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Stability analysis of golden-section adaptive control systems based on the characteristic model

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Abstract All-coefficient adaptive control theory and method based on characteristic models have already been applied successfully in the fields of astronautics and industry. However, the stability analysis of the characteristic model-based golden-section adaptive control systems is still an open question in both theory and practice. To investigate such stability issues, the author first provides a method for choosing initial parameter values and new performances for a projection algorithm with dead zone for adaptive parameter estimation, and develops some properties of time-varying matrices by utilizing some algebraic techniques. And then a new Lyapunov function with logarithmic form for time-varying discrete systems is constructed. Finally, the author transforms the characteristic models of some multi-input and multi-output (MIMO) controlled systems into their equivalent form, and proves the stability of the closed-loop systems formed by the golden-section adaptive control law based on the characteristic model using mathematical techniques.

Keywords characteristic model, golden-section control law, stability, time-varying system, nonlinear system

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1 Introduction

Substantial theoretical progress has been made in the area of adaptive control since the first design of autopilots for high performance aircrafts in the 1950s [1]. Despite the various applications, limitations of adaptive control system still exist, such as the poor transient response, difficulties in guaranteeing the convergence of the parameter estimation due to measurement inaccuracies, disturbances. Current modeling and control theory are based on accurate dynamics analysis and mathematical description, and the modeling and the control requirements are considered separately, which leads to the above mentioned problems.

In order to overcome the aforementioned issues, an all-coefficient adaptive control method based on characteristic model was proposed by Wu [2, 3] from the viewpoint of engineering applications. In the last 20 years, this theory has been successfully applied to more than 400 systems belonging to 10 kinds of engineering plants in the fields of astronautics and industry. It is worth mentioning that its application

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to the reentry adaptive control of a manned spaceship has achieved a parachute-opening point accuracy, which is comparable with the best ones in the world [2, 4, 5]. The uniqueness of this method is that does not depend on accurate dynamic models of controlled objects. It is a simple, adaptable, and robust approach that addresses the prevailing issues in the implementation of adaptive control systems in some way [6]. The key idea of the characteristic model is to use a low-order discrete time-varying system to capture the core dynamics of a high-order nonlinear/linear system based on the main features of the plant and the control demands [7].

This technique consists of three aspects [2]: (1) the all-coefficient adaptive control method, (2) the golden-section adaptive control law, and (3) the characteristic model. It is noted that the golden-section control law is a novel one, which has the prominent features such as simple construction and easy implementation.

This law uses the golden section ratio (0.382/0.618) to design controllers; details can be found in [2–4] or in the next section of this paper. The problem of the stability of the golden-section feedback control systems based on the characteristic model has received considerable attention since the all-coefficient adaptive control theory and method based on characteristic models were proposed.

The robust stability of the golden-section control law for the case of a second order single-input and single-output (SISO) linear invariant system was proved by Xie et al. [8]. The sufficient conditions for the stability of closed-loop systems formed by the characteristic model-based golden-section control law were provided by Qi et al. [9], Sun and Wu [10], and Sun [11], for SISO and 3-input-3-output linear time-invariant systems, respectively. However, it is difficult to verify these sufficient conditions in real systems. A stability analysis framework for adaptive control based on the characteristic model was proposed by Wang [12] after investigating the minimum phase with exponential stability and SISO nonlinear system with second relative degree. By using the stability results of the generalized least square control system and the Jury stability criteria, Meng et al. [13] investigated the properties of the closed-loop control system based on the golden-section control law for the characteristic model of a second-order linear time-invariant continuous SISO system. Based on the stability result of the generalized least-square control system [14] and the stability theory of matrix polynomial [15], Sun et al. [16] proved the asymptotic stability of the closed-loop system involving the characteristic model-based golden-section control law for a multi-input and multi-output (MIMO) minimum phase linear system. The coefficients of the characteristic models discussed in [13, 16] are constants, and this type of characteristic model may be merely suitable for second-order linear time-invariant continuous systems. However, the coefficients of the characteristic models are generally time-varying.

In summary, the stability of the characteristic model based golden-section adaptive control system has not yet been thoroughly understood due to two main factors, i.e., the time-varying nature of the coefficients of the characteristic models, and the limitations in the parameter estimation theory from the closed-loop system stability analysis viewpoint. The stability analysis, especially for MIMO systems, remains a challenging problem for all-coefficient adaptive control theory despite its successes in practice.

In this paper, we address the above-mentioned difficulties and prove the stability of the closed-loop systems formed by the characteristic model-based golden-section adaptive control law for MIMO systems with small sampling period.

The rest of the paper is organized as follows. Section 2 describes the characteristic model and the use of discrete orthogonal polynomials to approximate its time-varying coefficients. Furthermore, we provide a golden-section control law based on the characteristic model. The main result is presented in Section 3. In Section 4, we list the various auxiliary results used to proof of our main theorem. We provide a numerical example in Section 5. Section 6 concludes the paper with a brief summary of our contributions.

Throughout this paper, the notations used are mostly standard. I_n and O_n denote the $n \times n$ identity matrix and $n \times n$ zero matrix, respectively. I and O denote the $p \times p$ identity matrix and $p \times p$ zero matrix, respectively. The notation $A \geqslant B$ ($A > B$) indicates that $A-B$ is a positive semi-definite (positive definite) matrix. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent the minimum and maximum eigenvalues of the matrix A, respectively. $\rho(A)$ denotes the spectral radius of the matrix A. $\|\boldsymbol{a}\|$ represents the Euclidean norm of the vector $\boldsymbol{a} = (a_1,\ a_2, \ldots, a_p)^{\mathrm{T}} \in \mathbb{R}^p$, i.e., $\|\boldsymbol{a}\| = \|\boldsymbol{a}\|_2 = \sqrt{\sum_{i=1}^{p} |a_i|^2}$. Unless stated otherwise, $\|A\|$ denotes the Frobenius norm of the matrix $A = (a_{ij})_{p\times p} \in \mathbb{R}^{p\times p}$, i.e., $\|A\| = \|A\|_F = \sqrt{\sum_{i=1}^{p}\sum_{j=1}^{p} |a_{ij}|^2}$. All vectors and matrices are assumed to be compatible for algebraic operations when their dimensions are not explicitly stated.

Further notation will be explained when first introduced.

2 Problem formulation

Characteristic modeling is based on the plant dynamic characteristics and control performance requirements rather than accurate dynamic analysis of the controlled plant alone. The important features of the characteristic model are listed as follows [2, 4, 5, 17, 18].

(1) The output of the characteristic model and that of the practical plant are equivalent for the same input.

(2) The form and order of the characteristic model are mainly dependent on the control performance requirements except the characteristics of the controlled plant.

(3) The form of the characteristic model is simpler than the dynamical equation of the original plant.

(4) The characteristic model is different from the reduced-order model of a high-order system, in which all the information of the high-order model is compressed into several characteristic parameters. Generally, the characteristic model is described by time-varying difference equations.

Consider high-order MIMO linear time-invariant plants and some MIMO nonlinear plants that have a position keeping or tracking control requirement. Assume that the sampling period is less than a certain positive number (which is described quantitatively). Then, their characteristic model can be expressed by the following system of second-order linear time-varying difference equations [4, 5, 7]:

$$\boldsymbol{y}_{k+1} = F_{1,k}\boldsymbol{y}_k + F_{2,k}\boldsymbol{y}_{k-1} + G_k\boldsymbol{u}_k + \boldsymbol{E}_k, \tag{1}$$

where $\boldsymbol{y}_k \in \mathbb{R}^p$ and $\boldsymbol{u}_k \in \mathbb{R}^p$ are the system output and the control input, respectively; $F_{1,k},\ F_{2,k},\ G_k \in \mathbb{R}^{p\times p}$ are the coefficient matrices; $\boldsymbol{E}_k \in \mathbb{R}^p$ is the modeling error vector. And $\boldsymbol{y}_k = \boldsymbol{y}(k) = \boldsymbol{y}(kT),\ \boldsymbol{u}_k = \boldsymbol{u}(k) = \boldsymbol{u}(kT),\ F_{1,k} = F_1(k) = F_1(kT) = (f_{ij}(k))_{p\times p},\ F_{2,k} = F_2(k) = F_2(kT) = (f_{i,p+j}(k))_{p\times p}$, $G_k = G(k) = G(kT) = (g_{ij}(k))_{p\times p},\ \boldsymbol{E}_k = \boldsymbol{E}(k) = \boldsymbol{E}(kT)$. Here T is the sampling period. For simplicity, other time-varying vectors and time-varying matrices are also denoted by similar notations when no conflict is caused.

In engineering practice, characteristic models are often chosen shown in formula (1) [4, 5]. The following formulae give the limit characteristics of the coefficient matrices of the characteristic model (1) [5]:

$$\lim_{T\to 0} F_{1,k} = 2I, \quad \lim_{T\to 0} F_{2,k} = -I, \quad \lim_{T\to 0} G_k = O_p, \tag{2}$$

where I is the $p \times p$ identity matrix. For any prescribed positive constant ε_0, if the sampling period is less than certain positive number, the modeling error vector $\boldsymbol{E}_k = (E_1(k), E_2(k), \ldots, E_p(k))^{\mathrm{T}}$ satisfies $|E_i(k)| < \varepsilon_0$ $(i = 1, 2, \ldots, p)$. During the transient phase, we have $E_j(k) = O(T)$; after the steady state has been reached, $E_j(k) = O(T^2)$. Here, $O(T)$ and $O(T^2)$ denote infinitesimals of the same order for T and T^2, respectively. Formula (2) implies that the sum of all the coefficient matrices of the characteristic model is an identity matrix. This provides priori information for the designer of controller based on the characteristic model.

We consider a set-point control problem in this paper, i.e., our objective is to design an adaptive controller such that the controlled plant output $\boldsymbol{y}(t)$ is a constant vector $\boldsymbol{y}_d^*$ in steady state.

It is well-known that a continuous function defined over a closed interval can be approximated to a prescribed accuracy by a polynomial that consists of a linear combination of a finite number of orthogonal polynomial basis functions. The identification of time-varying parameters can be achieved by identifying the constant coefficients in the linear combinations, and this is an effective method to identify time-varying parameters [19]. We use the following transformation (one-to-one mapping):

$$z_{ijk} = z_{ij}(k) = f_{ij}(k)/[1 + |f_{ij}(k)|], \quad i = 1, 2, \ldots, p; \quad j = 1, 2, \ldots, 2p;$$

$$z_{ijk} = z_{ij}(k) = g_{ij}(k)/[1+|g_{ij}(k)|], \quad i=1,2,\ldots,p; \quad j=2p+1,\ldots,3p.$$

Thus, it is clear that $z_{ijk} \in (-1,1)$ for $k \in [1,+\infty)$, and $f_{ij}(k) = z_{ijk}/[1-|z_{ijk}|]$ can be regarded as a functions of z_{ijk}. This transformation ensures that the unknown functions $f_{ij}(k)$ and $g_{ij}(k)$ are approximated by polynomials defined on a fixed finite interval. We choose one among the Legendre, Chebyshev, Laguerre, and Hermite polynomials for clarity as the basis function denoted by $P_0(z_{ijk})$, $P_1(z_{ijk})$, ... Note that $P_0(z_{ijk}) = 1$ for all these basis functions.

