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Convergence analysis of ILC input sequence for underdetermined linear systems

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Dear editor,

Iterative learning control (ILC) is a kind of intelligent control strategy, applied to those systems that could complete a given task over a finite time interval and repeat it again and again. Since introduced in 1984, it has gained fast developments during the past three decades [1]. In order to analyze the tracking performance of ILC, it is conventional to prove that the actual input sequence converges to the desired input along the iteration axis in most literature. This idea is intuitive as the output is driven by the input, thus better input convergence to the desired input leads to better output tracking performance. However, to ensure the convergence of the input sequence, it is usually assumed that the correlation matrix from the input to the output, i.e., the coupling matrix CB, provided that the system is (A, B, C), is of full-column rank [2,3]. This condition means that a unique desired input could be solved according to the desired trajectory. Therefore, algorithms could be designed to find the unique solution. As a consequence of this requirement, the dimension of the outputs should be not less than the dimension of the inputs when a general multi-input-multi-output (MIMO) system is taken into account.

However, in many practical problems, the dimension condition is not satisfied. The system is referred to as a underdetermined system if the dimension of the inputs is larger than that of the outputs in this letter. For such kind of system, few results have been reported on the convergence of the input sequence along the iteration axis. In [4], it was proved that the state error converges to zero for the underdetermined system. In [5,6], convergence of the modified tracking errors, i.e., tracking errors at selected output positions, was given. In short, the reported results mainly focused on the convergence of the tracking error itself or on the state error rather than on the input error.

In this letter, in order to analyze the properties of the input convergence, an update law based on the stochastic Kaczmarz algorithm [7,8] is first introduced for underdetermined linear systems with stochastic noises. Then, the almost sure convergence of the input sequence is strictly proven with the help of stochastic approximation. In addition, a deterministic version of the update law is provided as an extension.

Notations. $\mathbb{R}$ denotes the set of real numbers. $H \in \mathbb{R}^{m\times n}$ denotes a matrix, while H_i denotes the transpose of its ith row. $\mathcal{R}_H$ is the row space of H. A superscript T denotes a transpose. Pr(event) denotes the occurrence probability of its indicated event. E denotes the mathematical expectation. For a vector $x = [x_1, \ldots, x_n]^{\mathrm{T}}$, $\|\cdot\|$ denotes its Euclidean norm, i.e., $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Problem formulation. Consider the following

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lifted linear system

$$Y_k = HU_k + Y_k^0 + \xi_k, \tag{1}$$

where k is the iteration index, $U_k \in \mathbb{R}^n$ and $Y_k \in \mathbb{R}^m$ denote the input and output at the kth iteration, respectively. $H \in \mathbb{R}^{m\times n}$ denotes the transfer matrix, where $m < n$ is assumed throughout this letter, which means that the dimension of the outputs is less than the dimension of the inputs. Y_k^0 is the initial response and ξ_k is the stochastic system noise.

We impose the following assumptions for further analysis. The transfer matrix H is of full-row rank, i.e., $\mathrm{rank}(H) = m$. The initial response Y_k^0 is a random variable such that $\mathrm{E}Y_k^0 = 0$, $\sup_k \mathrm{E}\|Y_k^0\|^2 < \infty$. The noise $\{\xi_k\}$ is an independent and identically distributed sequence of random variables along the iteration axis satisfying that $\mathrm{E}\xi_k = 0$ and $\sup_k \mathrm{E}\|\xi_k\|^2 < \infty$.

The desired trajectory is denoted by $Y_d \in \mathbb{R}^m$. The control objective of this letter is to find an input sequence $\{U_k\}$ such that $U_k \to U^*$ almost surely with $HU^* = Y_d$. Since the number of rows of H is less than the number of columns, there exist infinite solutions to $HU^* = Y_d$. Thus, a natural optimization problem is to consider $\min_{U\in\mathbb{R}^n} \|U\|$, s.t. $HU = Y_d$. Using the well-known KKT condition, the solution to the optimization problem is $U^* = H^{\mathrm{T}}(HH^{\mathrm{T}})^{-1}Y_d$.

