\mathrm{L}}$. As can be seen in Figures 6–9, the followers move toward the moving convex hull formed by the leaders while producing swarming behavior, keeping group dispersion, and maintaining connectivity.

## 6 Conclusion

In this paper, we have proposed distributed control strategies to drive a group of follower agents to the

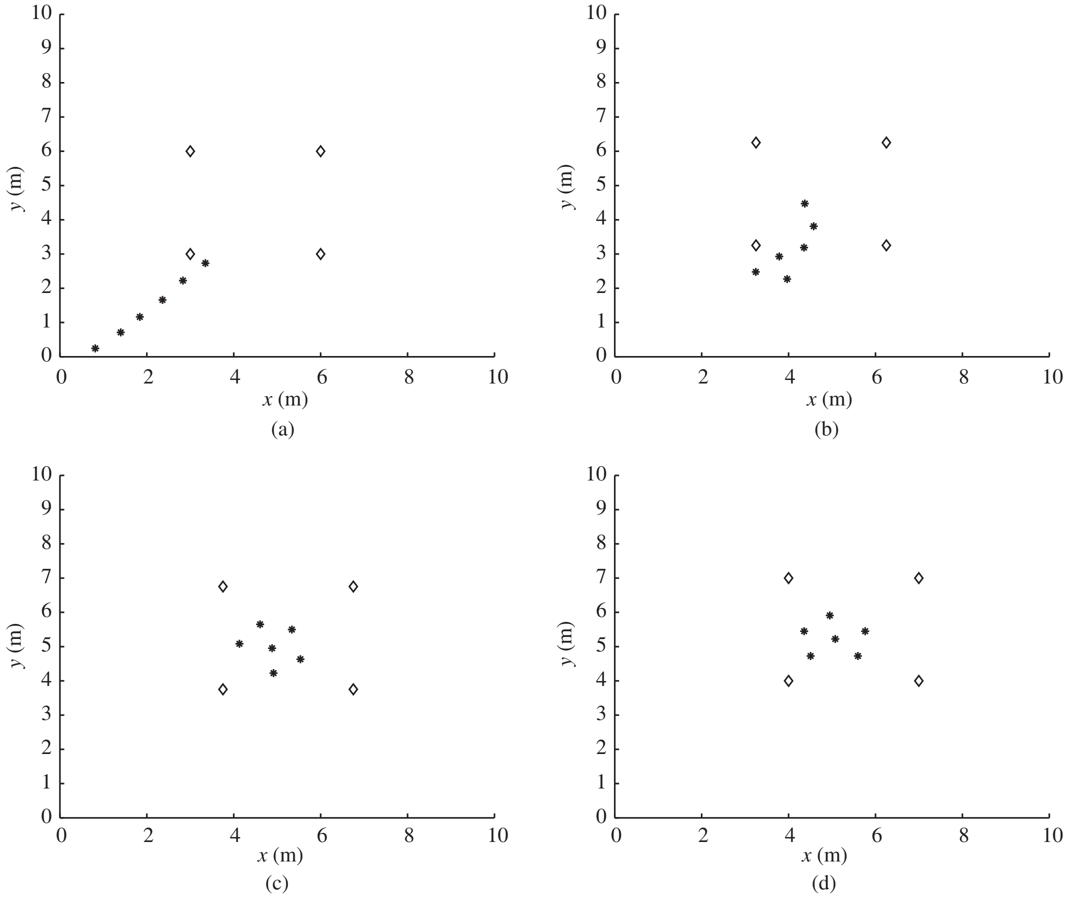


**Figure 6** Snapshots for the moving leaders case. (a) $t = 0$ s; (b) $t = 0.25$ s; (c) $t = 0.75$ s; (d) $t = 1$ s.

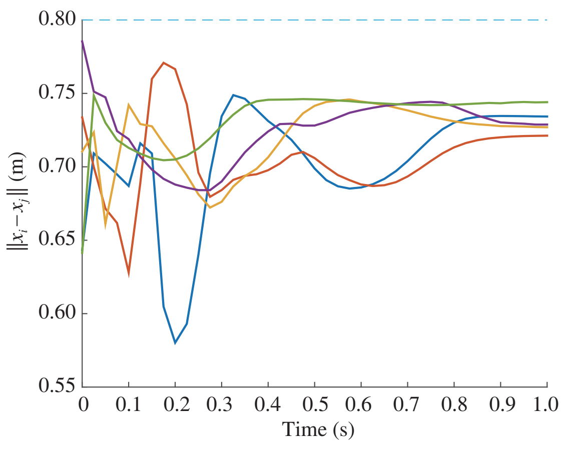


**Figure 7** (Color online) Moving leaders: the trajectory of $\|x_i - x_j\|$ for $(i, j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_\mathrm{F}$ and $j \in \mathcal{V}_\mathrm{F}$ with the dashed line $\|x_i - x_j\| = d_1$.

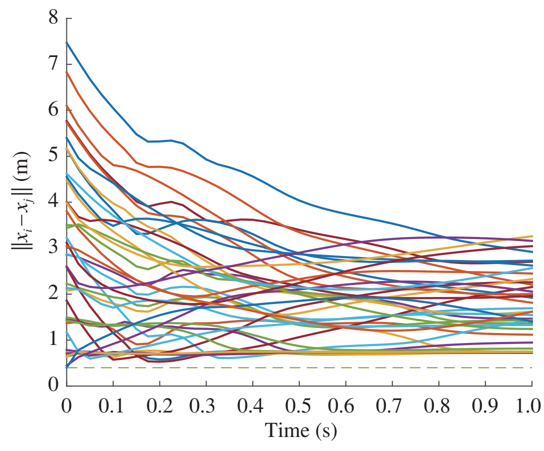


**Figure 8** (Color online) Moving leaders: the trajectory of $\|x_i - x_j\|$ for all $i, j \in \mathcal{V}_\mathrm{L} \cup \mathcal{V}_\mathrm{F}$ and $i \neq j$ with the dashed line $\|x_i - x_j\| = d_2$.

convex hull spanned by some leader agents, while producing swarming behavior and keeping group dispersion. In the stationary leaders case, we have designed a gradient-based continuous control algorithm. We show that, with the continuous algorithm, the control objective can be achieved and the tracking error bound is tunable through some control parameters. We applied the continuous control algorithm