Thus, the time-varying elements $f_{ij}(k)$, $f_{i,\,p+j}(k)$ and $g_{ij}(k)$ of $F_{1,k}, F_{2,k}$ and G_k are given by the following expressions:

$$f_{ij}(k) = \sum_{s=0}^{q} a_{ijs} P_s(z_{ijk}) + \omega_{fij}(z_{ijk}), \quad i=1,2,\ldots,p; \quad j=1,2,\ldots,2p.$$

$$g_{ij}(k) = \sum_{s=0}^{q} b_{ijs} P_s(z_{i,\,2p+j,\,k}) + \omega_{gij}(z_{i,\,2p+j,\,k}), \quad i=1,2,\ldots,p; \quad j=1,2,\ldots,p.$$

Here, $\omega_{fij}(z_{ijk})$ and $\omega_{gij}(z_{i,\,2p+j,k})$ are approximate errors, and they satisfy the inequalities:

$$|\omega_{fij}(z_{ijk})| < \varepsilon/\sqrt{3p}, \quad |\omega_{gij}(z_{i,\,2p+j,\,k})| < \varepsilon/\sqrt{3p}, \tag{3}$$

where ε is a prescribed positive constant.

For the sake of brevity, we assume that the number of required basis functions for approximating each $f_{ij}(k)$ and $g_{ij}(k)$, to achieve the prescribed approximation accuracy in (3), are all of order $q+1$. If the number of required basis functions is only $q_0(<q+1)$ in order to achieve the prescribed approximation accuracy for approximating some of $f_{ij}(k)$ or $g_{ij}(k)$, then we let $a_{ijs} = b_{ijs} = 0$ for $s \geqslant q_0$.

Denote that

$$\bar{f}_{ij}(k) = \sum_{s=0}^{q} a_{ijs} P_s(z_{ijk}), \quad i=1,2,\ldots,p; \quad j=1,2,\ldots,2p.$$

$$\bar{g}_{ij}(k) = \sum_{s=0}^{q} b_{ijs} P_s(z_{i,\,2p+j,\,k}), \quad i=1,2,\ldots,p; \quad j=1,2,\ldots,p.$$

$$\overline{F}_{1,k} = (\bar{f}_{ij}(z_{ijk}))_{p\times p}, \quad \overline{F}_{2,k} = (\bar{f}_{i,p+j}(z_{i,p+j,\,k}))_{p\times p}, \quad \overline{G}_k = (\bar{g}_{ij}(z_{i,\,2p+j,\,k}))_{p\times p},$$

$$\omega_{1F} = (\omega_{fij}(z_{ijk}))_{p\times p}, \quad \omega_{2F} = (\omega_{fi,p+j}(z_{i,p+j,\,k}))_{p\times p}, \quad \omega_G = (\omega_{gij}(z_{i,\,2p+j,\,k}))_{p\times p}. \tag{4}$$

Then, formula (1) can be rewritten as

$$\boldsymbol{y}_{k+1} = \overline{F}_{1,k}\boldsymbol{y}_k + \overline{F}_{2,k}\boldsymbol{y}_{k-1} + \overline{G}_k\boldsymbol{u}_k + \bar{\boldsymbol{\omega}}(k) + \boldsymbol{E}_k, \tag{5}$$

where

$$\bar{\boldsymbol{\omega}}(k) = (\bar{\omega}_1(k), \bar{\omega}_2(k), \ldots, \bar{\omega}_p(k))^{\mathrm{T}},$$

$$\bar{\omega}_i(k) = \boldsymbol{\omega}_i(k)^{\mathrm{T}}\boldsymbol{\phi}(k) = \boldsymbol{\phi}(k)^{\mathrm{T}}\boldsymbol{\omega}_i(k), \quad i=1,2,\ldots,p;$$

$$\begin{aligned}\boldsymbol{\omega}_i(k)^{\mathrm{T}} = {} & (\omega_{fi1}(z_{i1k}), \ldots, \omega_{fip}(z_{ipk}), \omega_{fi,p+1}(z_{i,p+1,\,k}), \ldots, \\ & \omega_{fi,2p}(z_{i,\,2p,k}), \omega_{gi1}(z_{i,\,2p+1,\,k}), \ldots, \omega_{gip}(z_{i,\,3p,\,k})),\end{aligned}$$

$$\boldsymbol{\phi}_k^{\mathrm{T}} = \boldsymbol{\phi}(k)^{\mathrm{T}} = (\boldsymbol{y}(k)^{\mathrm{T}}, \boldsymbol{y}(k-1)^{\mathrm{T}}, \boldsymbol{u}(k)^{\mathrm{T}}). \tag{6}$$

Also denote that

$$\begin{aligned}\boldsymbol{\theta}_i^{\mathrm{T}} = {} & (a_{i10}, \ldots, a_{i1q}, \ldots, a_{ip0}, \ldots, a_{ipq},\ a_{i,p+1,0}, \ldots, a_{i,p+1,q}, \ldots, a_{i,2p,0}, \ldots, \\ & a_{i,2p,q}, b_{i10}, \ldots, b_{i1q}, \ldots, b_{ip0}, \ldots, b_{ipq}),\end{aligned}$$

$$L_i(k) = \mathrm{diag}[\boldsymbol{l}_{i1}(k), \ldots, \boldsymbol{l}_{i,\,3p}(k)]_{[3p\,(q+1)]\times(3p)},$$

$$\boldsymbol{\varphi}_{i,k} = \boldsymbol{\varphi}_i(k) = L_i(k)\boldsymbol{\phi}(k), \tag{7}$$

where $\boldsymbol{l}_{ij}(k) = (P_0(z_{ijk}), \dots, P_q(z_{ijk}))^{\mathrm{T}}$. Thus, we have

$$y_i(k+1) = \boldsymbol{\theta}_i^{\mathrm{T}}\boldsymbol{\varphi}_i(k) + \bar{\omega}_i(k) + E_i(k) = \boldsymbol{\varphi}_i(k)^{\mathrm{T}}\boldsymbol{\theta}_i + \bar{\omega}_i(k) + E_i(k), \quad i = 1, \dots, p. \tag{8}$$

The estimation of $\boldsymbol{\theta}_i$ is denoted by $\hat{\boldsymbol{\theta}}_i(k)$, namely,

$$\begin{aligned}\hat{\boldsymbol{\theta}}_i(k)^{\mathrm{T}} = (&\hat{a}_{i10}(k), \dots, \hat{a}_{i1q}(k), \dots, \hat{a}_{ip0}(k), \dots, \hat{a}_{ipq}(k), \dots, \hat{a}_{i,p+1,0}(k), \dots, \hat{a}_{i,p+1,q}(k),\\ &\hat{a}_{i,2p,0}(k), \dots, \hat{a}_{i,2p,q}(k),\ \hat{b}_{i10}(k), \dots, \hat{b}_{i1q}(k), \dots, \hat{b}_{ip0}(k), \dots, \hat{b}_{ipq}(k)).\end{aligned}$$

Let

$$\hat{f}_{ij}(k) = \sum\nolimits_{s=0}^{q} \hat{a}_{ijs}(k)P_s(z_{ijk}), \quad i = 1, 2, \dots, p; \quad j = 1, 2, \dots, 2p.$$

$$\hat{g}_{ij}(k) = \sum\nolimits_{s=0}^{q} \hat{b}_{ijs}(k)P_s(z_{i,\,2p+j,\,k}), \quad i = 1, 2, \dots, p; \quad j = 1, 2, \dots, p.$$

$$\hat{F}_{1,k} = \hat{F}_1(k) = (\hat{f}_{ij}(k))_{p\times p}, \quad \hat{F}_{2,k} = \hat{F}_2(k) = (\hat{f}_{i,p+j}(k))_{p\times p}, \quad \hat{G}_k = (\hat{g}_{ij}(k))_{p\times p}.$$

Now, we can design the golden-section controller as follows:

$$\boldsymbol{u}_k = \boldsymbol{u}_{0,k} + \boldsymbol{u}_{g,k}, \tag{9}$$

where

$$\boldsymbol{u}_{0,k} = \hat{G}_k^{-1}\left(\boldsymbol{y}_d^* - \hat{F}_{1,k}\boldsymbol{y}_d^* - \hat{F}_{2,k}\boldsymbol{y}_d^*\right), \tag{10}$$

$$\boldsymbol{u}_{g,k} = \hat{G}_k^{-1}\left(L_1\hat{F}_{1,k}\tilde{\boldsymbol{y}}_k + L_2\hat{F}_{2,k}\tilde{\boldsymbol{y}}_{k-1}\right). \tag{11}$$

Here $L_1 = (3-\sqrt{5})/2 \approx 0.382$, $L_2 = (\sqrt{5}-1)/2 \approx 0.618$, $\tilde{\boldsymbol{y}}_k = \boldsymbol{y}_d^* - \boldsymbol{y}_k$, and $\hat{F}_{1,k}, \hat{F}_{2,k}$ and $\hat{G}_k$ are estimations of $\overline{F}_{1,k}, \overline{F}_{2,k}$ and $\overline{G}_k$, respectively.

Comments on a method to overcome the possible singular problem of the controller and the boundedness of $\hat{G}_k^{-1}$ are mentioned in Remark 5.

3 Main result

In order to obtain the controller (9), we must identify the parameters in the characteristics. The parameter vector adaptive laws are chosen using the following projection algorithm with dead zone:

$$\hat{\boldsymbol{\theta}}_i(k) = \hat{\boldsymbol{\theta}}_i(k-1) + \frac{a_i(k-1)\boldsymbol{\varphi}_i(k-1)[y_i(k) - \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(k-1)]}{c + \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}, \quad i = 1, 2, \dots, p, \tag{12}$$

where $c > 0$, and

$$a_i(k-1) = \begin{cases} 1, & \text{if } |y_i(k) - \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(k-1)| > 2\Delta(k);\\ 0, & \text{if } |y_i(k) - \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(k-1)| \leqslant 2\Delta(k),\end{cases} \tag{13}$$

$$\Delta(k) = \|\boldsymbol{\phi}(k-1)\|\varepsilon + \varepsilon_0.$$

Here the constant ε is the approximation error bound used in formula (3), and ε_0 is the prescribed positive constant defined below formula (2).

According to formula (2), the choices of all the initial vectors $\hat{\boldsymbol{\theta}}_i(0) (i = 1, 2, \dots, p)$ in formulas (10) and (11) satisfy the following property:

Property 1. $\hat{F}_1(0) \to 2I$, $\hat{F}_2(0) \to -I, \hat{G}(0) \to O$ as $T \to 0$, and $\hat{G}(0)$ is invertible.

Remark 1. As the controller design is directly based on the parameters in the characteristics, a $\hat{\boldsymbol{\theta}}_i(0)$ that satisfies Property 1 is easy to be obtained, regardless of whether the system parameters are known or not. For example, we can choose $\hat{\boldsymbol{\theta}}_i(0)$ as follows.

First, $\hat{a}_{ij0}(0), \ldots, \hat{a}_{ijq}(0)$ are determined using the expressions:

$$\hat{a}_{ii0}(0)P_0(z_{ii0}) + \cdots + \hat{a}_{iiq}(0)P_q(z_{ii0}) = 2, \tag{14}$$

$$\hat{a}_{ij0}(0) = \cdots = \hat{a}_{ijq}(0) = 0, \text{ and } j \neq i. \tag{15}$$

Also, note that $P_0(z_{ij0}) = 1$. By assuming $\hat{a}_{ii0}(0) = 2$ and $\hat{a}_{iis}(0) = 0$ for $s \neq 0$, it follows that formula (14) holds. Therefore, $\hat{F}_1(0) = 2I$.

Next, $\hat{a}_{i,p+j,0}(0), \ldots, \hat{a}_{i,p+j,q}(0)$ are determined by

$$\hat{a}_{i,p+i,0}(0)P_0(z_{i,p+i,\,0}) + \cdots + \hat{a}_{i,p+i,q}(0)P_q(z_{i,p+i,0}) = -1, \tag{16}$$

$$\hat{a}_{i,p+j,0}(0) = \cdots = \hat{a}_{i,p+j,q}(0) = 0, \text{ and } j \neq i. \tag{17}$$

Similarly, let $\hat{a}_{i,p+i,0}(0) = -1$ and $\hat{a}_{i,p+i,s}(0) = 0$ for $s \neq 0$, this satisfies formula (16). Therefore, $\hat{F}_2(0) = -I$.