Stochastic version. Denote the tracking error $E_k = Y_d - Y_k$. The ith element of E_k is denoted as $e_{i,k}$, $1 \leqslant i \leqslant m$. Let α_k be a random variable that takes value in the integer set $\{1, 2, \ldots, m\}$ with arbitrary probability distribution. Denote the probability of each case as $\Pr(\alpha_k = i) = \lambda_i$, $1 \leqslant i \leqslant m$. Besides, $\theta_i \in \mathbb{R}^m$ is defined as a unit vector with its ith position being occupied by 1 and 0 elsewhere. Without loss of generality, it is assumed that the rows of H are of unit norm. The stochastic update law is

$$U_{k+1} = U_k + a_k e_{\alpha_k,k} H_{\alpha_k}, \tag{2}$$

where $e_{\alpha_k,k}$ denotes the α_k-th entry of the tracking error E_k, while H_{α_k} denotes the transpose of the α_k-th row of the matrix H. Moreover, a_k is a learning stepsize satisfying that $\sum_{k=1}^{\infty} a_k = \infty$ and $\sum_{k=1}^{\infty} a_k^2 < \infty$. From (2) one has

$$U_{k+1} = U_k + a_k\theta_{\alpha_k}^{\mathrm{T}}(Y_d - HU_k)H_{\alpha_k} - a_k\theta_{\alpha_k}^{\mathrm{T}}\xi_k H_{\alpha_k}.$$

Set initial input $U_1 = 0$, then it is obvious that U_k belongs to the space $\mathcal{R}_H$, $\forall k$. Thus there is a unique vector ϕ_k such that $U_k = H^{\mathrm{T}}\phi_k$. Then, from the recursion of U_k, we can derive a recursion of ϕ_k as

$$\begin{aligned}\phi_{k+1} =& \phi_k + a_k\Gamma(Y_d - HH^{\mathrm{T}}\phi_k) - a_k M_{\alpha_k}\xi_k \\ &+ a_k(M_{\alpha_k} - \Gamma)(Y_d - HH^{\mathrm{T}}\phi_k),\end{aligned} \tag{3}$$

where $M_{\alpha_k} \in \mathbb{R}^{m\times m}$ is a matrix with 1 in its α_k-th diagonal position and zero elsewhere, and $\Gamma \triangleq \mathrm{E}M_{\alpha_k} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$.

Lemma 1. For recursion (3), ϕ_k converges to ϕ_d almost surely as k approaches to infinity, where $\phi_d = (HH^{\mathrm{T}})^{-1}Y_d$.

Theorem 1. For underdetermined linear system (1), the input U_k generated by update law (2) converges almost surely to U^* that solves the optimization problem.

Corollary 1. Apply update law (2) to the underdetermined linear system (1), then the corresponding output Y_k minimizes the averaged index $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\|Y_d - Y_k\|^2$. In other words, if we eliminate the current noise from the tracking error E_k and denote $\widehat{E}_k = Y_d - HU_k$, then we almost surely have $\widehat{E}_k \to 0$.

Deterministic version. As an analogue to the stochastic version, if we have full information of the tracking error, we may use the following deterministic update law:

$$U_{k+1} = U_k + a_k\frac{1}{m}\left(e_{1,k}H_1 + \cdots + e_{m,k}H_m\right), \tag{4}$$

where $e_{i,k}$ denotes the ith entry of tracking error E_k and H_i denotes the transpose of the ith row of the matrix H. Following basic calculations, one has $U_{k+1} = U_k + \frac{a_k}{m}H^{\mathrm{T}}E_k$. Similarly to the stochastic version, we can derive a recursion of ϕ_k,

$$\phi_{k+1} = \phi_k + \frac{a_k}{m}(Y_d - HH^{\mathrm{T}}\phi_k) - \frac{a_k}{m}\xi_k. \tag{5}$$

Lemma 2. For recursion (5), ϕ_k converges to ϕ_d almost surely as k approaches to infinity, where $\phi_d = (HH^{\mathrm{T}})^{-1}Y_d$.

Theorem 2. For underdetermined linear system (1), the input U_k generated by update law (4) converges almost surely to U^* that solves the optimization problem.