to the moving leaders case and showed that the tracking error bound is related to the velocities of the leaders. For the moving leaders case, the continuous algorithm has the restriction of the velocities of the

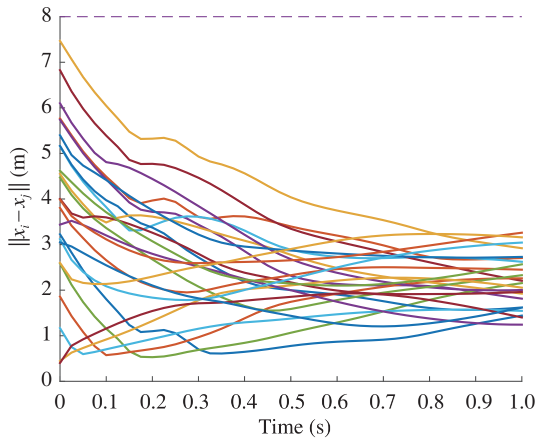


**Figure 9** (Color online) Moving leaders: the trajectory of $\|x_i - x_j\|$ for $(i,j) \in \mathcal{E}(0)$, $i \in \mathcal{V}_{\mathrm{L}}$, and $j \in \mathcal{V}_{\mathrm{F}}$ with the dashed line $\|x_i - x_j\| = r_{\mathrm{L}}$.

leaders being dependent on neighboring followers' velocities, which might not be desirable in some scenarios. Therefore, we proposed a nonsmooth algorithm, which works under the mild assumption on the boundedness of leaders' velocities. In all cases, we show that collision avoidance is guaranteed. Finally, we have provided numerical examples to show the validity of the derived results.

Future works may include: the design of continuous containment control algorithms that are capable of tracking time-varying leaders with bounded velocities, the consideration of the effect of time delay in collision avoidance and connectivity maintenance, the extension to the double-integrator agent model, and the discrete-time implementation of the proposed containment control algorithms.

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**Conflict of interest** The authors declare that they have no conflict of interest.