To choose $\hat{G}(0)$ as an invertible matrix that also satisfies $\hat{G}(0) \to O$ as $T \to 0$, let $\hat{G}(0) = TI$. $\hat{b}_{ij0}(0), \ldots, \hat{b}_{ijq}(0)$ are determined by using the expressions:

$$\hat{b}_{ii0}(0)P_0(z_{i,\,2p+i,\,0}) + \cdots + \hat{b}_{iiq}(0)P_q(z_{i,\,2p+i,\,0}) = T, \tag{18}$$

$$\hat{b}_{ij0}(0) = \cdots = \hat{b}_{ijq}(0) = 0, \text{ and } j \neq i. \tag{19}$$

Depending on which $P_0(z_{i,2p+i,\,0}) = 1$, by taking $\hat{b}_{ii0}(0) = T$ and $\hat{b}_{iis}(0) = 0$ for $s \neq 0$, it follows that formula (18) holds. Therefore, $\hat{G}(0) = TI$.

In the following theorem, we state the main result of this paper, which will be proved in the next section.

Theorem 1. Consider a system (1) with the golden-section controller (9)–(11), and let the parameter vector $\hat{\boldsymbol{\theta}}_i(k)$ be adjusted according to the adaptive law (12) with initial parameter vector $\hat{\boldsymbol{\theta}}_i(0)$ satisfying Property 1. Then the following properties are guaranteed:

(i) The estimates of parameter vectors are bounded.

(ii) The tracking error converges to a small neighborhood of the origin, i.e., there exists a infinite time instant $\bar{k}$ such that for all $k \geqslant \bar{k}$, the tracking error is given by

$$\|\boldsymbol{y}_k - \boldsymbol{y}_d^*\| \leqslant \sqrt{(\mathrm{e}^{\bar{p}/K} - 1)/(\mu\lambda_{\min}(\Lambda_{k+1}))},$$

where μ and K are all positive constants; $\bar{p}$ is a small positive constant number, and the size of $\bar{p}$ depends on the size of the sampling period T. Λ_k is a sequence of uniformly bounded and positive definite matrices satisfying the relationship $\hat{A}_k^{\mathrm{T}}\Lambda_{k+1}\hat{A}_k - \Lambda_k = -Q - I$ for a given positive definite matrix Q.

The expressions for the positive constants μ, K, $\bar{p}$ and the matrix $\hat{A}_k$ can be found in the proof of Theorem 1.

4 Proof of the main result

4.1 Important lemmas

Before presenting the proof of the main result, we consider the following linear time-varying discrete system and introduce some lemmas:

$$\boldsymbol{x}_{k+1} = A_k\boldsymbol{x}_k, \quad k \geqslant 1, \tag{20}$$

where $A_k \in \mathbb{R}^{p\times p}$, and $\boldsymbol{x}_k \in \mathbb{R}^p$.

Lemma 1 ([20]). Let $A_k \in \mathbb{R}^{p\times p}$, $k \geqslant 1$. We assume that A_k satisfies

(1) $\rho_0 = \lim_{k\to\infty} \sup \rho(A_k) < 1$,

(2) $a_M = \sup_{k\geqslant 0} \|A_k\| < \infty$.

Under these conditions, there exists a positive number δ_0 that depends only on p, ρ_0 and a_M, provided that

$$\lim_{k\to\infty} \sup \|A_k - A_{k-1}\| \leqslant \delta_0.$$

Then the equilibrium solution of the system (20) is exponentially stable.

It is well-known that the following assertion is true.

Lemma 2 ([21]). If the trivial solution of the system (20) is uniformly asymptotically stable and if Q_k is a sequence of uniformly bounded and positive definite matrices, then there is a sequence of positive definite matrices Λ_k satisfying the relationship:

$$A_k^{\mathrm{T}} \Lambda_{k+1} A_k - \Lambda_k = -Q_k,$$

and there exist constants $m > 0$ and $M > 0$ such that

$$mI \leqslant \Lambda_k \leqslant MI.$$

Now let us establish the following two results in connection with the theory of time-varying matrices.

Lemma 3. Let A_k, $H_k \in \mathbb{R}^{p\times p}$, and suppose that the entries of A_k are bounded and that each H_k is a symmetric matrix. Also, let $H_k \leqslant MI$, where M is a constant. Then,

(1) the supremum $\sup_{k\geqslant 1}\{\lambda_{\max}(H_k)\}$ of the largest eigenvalue $\lambda_{\max}(H_k)$ of H_k exists;

(2) the supremum $\sup_{k\geqslant 1}\{\lambda_{\max}\left(A_k^{\mathrm{T}} H_{k+1} A_k\right)\}$ of the largest eigenvalue $\lambda_{\max}\left(A_k^{\mathrm{T}} H_{k+1} A_k\right)$ of $A_k^{\mathrm{T}} H_{k+1} A_k$ exists, and

(3) for $\Gamma = \left[\begin{smallmatrix} O \\ I \end{smallmatrix}\right]$, the supremum $\sup_{k\geqslant 1}\{\lambda_{\max}(\Gamma^{\mathrm{T}} H_k \Gamma)\}$ exists, and $\sup_{k\geqslant 1}\{\lambda_{\max}(\Gamma^{\mathrm{T}} H_k \Gamma)\} > 0$.

Proof. Using the properties of the matrix eigenvalues, and noting the fact that the eigenvalues of a matrix are continuous functions of its entries, the proof is readily achieved by using the least upper bound axiom.

Lemma 4. Let $A_k, H_k \in \mathbb{R}^{p\times p}$. Suppose that all A_k and H_k are symmetric matrices and that the entries of A_k are bounded. Also, let $H_k \geqslant mI$, where $m > 0$ is a constant. Then,

(1) there exists a positive constant μ_0 such that $\mu_0 I > A_k$, i.e., for any nonzero vector $\boldsymbol{x} \in \mathbb{R}^p$, $\boldsymbol{x}^{\mathrm{T}}(\mu_0 I - A_k)\boldsymbol{x} > 0$;

(2) there exists a positive constant μ_1 such that $\mu_1 H_k > A_k$, i.e., for any nonzero vector $\boldsymbol{x} \in \mathbb{R}^p$, $\boldsymbol{x}^{\mathrm{T}}(\mu_1 H_k - A_k)\boldsymbol{x} > 0$.

Proof. Note that there exists an orthogonal matrix $T(k)$ such that $T(k)^{-1} A_k T(k) = \mathrm{diag}[\lambda_1(k), \ldots, \lambda_p(k)]$ for any fixed $k \geqslant 1$, where $\lambda_1(k), \ldots, \lambda_p(k)$ are all the eigenvalues of A_k. Let $\lambda_0 = \max_{0\leqslant i\leqslant p}\{\sup_{k\geqslant 1} |\lambda_i(k)|\}$ and $\mu_0 = \lambda_0 + 1$. Then we can easily prove this lemma.

We now generalize the result of a time-invariant system to that of a time-varying system, and correct a mistake in formula (A2) of [22].

Lemma 5. Let $A_k \in \mathbb{R}^{p\times p}$, $k \geqslant 1$, and suppose that the entries of A_k are bounded. If the equilibrium solution of the system

$$\boldsymbol{x}_{k+1} = A_k \boldsymbol{x}_k + \Gamma \boldsymbol{v}_k$$

is asymptotically stable when $\boldsymbol{v}_k = 0$, then for all positive definite matrices Q and all positive constants μ there exist a sequence of uniformly bounded and positive definite matrices Λ_k and a positive constant c_0 such that the function

$$V(\boldsymbol{x}_k) = \ln\left(1 + \mu \boldsymbol{x}_k^{\mathrm{T}} \Lambda_k \boldsymbol{x}_k\right)$$

satisfies

$$V\left(\boldsymbol{x}_{k+1}\right) - V\left(\boldsymbol{x}_k\right) < \mu \frac{-\boldsymbol{x}_k^{\mathrm{T}} Q \boldsymbol{x}_k + c_0^2 \boldsymbol{v}_k^{\mathrm{T}} \Gamma^{\mathrm{T}} \Lambda_{k+1} \Gamma \boldsymbol{v}_k}{1 + \mu \boldsymbol{x}_k^{\mathrm{T}} \Lambda_k \boldsymbol{x}_k},$$

where $c_0^2 = \tau_0 + 1 + \sup_{k\geqslant 1} \lambda_{\max}\left(A_k^{\mathrm{T}} \Lambda_{k+1} A_k\right)$. Here τ_0 is an arbitrary positive constant.

Remark 2. From Lemma 3, c_0^2 is well-defined.

Proof. See Appendix A.

Now we introduce the following notations:

$$\tilde{\boldsymbol{\theta}}_i(k) = \hat{\boldsymbol{\theta}}_i(k) - \boldsymbol{\theta}_i, \tag{21}$$

$$\begin{aligned} e_i(k) &= y_i(k) - \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(k-1) = \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\theta}_i + \bar{\omega}_i(k-1) + E_i(k-1) - \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(k-1) \\ &= -\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\tilde{\boldsymbol{\theta}}_i(k-1) + \bar{\omega}_i(k-1) + E_i(k-1). \end{aligned} \tag{22}$$

Lemma 6. Considering algorithm (12) with respect to (7) and (8), it follows that

(1)

$$\|\hat{\boldsymbol{\theta}}_i(k) - \boldsymbol{\theta}_i\| \leqslant \|\hat{\boldsymbol{\theta}}_i(k-1) - \boldsymbol{\theta}_i\| \leqslant \|\hat{\boldsymbol{\theta}}_i(0) - \boldsymbol{\theta}_i\|, \ k \geqslant 1. \tag{23}$$

This implies that each $\hat{\boldsymbol{\theta}}_i(k)$ is bounded.

(2) (i)

$$\frac{\Delta(k)^2}{c + \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)} = \frac{[\|\boldsymbol{\phi}(k-1)\|\varepsilon + \varepsilon_0]^2}{c + \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)} \leqslant s_{0i}^2, \tag{24}$$

where $s_{0i} = \frac{\varepsilon\sqrt{c}+\varepsilon_0}{\sqrt{c}} \cdot \sqrt{\max\{1,\ 1/\overline{\lambda}_{\min}(L_i(k)^{\mathrm{T}}L_i(k))\ \}}$, $\overline{\lambda}_{\min}(L_i(k)^{\mathrm{T}}L_i(k)) = \inf\{\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))\}$.

(ii) $\frac{a_i(k-1)e_i(k)}{[c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)]^{1/2}}$ is bounded, i.e., there exists a constant $M_{e\,i} > 0$ such that for any $k \geqslant 1$,

$$\frac{a_i(k-1)\cdot|e_i(k)|}{[c + \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)]^{1/2}} \leqslant M_{e\,i}, \tag{25}$$

where $M_{e\,i} = \sqrt{M + 4s_{0i}^2}$, and M is a positive constant.

(iii) For any $\in > 0$, there exists an integer $k_0 \geqslant 1$ such that when $k > k_0$,

$$\frac{a_i(k-1)\cdot|e_i(k)|}{[c + \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)]^{1/2}} \leqslant \overline{M}_{ei}, \tag{26}$$

where $\overline{M}_{ei} = \sqrt{\in + 4s_{0i}^2}$. By taking $\in = 5s_{0i}^2$ in $\overline{M}_{ei}$, we have

$$\frac{a_i(k-1)\cdot|e_i(k)|}{(c + \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1))^{1/2}} \leqslant 3\,s_{0i}. \tag{27}$$

In this case, $\overline{M}_{ei} = 3\,s_{0i}$.

(3)

$$\lim_{k\to\infty} \sup \|\hat{\boldsymbol{\theta}}_i(k) - \hat{\boldsymbol{\theta}}_i(k-1)\| \leqslant \overline{M}_{ei}, \tag{28}$$

where $\overline{M}_{e\,i}$ is the above given positive constant.