Corollary 2. Apply update law (4) to the underdetermined linear system (1), then the corresponding output Y_k minimizes the averaged index $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\|Y_d - Y_k\|^2$. In other words, if we eliminate the current noise from the tracking error E_k and denote $\widehat{E}_k = Y_d - HU_k$, then we almost surely have $\widehat{E}_k \to 0$.

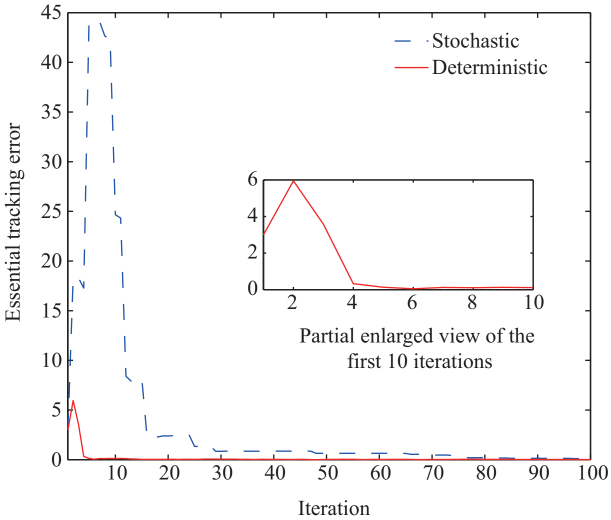

Figure 1 (Color online) Essential tracking error $\|Y_d - \Phi G U_k\|$.

Illustrative simulations. We consider a linear system $Y_k = GU_k + \eta_k^0 + \epsilon_k$, where $Y_k \in \mathbb{R}^{12}$ is the output, $U_k \in \mathbb{R}^{18}$ the input, η_k^0 the initial response, and ϵ_k the stochastic noise. In the simulation, we simply let $\eta_k^0 = 0$. The expression of $G \in \mathbb{R}^{12\times 18}$ is given in the supplementary file. We consider a point-to-point tracking problem, that is, suppose that only the points at the 2nd, 4th, 5th, and 12th dimensions of Y_k are required to track. Y_k^{ob} denotes the vector stacked by these four points, $Y_k^{ob} \in \mathbb{R}^4$. To describe the relationship between Y_k^{ob} and Y_k, a matrix Φ is introduced with $\Phi_{1,2}$, $\Phi_{2,4}$, $\Phi_{3,5}$, and $\Phi_{4,12}$ being 1 and other elements being 0. Then, we have the following formulation:

$$Y_k^{ob} = \Phi G U_k + \Phi \epsilon_k.$$

Note that $\Phi G \in \mathbb{R}^{4\times 18}$, thus the underdetermined linear system (1) has been obtained. The arbitrary given reference trajectory is $Y_d = [3\ 2\ 1.5\ 1]^{\mathrm{T}}$. The initial input U_1 is simply set zero. The learning step is $a_k = 6.5/k$.

The convergence property for tracking error is shown in Figure 1, where the essential tracking error is defined as $\|Y_d - \Phi G U_k\|$ to eliminate the influence of stochastic noises and to display the inherent tracking capacity of the proposed learning algorithms. To allow for a clear comparison, the error of deterministic version for the first ten iterations is zoomed out as a subplot. As can be seen, both algorithms would have ideal convergence. However, the stochastic version generates a distinctly larger error during the first several iterations. One conceivable reason is that the stochastic version loses more information, which results in larger errors for the points that are not selected during learning. It should be pointed out that the stochastic version would save many computation efforts and we believe this is a major advantage of the stochastic update law. In addition, all inputs converges to the corresponding values of U^* that solves the optimization problem. This performance is shown in the supplementary file.

Conclusion. In this letter, the convergence properties of ILC input sequence for the underdetermined linear stochastic system are discussed. With the help of Kaczmarz algorithm, a stochastic version and a deterministic version of the update algorithms are proposed. The almost sure convergence of the input sequence is strictly proved and then verified by illustrative simulations. In addition, it is also observed that more information used in the deterministic update algorithm results in better performance. However, the stochastic version has its significance since it may be difficult to acquire full information under some harsh situations such as data dropouts and data transmission delays.

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