## References


1 Qu Z. Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles. Berlin: Springer, 2009
2 Bullo F, Cortés J, Martínez S. Distributed Control of Robotic Networks. Princeton: Princeton University Press, 2009
3 Yu W, Wen G, Chen G, et al. Distributed Cooperative Control of Multi-agent Systems. Hoboken: John Wiley & Sons, 2016
4 Chen F, Ren W. Distributed Consensus in Networks. Hoboken: John Wiley & Sons, 2016
5 Ji M, Ferrari-Trecate G, Egerstedt M, et al. Containment control in mobile networks. IEEE Trans Autom Control, 2008, 53: 1972–1975
6 Rui L, Shi Y, Teo K. Coordination arrival control for multi-agent systems. Int J Robust Nonlinear Control, 2016, 26: 1456–1474
7 Engelberger J F. Robotics in Practice: Management and Applications of Industrial Robots. Berlin: Springer, 2012
8 Iyengar S, Brooks R. Distributed Sensor Networks: Sensor Networking and Applications. Boca Raton: CRC Press, 2016
9 Cao Y, Ren W, Egerstedt M. Distributed containment control with multiple stationary or dynamic leaders in fixed and switching directed networks. Automatica, 2012, 48: 1586–1597
10 Li J, Ren W, Xu S. Distributed containment control with multiple dynamic leaders for double-integrator dynamics using only position measurements. IEEE Trans Autom Control, 2012, 57: 1553–1559
11 Wang X, Li S, Shi P. Distributed finite-time containment control for double-integrator multiagent systems. IEEE Trans Cyber, 2014, 44: 1518–1528
12 Mei J, Ren W, Ma G. Distributed containment control for Lagrangian networks with parametric uncertainties under a directed graph. Automatica, 2012, 48: 653–659
13 Meng Z, Lin Z, Ren W. Leader-follower swarm tracking for networked Lagrange systems. Syst Control Lett, 2012, 61: 117–126
14 Hong Y, Chen G, Bushnell L. Distributed observers design for leader-following control of multi-agent networks. Automatica, 2008, 44: 846–850

15 Chen F, Xiang L, Lan W, et al. Coordinated tracking in mean square for a multi-agent system with noisy channels and switching directed network topologies. IEEE Trans Circ Syst II Express Briefs, 2012, 59: 835–839

16 Chen F, Ren W, Cao Y. Surrounding control in cooperative agent networks. Syst Control Lett, 2010, 59: 704–712

17 Shi Y J, Li R, Teo K L. Cooperative enclosing control for multiple moving targets by a group of agents. Int J Control, 2015, 88: 80–89

18 Chen F, Ren W, Lin Z L. Multi-agent coordination with cohesion, dispersion, and containment control. In: Proceedings of American Control Conference, Baltimore, 2010

19 Lalish E, Morgansen K A, Tsukamaki T. Decentralized reactive collision avoidance for multiple unicycle-type vehicles. In: Proceedings of American Control Conference, Seattle, 2008

20 Su H, Wang X, Chen G. A connectivity-preserving flocking algorithm for multi-agent systems based only on position measurements. Int J Control, 2009, 82: 1334–1343

21 Ji M, Egerstedt E. Distributed coordination control of multiagent systems while preserving connectedness. IEEE Trans Robot, 2007, 23: 693–703

22 Sabattini L, Secchi C, Chopra N. Distributed control of multirobot systems with global connectivity maintenance. IEEE Trans Robot, 2013, 29: 1326–1332

23 Olfati-Saber R. Flocking for multi-agent dynamic systems: algorithms and theory. IEEE Trans Autom Control, 2006, 51: 401–420

24 Tanner H G, Jadbabaie A, Pappas G J. Flocking in fixed and switching networks. IEEE Trans Autom Control, 2007, 52: 863–868

25 Zavlanos M M, Pappas G J. Distributed connectivity control of mobile networks. IEEE Trans Robot, 2008, 24: 1416–1428

26 Zhan J, Li X. Flocking of multi-agent systems via model predictive control based on position-only measurements. IEEE Trans Ind Inf, 2013, 9: 377–385

27 Zhang H, Cheng Z, Chen G, et al. Model predictive flocking control for second-order multi-agent systems with input constraints. IEEE Trans Circuit Syst I Regul Pap, 2015, 62: 1599–1606