(4) By choosing one among the Legendre, Chebyshev, Laguerre, or Hermite polynomials as the basis function, we can set $s_{0i} = s_0 = \frac{\varepsilon\sqrt{c}+\varepsilon_0}{\sqrt{c}}$.

Remark 3. From the proof of Lemma 6, it can be seen that the parameter vector adaptive laws (12) can guarantee that $\|\hat{\boldsymbol{\theta}}_i(k) - \boldsymbol{\theta}_i\|$ is non-increasing.

Proof. See Appendix B.

4.2 Proof of Theorem 1

From the first assertion in Lemma 6, the estimates of the parameter vectors are bounded, and thus we can establish (i).

In the following discussion, we shall prove the second assertion. The proof will be carried out in three steps.

Step 1. Transforming the characteristic model (1) into its equivalent form.

For the convenience of stability analysis, we can rewrite (1) as

$$-\boldsymbol{y}_{k+1} = \left(\hat{F}_{1,k} - F_{1,k}\right)\boldsymbol{y}_k + \left(\hat{F}_{2,k} - F_{2,k}\right)\boldsymbol{y}_{k-1} + \left(\hat{G}_k - G_k\right)\boldsymbol{u}_k - \boldsymbol{E}_k - \hat{F}_{1,k}\boldsymbol{y}_k - \hat{F}_{2,k}\boldsymbol{y}_{k-1} - \hat{G}_k\boldsymbol{u}_k.$$

Substituting (9)–(11) into the above equality and noting that $1 - L_1 = L_2$ and $1 - L_2 = L_1$, we have

$$\tilde{\boldsymbol{y}}_{k+1} = L_2\hat{F}_{1,k}\tilde{\boldsymbol{y}}_k + L_1\hat{F}_{2,k}\tilde{\boldsymbol{y}}_{k-1} + (\hat{F}_{1,k} - F_{1,k})\boldsymbol{y}_k + (\hat{F}_{2,k} - F_{2,k})\boldsymbol{y}_{k-1} + (\hat{G}_k - G_k)\boldsymbol{u}_k - \boldsymbol{E}_k. \tag{29}$$

Let

$$\boldsymbol{v}_k = \left(\hat{F}_{1,k} - F_{1,k}\right)\boldsymbol{y}_k + \left(\hat{F}_{2,k} - F_{2,k}\right)\boldsymbol{y}_{k-1} + \left(\hat{G}_k - G_k\right)\boldsymbol{u}_k - \boldsymbol{E}_k. \tag{30}$$

Then Eq. (29) can be rewritten as

$$\tilde{\boldsymbol{y}}_{k+1} = L_2\hat{F}_{1,k}\tilde{\boldsymbol{y}}_k + L_1\hat{F}_{2,k}\tilde{\boldsymbol{y}}_{k-1} + \boldsymbol{v}_k. \tag{31}$$

Let $\boldsymbol{x}_k = \left[\begin{smallmatrix}\tilde{\boldsymbol{y}}_{k-1}\\ \tilde{\boldsymbol{y}}_k\end{smallmatrix}\right]$, then Eq. (31) becomes

$$\boldsymbol{x}_{k+1} = \hat{A}_k\boldsymbol{x}_k + \varGamma\boldsymbol{v}_k, \tag{32}$$

where

$$\hat{A}_k = \begin{bmatrix} O & I \\ L_1\hat{F}_{2,k} & L_2\hat{F}_{1,k}\end{bmatrix}, \quad \varGamma = \begin{bmatrix} O \\ I \end{bmatrix}. \tag{33}$$

Step 2. Proving the uniformly asymptotic stability of the system (32) when $\boldsymbol{v}_k = 0$.

To obtain the Lyapunov function for analyzing the stability, we need first to verify that $\hat{A}_k$ satisfies the conditions in Lemma 1. Using the fact that $L_2^2 = [(\sqrt{5}-1)/2]^2 = L_1$, and according to [23], we can obtain the characteristic polynomial of $\hat{A}_k$ as

$$\det\left[\lambda I_{2p} - \hat{A}_k\right] = L_1^p \det\left[\bar{\lambda}^2 I_p - \bar{\lambda}\hat{F}_{1,k} - \hat{F}_{2,k}\right], \tag{34}$$

where $\bar{\lambda} = \lambda/L_2$.

It can be proved that $\hat{F}_{1,k} \to 2I_p$, $\hat{F}_{2,k} \to -I_p$ as $T \to 0$ and approximate errors $\omega_{fij}(z_{ijk}) \to 0$, $\omega_{gij}(z_{i,\,2p+j,\,k}) \to 0$. In fact,

$$\|\hat{F}_{1,k} - 2I_p\| \leqslant \|\hat{F}_{1,k} - (\overline{F}_{1,k} + \omega_{1F})\| + \|F_{1,k} - 2I_p\| \leqslant \|\hat{F}_{1,k} - \overline{F}_{1,k}\| + \|\omega_{1F}\| + \|F_{1,k} - 2I_p\|, \tag{35}$$

where the definition of ω_{1F} is as in (4), and

$$\|\hat{F}_{1,k} - \overline{F}_{1,k}\| = \|\hat{F}_{1,k} - \overline{F}_{1,k}\|_F \leqslant p\sqrt{\max_{1\leqslant i,j\leqslant p}\{|\hat{f}_{ij}(k) - \bar{f}_{ij}(k)|^2\}}, \tag{36}$$

$$\hat{f}_{ij}(k) - \bar{f}_{ij}(k) = (\hat{a}_{ij0}(k) - a_{ij0}, \ldots, \hat{a}_{ijq}(k) - a_{ijq})(\,P_0(z_{ijk}), \cdots, P_q(z_{ijk})\,)^{\mathrm{T}}. \tag{37}$$

Thus, $|\hat{f}_{ij}(k) - \bar{f}_{ij}(k)| \leqslant \|\hat{\boldsymbol{\theta}}_i(k) - \boldsymbol{\theta}_i\|\sqrt{P_0(z_{ijk})^2 + \cdots + P_q(z_{ijk})^2}$. By using (23), we have $\|\hat{\boldsymbol{\theta}}_i(k) - \boldsymbol{\theta}_i\| \leqslant \|\hat{\boldsymbol{\theta}}_i(0) - \boldsymbol{\theta}_i\|$. According to the above formula , it is clear that

$$|\hat{f}_{ij}(k) - \bar{f}_{ij}(k)| \leqslant \|\hat{\boldsymbol{\theta}}_i(0) - \boldsymbol{\theta}_i\|\sqrt{P_0(z_{ijk})^2 + \cdots + P_q(z_{ijk})^2}. \tag{38}$$

Notice that $F_{1,k} \to 2I_p$, $F_{2,k} \to -I_p$ and $G_k \to O$ as $T \to 0$. Therefore, as $T \to 0$ and $\omega_{1F} \to O$,

$$\|\overline{F}_{1,k} - 2I_p\| = \|F_{1,k} - \omega_{1F} - 2I_p\| \leqslant \|F_{1,k} - 2I_p\| + \|\omega_{1F}\| \to 0.$$

In other words, $\overline{F}_{1,k} \to 2I_p$ as $T \to 0$ and $\omega_{1F} \to O$. Similarly, $\overline{F}_{2.k} \to -I_p$ as $T \to 0$ and $\omega_{2F} \to O$, and $\overline{G}_k \to O$ as $T \to 0$ and $\omega_G \to O$.

Therefore, using the definition of $\boldsymbol{\theta}_i$ introduced in Section 2, and the fact that $\overline{F}_{1,k} \to 2I_p$, $\overline{F}_{2,k} \to -I_p$, and $\overline{G}_k \to O$ as $T \to 0$, $\omega_{1F} \to O$, $\omega_{2F} \to O$ and $\omega_G \to O$, we can conclude that if T and ε are all sufficiently small, we obtain $\|\hat{\boldsymbol{\theta}}_i(0) - \boldsymbol{\theta}_i\| \to 0$ by using $\hat{\boldsymbol{\theta}}_i(0)$ that satisfy Property 1. Also, notice that $\sqrt{P_0(z_{ijk})^2 + \cdots + P_q(z_{ijk})^2}$ is bounded (see the proof of Lemma 6). From (38) it follows that $|\hat{f}_{ij}(k) - \bar{f}_{ij}(k)| \to 0$ as $T \to 0$, $\omega_{1F} \to O$, $\omega_{2F} \to O$, and $\omega_G \to O$. Thus, according to (36), we have $\|\hat{F}_{1,k} - \overline{F}_{1,k}\| \to 0$ as $T \to 0$, $\omega_{1F} \to O$, $\omega_{2F} \to O$, and $\omega_G \to O$. Also, since $F_{1,k} \to 2I_p$ as $T \to 0$, it follows from (35) that $\hat{F}_{1,k} \to 2I_p$ as $T \to 0$, and the approximate errors $\omega_{fij}(z_{ijk})$, $\omega_{gij}(z_{i,\,2p+j,\,k}) \to 0$. Similarly, we can prove that $\hat{F}_{2,k} \to -I_p$ and $\hat{G}_k \to O$ as $T \to 0$ and the approximate errors $\omega_{fij}(z_{ijk})$, $\omega_{gij}(z_{i,\,2p+j,\,k}) \to 0$. Therefore, from (34) it follows that

$$\det(\lambda I_{2p} - \hat{A}_k) \to L_1^p \det\left(\bar{\lambda}^2 I_p - 2\bar{\lambda} I_p + I_p\right), \tag{39}$$

as $T \to 0$ and the approximate errors $\omega_{fij}(z_{ijk})$, $\omega_{gij}(z_{i,\,2p+j,\,k}) \to 0$. It is easy to see that

$$\det\left(\bar{\lambda}^2 I_p - 2\bar{\lambda} I_p + I_p\right) = \left(\bar{\lambda}^2 - 2\bar{\lambda} + 1\right)^p = \left(\bar{\lambda} - 1\right)^{2p}. \tag{40}$$

As the eigenvalues of a matrix are continuous functions of its entries, by combining (39) and (40), we can conclude that the eigenvalue of the matrix $\hat{A}_k$ is $\lambda = L_2\bar{\lambda} \to L_2 \times 1 = L_2$ as $T \to 0$ and the approximate errors $\omega_{fij}(z_{ijk})$, $\omega_{gij}(z_{i,\,2p+j,\,k}) \to 0$. Therefore, $\rho(\hat{A}_k) \to L_2$ as $T \to 0$ and the approximate errors $\omega_{fij}(z_{ijk})$, $\omega_{gij}(z_{i,\,2p+j,\,k}) \to 0$. So, $\rho_0 = \lim_{k\to\infty} \sup \rho(\hat{A}_k) = L_2 < 1$, as $T \to 0$ and the approximate errors $\omega_{fij}(z_{ijk})$, $\omega_{gij}(z_{i,\,2p+j,\,k}) \to 0$.

By Lemma 6, $\hat{\boldsymbol{\theta}}_i(k)$ is bounded, and since $P_0(z_{ijk})$, $\ldots$, and $P_q(z_{ijk})$ are all bounded in $(0,\ 1)$, $\|\hat{A}_k\|$ is bounded. This implies that $a_M = \sup_{k\geqslant 0} \|\hat{A}_k\| < \infty$. Therefore, the system $\boldsymbol{x}_{k+1} = \hat{A}_k \boldsymbol{x}_k$ satisfies the two conditions in Lemma 1. Hence, there exists a positive number δ_0 that only depends on p, ρ_0 and a_M, provided that

$$\lim_{k\to\infty} \sup \|\hat{A}_k - \hat{A}_{k-1}\| \leqslant \delta_0, \tag{41}$$

the system $\boldsymbol{x}_{k+1} = \hat{A}_k \boldsymbol{x}_k$ is exponentially stable. In the following discussion, we prove that Eq. (41) does hold true. From (33), it follows that

$$\|\hat{A}_k - \hat{A}_{k-1}\| = \left\| \begin{bmatrix} O & O \\ L_1(\hat{F}_{2,k} - \hat{F}_{2,k-1}) & L_2(\hat{F}_{1,k} - \hat{F}_{1,k-1}) \end{bmatrix} \right\|.$$

Thus, $\|\hat{A}_k - \hat{A}_{k-1}\| \to 0$ as $T \to 0$ and approximate errors $\omega_{fij}(z_{ijk})$, $\omega_{gij}(z_{i,\,2p+j,\,k}) \to 0$. Therefore, it is easy to see that as $T \to 0$ and approximate errors $\omega_{fij}(z_{ijk})$, $\omega_{gij}(z_{i,\,2p+j,\,k}) \to 0$, formula (41) holds. Hence, by Lemma 1, the equilibrium solution of the system $\boldsymbol{x}_{k+1} = \hat{A}_k \boldsymbol{x}_k$ is exponentially stable, and thus the equilibrium solution of the system is uniformly asymptotically stable.

Step 3. Constructing the Lyapunov function and proving the convergence of the tracking errors.

In this step, we construct a Lyapunov function based on Step 2 to analyze stability, and provide the convergence radius of the tracking error. By using the results from Step 2 and Lemma 5, we can see that for a given positive definite matrix Q and all positive constants μ, there exists a sequence of uniformly bounded and positive definite matrices Λ_k such that

$$V(\boldsymbol{x}_k) = \ln\left(1 + \mu \boldsymbol{x}_k^{\mathrm{T}} \Lambda_k \boldsymbol{x}_k\right), \tag{42}$$

which satisfies

$$V\left(\boldsymbol{x}_{k+1}\right) - V\left(\boldsymbol{x}_k\right) < \mu \frac{-\boldsymbol{x}_k^{\mathrm{T}} Q \boldsymbol{x}_k + c_0^2 \boldsymbol{v}_k^{\mathrm{T}} \Gamma^{\mathrm{T}} \Lambda_{k+1} \Gamma \boldsymbol{v}_k}{1 + \mu \boldsymbol{x}_k^{\mathrm{T}} \Lambda_k \boldsymbol{x}_k}, \tag{43}$$

where $c_0^2 = \tau_0 + 1 + \sup_{k\geqslant 1} \lambda_{\max}(\hat{A}_k^{\mathrm{T}} \Lambda_{k+1} \hat{A}_k)$. From (A1), Λ_k satisfies the discrete type Lyapunov matrix equation:

$$\hat{A}_k^{\mathrm{T}} \Lambda_{k+1} \hat{A}_k - \Lambda_k = -Q - I.$$

Since the entries of $\hat{A}_k$ are bounded according to (23), c_0^2 is well-defined by Lemma 3.

Let $\tilde{\boldsymbol{\theta}}(k) = (\tilde{\boldsymbol{\theta}}_1(k), \ldots, \tilde{\boldsymbol{\theta}}_p(k))^{\mathrm{T}}$ and $\boldsymbol{x}_{\boldsymbol{\theta}}(k) = [\boldsymbol{x}_k^{\mathrm{T}}\ \tilde{\boldsymbol{\theta}}(k)^{\mathrm{T}}]^{\mathrm{T}}$.

Choose a Lyapunov function of the form

$$V(\boldsymbol{x}_\theta(k)) = \tilde{\boldsymbol{\theta}}(k)^{\mathrm{T}}\tilde{\boldsymbol{\theta}}(k) + K\ln(1+\mu\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k), \tag{44}$$

where μ and K are positive constants. From (12) and (43), we have

$$\begin{aligned}\Delta V &= V(\boldsymbol{x}_\theta(k+1)) - V(\boldsymbol{x}_\theta(k))\\ &< \sum_{i=1}^{p}\frac{2a_i(k)\tilde{\boldsymbol{\theta}}_i(k)^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}e_i(k+1)}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}} + \sum_{i=1}^{p}\frac{a_i(k)^2e_i(k+1)^2\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}}{[c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}]^2}\\ &\quad+\mu K\frac{-\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k + c_0^2\boldsymbol{v}_k^{\mathrm{T}}\Gamma^{\mathrm{T}}\Lambda_{k+1}\Gamma\boldsymbol{v}_k}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k}.\end{aligned}$$

Applying (22) and substituting $a_i(k)^2 = a_i(k)$, the above formula becomes

$$\begin{aligned}\Delta V &< \sum_{i=1}^{p}\frac{-a_i(k)e_i(k+1)^2}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}} + \sum_{i=1}^{p}\frac{2a_i(k)[\bar{\omega}_i(k)+E_i(k)]e_i(k+1)}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}}\\ &\quad+\mu K\frac{-\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k + c_0^2\boldsymbol{v}_k^{\mathrm{T}}\Gamma^{\mathrm{T}}\Lambda_{k+1}\Gamma\boldsymbol{v}_k}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k}.\end{aligned} \tag{45}$$

In order to further enlarge inequality (45), we use the expressions:

$$\boldsymbol{\phi}_k = \left(\boldsymbol{y}_k^{\mathrm{T}}, \boldsymbol{y}_{k-1}^{\mathrm{T}}, \boldsymbol{u}_k^{\mathrm{T}}\right)^{\mathrm{T}} = \left((\boldsymbol{y}_k-\boldsymbol{y}_d^*)^{\mathrm{T}}, (\boldsymbol{y}_{k-1}-\boldsymbol{y}_d^*)^{\mathrm{T}},\ \boldsymbol{u}_{g,k}^{\mathrm{T}}\right)^{\mathrm{T}} + \left(\boldsymbol{y}_d^{*\mathrm{T}}, \boldsymbol{y}_d^{*\mathrm{T}}, \boldsymbol{u}_{0,k}^{\mathrm{T}}\right)^{\mathrm{T}},$$

and

$$\overline{\boldsymbol{x}}_k = \left((\boldsymbol{y}_k-\boldsymbol{y}_d^*)^{\mathrm{T}}, (\boldsymbol{y}_{k-1}-\boldsymbol{y}_d^*)^{\mathrm{T}}\ \right)^{\mathrm{T}}, \quad \boldsymbol{\phi}_k^* = \left(\boldsymbol{y}_d^{*\mathrm{T}}, \boldsymbol{y}_d^{*\mathrm{T}}, \boldsymbol{u}_{0,k}^{\mathrm{T}}\right)^{\mathrm{T}}. \tag{46}$$

Thus, we have

$$\boldsymbol{\phi}_k = \left[\overline{\boldsymbol{x}}_k^{\mathrm{T}}, \boldsymbol{u}_{g,k}^{\mathrm{T}}\right] + \boldsymbol{\phi}_k^*, \tag{47}$$

We further notice that

$$\begin{bmatrix}\overline{\boldsymbol{x}}_k\\ \boldsymbol{u}_{g,k}\end{bmatrix} = \begin{bmatrix}\overline{\boldsymbol{x}}_k\\ \hat{G}_k^{-1}(L_1\hat{F}_{1,k}\tilde{\boldsymbol{y}}_k + L_2\hat{F}_{2,k}\tilde{\boldsymbol{y}}_{k-1})\end{bmatrix} = \begin{bmatrix}\overline{\boldsymbol{x}}_k\\ -\hat{G}_k^{-1}[L_1\hat{F}_{1,k}\ \ L_2\hat{F}_{2,k}]\overline{\boldsymbol{x}}_k\end{bmatrix} = B_k\,\overline{\boldsymbol{x}}_k, \tag{48}$$

where $B_k = \left[\begin{smallmatrix} I_{2p}\\ -\hat{G}_k^{-1}[L_1\hat{F}_{1,k}\ \ L_2\hat{F}_{2,k}]\end{smallmatrix}\right]$.

By using (47) and (48), we obtain

$$\begin{aligned}\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k} &= \boldsymbol{\phi}_k^{\mathrm{T}}L_i(k)^{\mathrm{T}}L_i(k)\boldsymbol{\phi}_k = [(\overline{\boldsymbol{x}}_k^{\mathrm{T}}\ \boldsymbol{u}_{g,k}^{\mathrm{T}}) + \boldsymbol{\phi}_k^{*\mathrm{T}}]L_i(k)^{\mathrm{T}}L_i(k)\cdot\left[\begin{pmatrix}\overline{\boldsymbol{x}}_k\\ \boldsymbol{u}_{g,k}\end{pmatrix} + \boldsymbol{\phi}_k^*\right]\\ &= \overline{\boldsymbol{x}}_k^{\mathrm{T}}B_k^{\mathrm{T}}L_i(k)^{\mathrm{T}}L_i(k)B_k\overline{\boldsymbol{x}}_k + (L_i(k)B_k\overline{\boldsymbol{x}}_k)^{\mathrm{T}}L_i(k)\boldsymbol{\phi}_k^*\\ &\quad+(L_i(k)\boldsymbol{\phi}_k^*)^{\mathrm{T}}L_i(k)B_k\overline{\boldsymbol{x}}_k + (L_i(k)\boldsymbol{\phi}_k^*)^{\mathrm{T}}L_i(k)\boldsymbol{\phi}_k^*.\end{aligned}$$

Since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} \leqslant \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}+\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}}{2}$ (The column vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same number of dimensions), it is easy to see that

$$\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k} \leqslant 2\overline{\boldsymbol{x}}_k^{\mathrm{T}}B_k^{\mathrm{T}}L_i(k)^{\mathrm{T}}L_i(k)B_k\overline{\boldsymbol{x}}_k + 2\boldsymbol{\phi}_k^{*\mathrm{T}}L_i(k)^{\mathrm{T}}L_i(k)\boldsymbol{\phi}_k^*.$$

It is obvious that

$$\overline{\boldsymbol{x}}_k = \begin{bmatrix} O & -I\\ -I & O\end{bmatrix}\begin{pmatrix}\boldsymbol{y}_d^*-\boldsymbol{y}_{k-1}\\ \boldsymbol{y}_d^*-\boldsymbol{y}_k\end{pmatrix} = \begin{bmatrix} O & -I\\ -I & O\end{bmatrix}\boldsymbol{x}_k = \tilde{I}\,\boldsymbol{x}_k, \quad \tilde{I} = \begin{bmatrix} O & -I\\ -I & O\end{bmatrix}.$$

From this, we have

$$\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k} \leqslant 2\boldsymbol{x}_k^{\mathrm{T}}\tilde{I}^{\mathrm{T}}B_k^{\mathrm{T}}L_i(k)^{\mathrm{T}}L_i(k)B_k\tilde{I}\,\boldsymbol{x}_k + 2\boldsymbol{\phi}_k^{*\mathrm{T}}L_i(k)^{\mathrm{T}}L_i(k)\boldsymbol{\phi}_k^*.$$

Using Lemma 4, we can introduce a constant $\mu_1 > 0$, such that $\mu_1 \Lambda_k > (L_i(k)B_k\tilde{I})^{\mathrm{T}} L_i(k)B_k\tilde{I}$. As the components of $\boldsymbol{\phi}_k^*$ are all bounded and $L_i(k)$ defined over $[1, +\infty]$ is also bounded, there exists positive constants c_{ri}^2 such that $\boldsymbol{\phi}_k^{*\mathrm{T}} L_i(k)^{\mathrm{T}} L_i(k)\boldsymbol{\phi}_k^* \leqslant c_{ri}^2$. Let $c_r^2 = \max\{c_{r1}^2, \ldots, c_{rp}^2\}$. From these fact, we have $\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k} \leqslant 2\mu_1 \boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k + 2c_r^2$. Hence,

$$\frac{-e_i(k+1)^2}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}} \leqslant \frac{-e_i(k+1)^2}{c+2c_r^2+2\mu_1\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k}. \tag{49}$$

Let

$$\mu = \frac{2\mu_1}{c+2c_r^2}, \quad K = \frac{1}{2\mu_1 c_0^2 M_1}, \quad M_1 = \sup_{k\geqslant 1}\{\lambda_{\max}(\Gamma^{\mathrm{T}}\Lambda_{k+1}\Gamma)\}.$$

We know that M_1 is well-defined, and $M_1 > 0$ by Lemma 3. Thus, we obtain the expression

$$\mu K \cdot \frac{c_0^2\boldsymbol{v}_k^{\mathrm{T}}\Gamma^{\mathrm{T}}\Lambda_{k+1}\Gamma\boldsymbol{v}_k}{1+\mu\boldsymbol{x}_k^{T}\Lambda_k\boldsymbol{x}_k} \leqslant \frac{1}{c+2c_r^2}\cdot\frac{1}{M_1}\cdot\frac{\sup_{k\geqslant 1}\{\lambda_{\max}(\Gamma^{\mathrm{T}}\Lambda_{k+1}\Gamma)\}\boldsymbol{v}_k^{\mathrm{T}}\boldsymbol{v}_k}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k} = \frac{\boldsymbol{v}_k^{\mathrm{T}}\boldsymbol{v}_k}{c+2c_r^2+2\mu_1\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k}. \tag{50}$$

According to (45), (49), and (50), it is clear that

$$\begin{aligned}\Delta V < &\sum_{i=1}^{p}\frac{-a_i(k)e_i(k+1)^2}{c+2c_r^2+2\mu_1\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k} + \sum_{i=1}^{p}\frac{2a_i(k)[\bar{\omega}_i(k)+E_i(k)]e_i(k+1)}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}}\\ &+\mu K\frac{-\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k} + \frac{\boldsymbol{v}_k^{\mathrm{T}}\boldsymbol{v}_k}{c+2c_r^2+2\mu_1\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k}.\end{aligned} \tag{51}$$

We need to find the relationship between $\boldsymbol{e}(k+1)$ and $\boldsymbol{v}_k$, where $\boldsymbol{e}(k) = (e_1(k), \ldots, e_p(k))^{\mathrm{T}}$. Note that $\boldsymbol{e}(k+1) = F_{1,k}\boldsymbol{y}_k + F_{2,k}\boldsymbol{y}_{k-1} + G_k\boldsymbol{u}_k + \boldsymbol{E}_k - \hat{F}_{1,k}\boldsymbol{y}_k - \hat{F}_{2,k}\boldsymbol{y}_{k-1} - \hat{G}_k\boldsymbol{u}_k$. From (30), it can be seen that

$$\boldsymbol{e}(k+1) = -\boldsymbol{v}_k. \tag{52}$$

Case 1. Each $a_i(k) = 1$.

From (51) and (52), we have

$$\Delta V < \sum_{i=1}^{p}\frac{2a_i(k)[\bar{\omega}_i(k)+E_i(k)]e_i(k+1)}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}} + \mu K\frac{-\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\Lambda_k\boldsymbol{x}_k}. \tag{53}$$

Before further analysis, inequality (53) must be enlarged. First, we note that $|\bar{\omega}_i(k-1) + E_i(k-1)| < \|\boldsymbol{\phi}_{k-1}\| \cdot \varepsilon + \varepsilon_0 = \Delta(k)$. Therefore,

$$\frac{a_i(k)[\bar{\omega}_i(k)+E_i(k)]e_i(k+1)}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}} < \frac{a_i(k)\Delta(k+1)|e_i(k+1)|}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}} = \frac{a_i(k)\,|e_i(k+1)|}{(c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k})^{1/2}}\cdot\frac{\Delta(k+1)}{(c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k})^{1/2}}. \tag{54}$$

It follows from (24) that

$$\frac{\Delta(k+1)^2}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}} \leqslant s_{0i}^2. \tag{55}$$

When we choose one among the Legendre, Chebyshev, Laguerre, and Hermite polynomials as the basis function, we can take $s_{0i} = s_0$ from (4) in Lemma 6.

From (25), we have $[a_i(k)|e_i(k+1)|]/[(c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k})^{1/2}] \leqslant M_{ei}$. According to (27), M_{ei} can be taken as $\overline{M}_{ei} = 3s_{0i} = 3s_0$ (In the following derivation process, M_{ei} is taken as $\overline{M}_{ei}$). Using (54) and (55), and taking $\varepsilon = \varepsilon_0$, it is obvious that

$$\frac{a_i(k)[\bar{\omega}_i(k)+E_i(k)]e_i(k+1)}{c+\boldsymbol{\varphi}_{i,k}^{\mathrm{T}}\boldsymbol{\varphi}_{i,k}} < \overline{M}_{ei}s_0 = 3s_0^2 = 3\frac{[\varepsilon\sqrt{c}+\varepsilon_0]^2}{c} = 3\varepsilon_0^2\frac{[\sqrt{c}+1]^2}{c} = 3\varepsilon_0^2 s_1, \tag{56}$$

where $s_1 = \frac{[\sqrt{c}+1]^2}{c}$.

Therefore, from (53) and (56), we obtain the following expression:

$$\Delta V < 6ps_1\varepsilon_0^2 + \mu K\frac{-\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k} \leqslant \frac{-\mu[K\lambda_{\min}(Q)-6p\mu s_1\varepsilon_0^2\cdot\sup_{k\geqslant 1}\{\lambda_{\max}(\varLambda_k)\}]\cdot\|\boldsymbol{x}_k\|^2+6ps_1\varepsilon_0^2}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k}.$$

Let $d_0 = K\lambda_{\min}(Q) - 6p\mu s_1\varepsilon_0^2\cdot\sup_{k\geqslant 1}\{\lambda_{\max}(\varLambda_k)\}$. Then the above formula can be written as

$$\Delta V < \frac{-\mu d_0\|\boldsymbol{x}_k\|^2+6ps_1\varepsilon_0^2}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k}.$$

Note that $\lambda_{\min}(Q) > 0$ and $\sup_{k\geqslant 1}\{\lambda_{\max}(\varLambda_k)\} > 0$. Therefore, when $\varepsilon_0^2 < \frac{K\lambda_{\min}(Q)}{6p\mu s_1\cdot\sup_{k\geqslant 1}\{\lambda_{\max}(\varLambda_k)\}}$, the coefficient of $\|\boldsymbol{x}_k\|^2$ is negative. Thus, when $\|\boldsymbol{x}_k\|^2 > \frac{6ps_1\varepsilon_0^2}{\mu d_0}$, $\Delta V < 0$.

Case 2. Each $a_i(k) = 0$.

In this case, from (13) and (22) it is obvious that $|e_i(k)| = |y_i(k) - \boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\hat{\boldsymbol{\theta}}_i(k-1)| \leqslant 2\Delta(k)$. It follows from (51) and (52) that

$$\Delta V < \mu K\frac{-\lambda_{\min}(Q)\|\boldsymbol{x}_k\|^2 + c_0^2\sup_{k\geqslant 1}\{\lambda_{\max}(\varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma)\}\cdot 4\Delta(k+1)^2}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k}. \tag{57}$$

For further analysis, inequality (57) must be enlarged. Since $\boldsymbol{\phi}_k = \left(\overline{\boldsymbol{x}}_k^{\mathrm{T}},\ \boldsymbol{0}_{1\times p}\right)^{\mathrm{T}} + \left(\boldsymbol{y}_d^{*\mathrm{T}}, \boldsymbol{y}_d^{*\mathrm{T}}, \boldsymbol{u}_k^{\mathrm{T}}\right)^{\mathrm{T}}$, we get $\|\boldsymbol{\phi}_k\| \leqslant \|\overline{\boldsymbol{x}}_k\| + \|\left(\boldsymbol{y}_d^{*\mathrm{T}}, \boldsymbol{y}_d^{*\mathrm{T}}\right)^{\mathrm{T}}\| + \|\boldsymbol{u}_{g,k}\| + \|\boldsymbol{u}_{0,k}\|$. Note that $\boldsymbol{u}_{g,k} = \hat{G}_k^{-1}(L_1\hat{F}_{1,k}\tilde{\boldsymbol{y}}_k + L_2\hat{F}_{2,k}\tilde{\boldsymbol{y}}_{k-1}) = -\hat{G}_k^{-1}(L_1\hat{F}_{1,k}\quad L_2\hat{F}_{2,k})\overline{\boldsymbol{x}}_k$. We have $\|\boldsymbol{u}_{g,k}\| \leqslant \|\hat{G}_k^{-1}(L_1\hat{F}_{1,k}\quad L_2\hat{F}_{2,k})\|\cdot\|\overline{\boldsymbol{x}}_k\|$. Because the estimation of the parameter $\hat{\boldsymbol{\theta}}_i(k)$ is bounded according to Lemma 6, $\|\hat{G}_k^{-1}(L_1\hat{F}_{1,k}\quad L_2\hat{F}_{2,k})\|$ is also bounded, and $\|\left(\boldsymbol{y}_d^{*\mathrm{T}}, \boldsymbol{y}_d^{*\mathrm{T}}\right)^{\mathrm{T}}\|$ and $\|\boldsymbol{u}_{0,k}\|$ are all bounded quantities. Then, we have

$$\|\boldsymbol{\phi}_k\| \leqslant M_2\|\overline{\boldsymbol{x}}_k\| + M_3, \tag{58}$$

where M_2 and M_3 are all positive constants. Note that, in the expression $\Delta(k+1) = \|\boldsymbol{\phi}_k\|\varepsilon + \varepsilon_0$, ε has been taken as ε_0, and $\|\overline{\boldsymbol{x}}_k\| = \|\boldsymbol{x}_k\|$ (The components of $\overline{\boldsymbol{x}}_k$ and $\boldsymbol{x}_k$ differ in order, and are opposite in sign). From (58), we have

$$\Delta(k+1)^2 \leqslant [(M_2\|\boldsymbol{x}_k\| + M_3)\varepsilon + \varepsilon_0]^2 \leqslant [M_2^2\|\boldsymbol{x}(k)\|^2 + 2M_2(M_3+1)\|\boldsymbol{x}_k\| + (M_3+1)^2]\varepsilon_0^2. \tag{59}$$

Thus, it follows from (57) and (59) that

$$\Delta V < \mu K\frac{-d_1\|\boldsymbol{x}_k\|^2 + 8d_2\|\boldsymbol{x}_k\| + 4d_3}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k},$$

where

$$\begin{aligned} d_1 &= \lambda_{\min}(Q) - 4c_0^2\varepsilon_0^2M_2^2\cdot\sup_{k\geqslant 1}\{\lambda_{\max}(\varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma)\},\\ d_2 &= c_0^2M_2(M_3+1)\varepsilon_0^2\cdot\sup_{k\geqslant 1}\{\lambda_{\max}(\varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma)\},\\ d_3 &= c_0^2(M_3+1)^2\varepsilon_0^2\cdot\sup_{k\geqslant 1}\{\lambda_{\max}(\varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma)\}. \end{aligned}$$

It is clear that the coefficient of $\|\boldsymbol{x}_k\|^2$ is negative when $\varepsilon_0^2 < \frac{\lambda_{\min}(Q)}{4c_0^2\sup_{k\geqslant 1}\{\lambda_{\max}(\varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma)\}\cdot M_2^2}$. Then when $\|\boldsymbol{x}_k\| > \frac{4d_2+2\sqrt{4d_2^2+d_1d_3}}{d_1}$, we have $\Delta V < 0$.

Case 3. Some $a_i(k) = 0$, and the others $a_i(k) = 1$.

Without loss of generality, we can assume that $a_i(k) = 1$, $i = 1,\ldots,m$, $m < p$; and $a_i(k) = 0$, $i = m+1,\ldots,p$.

Let the i-th component of v_k (i.e., $v(k)$) be denoted by $v_i(k)$. It follows from (51), (52) and (56) that

$$\Delta V < \frac{-2\mu_1\mu[K\lambda_{\min}(Q) - 6ms_1\cdot\sup_{k\geqslant 1}\{\lambda_{\max}(\Lambda_k)\}\cdot\varepsilon_0^2]\cdot\|\boldsymbol{x}_k\|^2 + 12m\mu_1s_1\varepsilon_0^2 + \mu\sum_{i=m+1}^{p}v_i(k)^2}{2\mu_1(1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k)}.$$

From (52), we have $v_i(k) = -e_i(k+1)$. Thus, for $i = m+1,\ldots,p$, we have $|v_i(k)| \leqslant 2\Delta(k+1)$, since $|e_i(k)| \leqslant 2\Delta(k)$. Therefore, from (59), we obtain the following inequality:

$$\Delta V < \frac{-\mu d_4\|\boldsymbol{x}(k)\|^2 + 2d_5\|\boldsymbol{x}_k\| + d_6}{\mu_1(1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k)},$$

where $d_4 = \mu_1 K\lambda_{\min}(Q) - 6\mu_1 m s_1\varepsilon_0^2 \cdot \sup_{k\geqslant 1}\{\lambda_{\max}(\varLambda_k)\} - 2(p-m)M_2^2\varepsilon_0^2$, $d_5 = 2(p-m)\mu M_2(M_3+1)\varepsilon_0^2$, $d_6 = 6m\mu_1 s_1\varepsilon_0^2 + 2(p-m)\mu(M_3+1)^2\varepsilon_0^2$.

Thus, when $\|\boldsymbol{x}_k\| > \frac{d_5+\sqrt{d_5^2+\mu d_4 d_6}}{\mu d_4}$, $\Delta V < 0$.

In the following discussion, we prove that the tracking error converges to a small neighborhood of the origin, and present the convergence radius.

Let $\bar{\boldsymbol{e}}(k) = [\boldsymbol{x}_k^{\mathrm{T}}, \tilde{\boldsymbol{\theta}}(k)^{\mathrm{T}})]^{\mathrm{T}}$. For Cases 1–3, let $\delta_1 = \varepsilon_0\sqrt{\frac{6ps_1}{\mu d_0}}$, $\delta_2 = \frac{4d_2+2\sqrt{4d_2^2+d_1d_3}}{d_1}$, and $\delta_3 = \frac{d_5+\sqrt{d_5^2+\mu d_4 d_6}}{\mu d_4}$, respectively. It is easily shown by differentiating δ_i with respect to ε_0 that δ_1, δ_2 and δ_3 decrease with decreasing ε_0. It follows from (23) that $\|\tilde{\boldsymbol{\theta}}_i(k)\| \leqslant \|\hat{\boldsymbol{\theta}}_i(0) - \boldsymbol{\theta}_i\|$, i.e., $\tilde{\boldsymbol{\theta}}_i(k)$ is bounded. Thus, we can suppose that $\|\tilde{\boldsymbol{\theta}}_i(k)\| \leqslant W_i$, where W_i is a positive constant. We can define the following closed set $B_j = \{\bar{\boldsymbol{e}}(k) \,|\, \|\boldsymbol{x}_k\| \leqslant \delta_j,\ \|\tilde{\boldsymbol{\theta}}_i(k)\| \leqslant W_i,\ i=1,\ldots,p\},\ j=1,2,3$. Let

$$S(p_{0j}) = \{\bar{\boldsymbol{e}}(k) \mid V(\boldsymbol{x}_\theta(k)) \leqslant p_{0j},\ \|\tilde{\boldsymbol{\theta}}_i(k)\| \leqslant W_i,\ i=1,\ldots,p\}, \tag{60}$$

where p_{0j} is a positive constant. Also, define the closed set $S(\underline{p}_j) = \{\bar{\boldsymbol{e}}(k) \,|\, V(\boldsymbol{x}_\theta(k)) \leqslant \underline{p}_j,\ \|\tilde{\boldsymbol{\theta}}_i(k)\| \leqslant W_i,\ i=1,\ldots,p\}$. Here $\underline{p}_j = \sum_{i=1}^p W_i^2 + K\ln(1+\mu\,\sup_{k\geqslant 1}\{\lambda_{\max}(\varLambda_k)\}\delta_j^2)$. $S(\underline{p}_j)$ is the smallest closed set containing the closed set B_j with the form (60). Now we choose an arbitrary constant $\bar{p}_j > \underline{p}_j$. According to [24], $\bar{\boldsymbol{e}}(k)$ is uniformly and ultimately bounded with respect to $S(\bar{p}_j)$, i.e., there exists a non-negative constant $T(\bar{\boldsymbol{e}}(k_j), S(\bar{p}_j))$ such that

$$\bar{\boldsymbol{e}}(k) \in S(\bar{p}_j), \tag{61}$$

for all $k \geqslant k_j + T(\bar{\boldsymbol{e}}(k_j), S(\bar{p}_j))$. Here k_j is a positive integer.

From $K\ln(1+\mu\lambda_{\min}(\varLambda_{k+1})\|\boldsymbol{x}_k\|^2) \leqslant K\ln(1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k) \leqslant V(\boldsymbol{x}_\theta(k)) \leqslant \bar{p}_j$, we have

$$\|\boldsymbol{y}_k - \boldsymbol{y}_d^*\| \leqslant \|\boldsymbol{x}_k\| \leqslant \sqrt{(\mathrm{e}^{\bar{p}_j/K} - 1)/(\mu\lambda_{\min}(\varLambda_{k+1}))}.$$

We have proved that if the sampling period T is sufficiently small, then $\|\hat{\boldsymbol{\theta}}_i(0) - \boldsymbol{\theta}_i\| \to 0$ can be ensured by choosing a $\hat{\boldsymbol{\theta}}_i(0)$ that satisfies Property 1 (see Step 2). Thus, we can assume W_i to be a sufficiently small positive number. When T is sufficiently small, ε_0 will be sufficiently small [4,6]). Thus, δ_j will also be sufficiently small, and so will be $\underline{p}_j$. Therefore, $\bar{p}_j$ can be taken as a small positive number. Consequently, the tracking error converges to a small neighborhood of the origin, and the radius of the neighborhood depends on the sampling period T. The result is proved by taking $\bar{p} = \max\{\bar{p}_1, \bar{p}_2, \bar{p}_3\}$ and $\bar{k} = \max\{k_1, k_2, k_3\}$.

Remark 4. It is readily seen from the proof of Theorem 1 that our proposed proof methodology is also suitable for tracking bounded time-varying reference signals.

Remark 5. It should be mentioned that to avoid possible controller singularity problem, $\hat{G}_k$ in (10) and (11) can be replaced by $\sigma_0 I + \hat{G}_k^{\mathrm{T}}\hat{G}_k$, where σ_0 is a prescribed positive constant. Theorem 1 would still holds in this case. The upper bound for $\|(\sigma_0 I + \hat{G}_k^{\mathrm{T}}\hat{G}_k)^{-1}\|$ is determined by the following discussion.

Let $\hat{G}_k^{\mathrm{T}}\hat{G}_k = (b_{ij}(k))$. Since the estimated parameters are bounded according to Conclusion 1 of Lemma 6, we can assume that $|b_{ij}(k)| \leqslant \bar{b}_{ij}$, where each $\bar{b}_{ij}$ is a positive constant. Let σ_0 be a positive constant that satisfies the inequality $\sigma_0 > \sum_{j=1,j\neq i}^p \bar{b}_{ij} + \tau$, $i = 1,2,\ldots,p$; where τ is a prescribed positive constant. Therefore, we have $\sigma_0 + b_{ii}(k) \geqslant \sigma_0 > \sum_{j=1,j\neq i}^p \bar{b}_{ij} + \tau \geqslant \sum_{j=1,j\neq i}^p |b_{ij}(k)| + \tau \geqslant \tau > 0$, which implies that $\sigma_0 I + \hat{G}_k^{\mathrm{T}}\hat{G}_k$ is a strictly diagonally dominant matrix. From formula 1 in [25], we have $\|(\sigma_0 I + \hat{G}_k^{\mathrm{T}}\hat{G}_k)^{-1}\|_\infty \leqslant 1/\tau$. Therefore, $\|(\sigma_0 I + \hat{G}_k^{\mathrm{T}}\hat{G}_k)^{-1}\|_F \leqslant p \cdot \|(\sigma_0 I + \hat{G}_k^{\mathrm{T}}\hat{G}_k)^{-1}\|_\infty \leqslant p/\tau$.

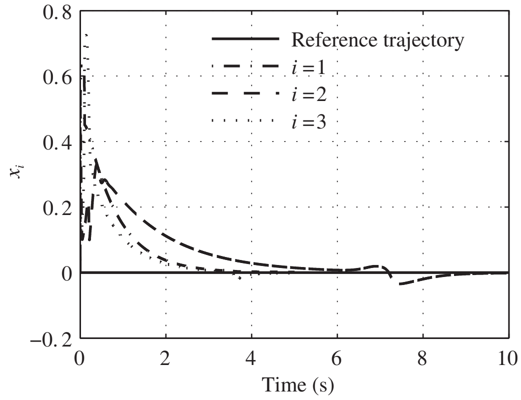

Figure 1 Reference trajectory and system outputs (time $t = 10$ s).

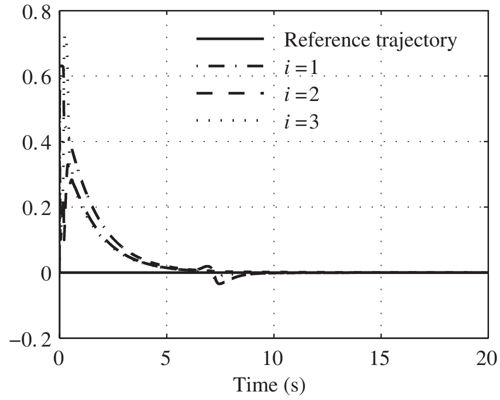

Figure 2 Reference trajectory and system outputs (time $t = 20$ s).

In summary, the above-mentioned method can not only avoid possible controller singularity problem, but also conveniently estimate the upper bound for $\|(\sigma_0 I + \hat{G}_k^{\mathrm{T}}\hat{G}_k)^{-1}\|$. By analyzing the expression (9)–(11) for $\boldsymbol{u}_k$, we conclude that the control input is bounded.

From the proof of Theorem 1, we can guarantee the convergence of the estimations of the coefficient matrices of the characteristic model (1) if the initial values of parameter estimation satisfy the Property 1 and the sampling period and approximate errors are all sufficiently small. From this viewpoint, we can avoid non-convergence of parameter estimation by considering modeling and control simultaneously.

Finally, note that the results from Theorem 1 can be applied to the SISO case. In this scenario, Case 3 will vanish from the proof of Theorem 1.

5 Simulation

In this section, we will validate the efficiency of the golden-section controller (9) based on the characteristic model by using a numerical example. Consider the following non-affine nonlinear system.

$$\begin{aligned}
\dot{x}_1 &= x_1u_1u_2 + 0.2u_3,\\
\dot{x}_2 &= x_1 + x_2^2 + x_3 + 3u_1 + u_2,\\
\dot{x}_3 &= x_1 + 2x_2 + 3x_1x_3 + u_1 + 2(2 + 0.5\sin x_1)u_2.
\end{aligned}$$

Here, u_1 and u_2 are the control inputs. We use law (9) to control the states x_1, x_2 and x_3 of the system to approach zero signal. The initial state is $[x_1, x_2, x_3] = [0, 0, 0]$. Let q =0, and the initial parameter vectors in controller (9) be $\hat{F}_1(0) = 2I$, $\hat{F}_2(0) = -I$, and $\hat{G}(0) = 500TI$. Thus, the initial parameter vectors satisfy Property 1. Let $c = 0.5$ in (12), and $a_i(k-1) = 1$ for simplify. Taking $T = 0.001$, the corresponding simulation results are shown in Figures 1 and 2. It can be observed from the simulation results that the proposed method is effective. It is well-known that it is difficult to design a controller for non-affine nonlinear system because the nonlinear function of its state equation implies control input. When the controller is designed based on characteristic model, we can overcome the above-mentioned difficulty and achieve good control performance.

6 Conclusion

In this study, we undertook the stability analysis of the characteristic model-based golden-section feedback control system, which is an unsolved problem in all-coefficient adaptive control theory. First, we studied the properties of the parameter vector adaptive laws and time-varying matrices. A new Lyapunov function was introduced to study this problem. Using this, we proved the stability of the closed-loop systems formed by the golden-section adaptive control law based on the characteristic model for MIMO controlled systems. The proof was carried out in three steps. In the first step, we transformed the characteristic model (1) into its equivalent form. In the second step, we proved the uniformly asymptotic stability of the system when the nonlinear term is zero. In the last step, we proved the convergence of the tracking errors by using the constructed Lyapunov function and some mathematical techniques. In addition, we discussed a method to avoid possible controller singularity problem. The effectiveness of the proposed method is verified by a numerical simulation.

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Appendix A Proof of Lemma 5

Since the system $\boldsymbol{x}_{k+1} = A_k\boldsymbol{x}_k$ is uniformly asymptotically stable, by Lemma 2, for any given positive definite matrix Q, there is a sequence of uniformly bounded and positive definite matrices $\varLambda_k$ satisfying the relationship:

$$A_k^{\mathrm{T}}\varLambda_{k+1}A_k - \varLambda_k = -Q - I. \tag{A1}$$

Let $c_1^2 = \sup_{k\geqslant 1}\lambda_{\max}\left(A_k^{\mathrm{T}}\varLambda_{k+1}A_k\right)$ and $c_2 = \sqrt{c_1^2+\tau_0}$. From Lemma 3, c_1^2 and c_2 are well-defined, and $c_2 > 0$. As $\varLambda_k$ is a positive definite matrix, according to Corollary 7.2.9[1)], there exists a unique nonsingular lower triangular matrix U_k with positive diagonal entries such that $\varLambda_k = U_k^{\mathrm{T}}U_k$, which is the Cholesky factorization of $\varLambda_k$. Let $F_k = \frac{1}{c_2}U_{k+1}A_k$ and $D_k = c_2U_{k+1}\varGamma$. Thus we have

$$\begin{aligned}\boldsymbol{x}_{k+1}^{\mathrm{T}}\varLambda_{k+1}\boldsymbol{x}_{k+1} - \boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k &= \left(\boldsymbol{x}_k^{\mathrm{T}}A_k^{\mathrm{T}} + \boldsymbol{v}_k^{\mathrm{T}}\varGamma^{\mathrm{T}}\right)\varLambda_{k+1}\left(A_k\boldsymbol{x}_k + \varGamma\boldsymbol{v}_k\right) - \boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k\\ &= \boldsymbol{x}_k^{\mathrm{T}}\left(A_k^{\mathrm{T}}\varLambda_{k+1}A_k - \varLambda_k + F_k^{\mathrm{T}}F_k\right)\boldsymbol{x}_k - (F_k\boldsymbol{x}_k - D_k\boldsymbol{v}_k)^{\mathrm{T}}\cdot(F_k\boldsymbol{x}_k - D_k\boldsymbol{v}_k)\\ &\quad +\boldsymbol{v}_k^{\mathrm{T}}\left(D_k^{\mathrm{T}}D_k + \varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma\right)\boldsymbol{v}_k.\end{aligned}$$

Hence it is can be seen that

$$\boldsymbol{x}_{k+1}^{T}\varLambda_{k+1}\boldsymbol{x}_{k+1} - \boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k \leqslant \boldsymbol{x}_k^{\mathrm{T}}\left(A_k^{\mathrm{T}}\varLambda_{k+1}A_k - \varLambda_k + F_k^{\mathrm{T}}F_k\right)\boldsymbol{x}_k + \boldsymbol{v}_k^{\mathrm{T}}\left(D_k^{\mathrm{T}}D_k + \varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma\right)\boldsymbol{v}_k. \tag{A2}$$

Notice that $c_2^2 = c_1^2 + \tau_0$. Applying this, we get

$$\boldsymbol{x}_k^{\mathrm{T}}F_k^{\mathrm{T}}F_k\boldsymbol{x}_k = \frac{1}{c_2^2}\boldsymbol{x}_k^{\mathrm{T}}A_k^{\mathrm{T}}\varLambda_{k+1}A_k\boldsymbol{x}_k \leqslant \frac{1}{c_2^2}\lambda_{\max}\left(A_k^{\mathrm{T}}\varLambda_{k+1}A_k\right)\|\boldsymbol{x}_k\|^2 \leqslant \frac{1}{c_2^2}c_1^2\|\boldsymbol{x}_k\|^2 = \frac{c_1^2}{c_1^2+\tau_0}\|\boldsymbol{x}_k\|^2 < \|\boldsymbol{x}_k\|^2. \tag{A3}$$

From (A1), it follows that

$$\boldsymbol{x}_k^{\mathrm{T}}\left(A_k^{\mathrm{T}}\varLambda_{k+1}A_k - \varLambda_k\right)\boldsymbol{x}_k = \boldsymbol{x}_k^{\mathrm{T}}(-Q-I)\boldsymbol{x}_k = -\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k - \boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{x}_k. \tag{A4}$$

Using (A3) and (A4), we deduce that

$$\boldsymbol{x}_k^{\mathrm{T}}\left(A_k^{\mathrm{T}}\varLambda_{k+1}A_k - \varLambda_k + F_k^{\mathrm{T}}F_k\right)\boldsymbol{x}_k = -\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k - \boldsymbol{x}_k^{\mathrm{T}}\boldsymbol{x}_k + \boldsymbol{x}_k^{\mathrm{T}}F_k^{\mathrm{T}}F_k\boldsymbol{x}_k < -\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k. \tag{A5}$$

We note that

$$\boldsymbol{v}_k^{\mathrm{T}}\left(D_k^{\mathrm{T}}D_k + \varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma\right)\boldsymbol{v}_k = \boldsymbol{v}_k^{\mathrm{T}}c_2^2\varGamma^{\mathrm{T}}U_{k+1}^{\mathrm{T}}U_{k+1}\varGamma\boldsymbol{v}_k + \boldsymbol{v}_k^{\mathrm{T}}\varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma\boldsymbol{v}_k = c_0^2\boldsymbol{v}_k^{\mathrm{T}}\varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma\boldsymbol{v}_k. \tag{A6}$$

According to (A2), (A5) and (A6), we have

$$\boldsymbol{x}_{k+1}^{\mathrm{T}}\varLambda_{k+1}\boldsymbol{x}_{k+1} - \boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k < -\boldsymbol{x}_k^{\mathrm{T}}Qx_k + c_0^2\,\boldsymbol{v}_k^{\mathrm{T}}\varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma\boldsymbol{v}_k. \tag{A7}$$

Applying (A7), we obtain the following expression:

$$V(\boldsymbol{x}_{k+1}) - V(\boldsymbol{x}_k) = \ln\left(1 + \mu\frac{\boldsymbol{x}_{k+1}^{\mathrm{T}}\varLambda_{k+1}\boldsymbol{x}_{k+1} - \boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k}\right) < \mu\frac{-\boldsymbol{x}_k^{\mathrm{T}}Q\boldsymbol{x}_k + c_0^2\,\boldsymbol{v}_k^{\mathrm{T}}\varGamma^{\mathrm{T}}\varLambda_{k+1}\varGamma\boldsymbol{v}_k}{1+\mu\boldsymbol{x}_k^{\mathrm{T}}\varLambda_k\boldsymbol{x}_k}.$$

The proof is completed.

Appendix B Proof of Lemma 6

(1) Using a manner similar to that of the proof of (3.6.12)[2)], we obtain the following expression:

$$\left\|\tilde{\boldsymbol{\theta}}_i(k)\right\|^2 \leqslant \|\tilde{\boldsymbol{\theta}}_i(k-1)\|^2 - \frac{1}{2}\frac{a_i(k-1)e_i(k)^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)} + \frac{2a_i(k-1)[\bar{\omega}_i(k-1)+E_i(k-1)]^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}.$$

Since $|\omega_{fij}(z_{ijk})| < \frac{\varepsilon}{\sqrt{3p}}$ and $|\omega_{gij}(z_{i,\,2p+j,\,k})| < \frac{\varepsilon}{\sqrt{3p}}$, we have $\|\omega_i(k-1)\| < \varepsilon$. Note that $|E_i(k)| < \varepsilon_0$. Thus, we have

$$|\bar{\omega}_i(k-1) + E_i(k-1)| \leqslant |\boldsymbol{\phi}_{k-1}^{\mathrm{T}}\omega_i(k-1)| + |E_i(k-1)| < \|\boldsymbol{\phi}_{k-1}\|\cdot\|\omega_i(k-1)\| + \varepsilon_0 < \|\boldsymbol{\phi}_{k-1}\|\cdot\varepsilon + \varepsilon_0 = \Delta(k). \tag{B1}$$

Hence,

$$\|\tilde{\boldsymbol{\theta}}_i(k)\|^2 \leqslant \|\tilde{\boldsymbol{\theta}}_i(k-1)\|^2 - \frac{1}{2}\frac{a_i(k-1)[e_i(k)^2 - 4\Delta(k)^2]}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}.$$

In view of (13), $\{\|\tilde{\boldsymbol{\theta}}_i(k)\|^2\}$ is a non-increasing sequence bounded below by zero. This establishes (23).

It is obvious that

$$\|\hat{\boldsymbol{\theta}}_i(k)\| = \|\hat{\boldsymbol{\theta}}_i(k) - \boldsymbol{\theta}_i + \boldsymbol{\theta}_i\| \leqslant \|\hat{\boldsymbol{\theta}}_i(k) - \boldsymbol{\theta}_i\| + \|\boldsymbol{\theta}_i\| \leqslant \|\hat{\boldsymbol{\theta}}_i(0) - \boldsymbol{\theta}_i\| + \|\boldsymbol{\theta}_i\|, \quad k \geqslant 1.$$

1) Horn R A, Johnson C R. Matrix Analysis. Cambridge: Cambridge University Press, 1985. 345–407.
2) Goodwin G C, Sin K S. Adaptive Filtering Prediction and Control. New Jersey: Prentice-Hall Inc, 1984. 47–105.

From this, it is easy to see that $\hat{\boldsymbol{\theta}}_i(k)$ is bounded.

(2) We proceed to (24)–(27). Note first that

$$\frac{\Delta(k)^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}=\frac{[\|\boldsymbol{\phi}_{k-1}\|\cdot\varepsilon+\varepsilon_0]^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}\leqslant\frac{\|\boldsymbol{\phi}_{k-1}\|^2\varepsilon^2+2\|\boldsymbol{\phi}_{k-1}\|\cdot\varepsilon\varepsilon_0+\varepsilon_0^2}{c+\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))\|\boldsymbol{\phi}_{k-1}\|^2}. \tag{B2}$$

When $\|\boldsymbol{\phi}_{k-1}\|\leqslant\sqrt{c}$, it follows from (B2) that

$$\frac{\Delta(k)^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}\leqslant\frac{c\varepsilon^2+2\varepsilon\varepsilon_0\sqrt{c}+\varepsilon_0^2}{c}=\frac{[\varepsilon\sqrt{c}+\varepsilon_0]^2}{c}. \tag{B3}$$

When $\|\boldsymbol{\phi}_{k-1}\|\geqslant\sqrt{c}$, noting that $L_i(k)^{\mathrm{T}}L_i(k)=[\boldsymbol{l}_{i1}(k)^{\mathrm{T}}\boldsymbol{l}_{i1}(k)+\cdots+\boldsymbol{l}_{i,\,3p}(k)^{\mathrm{T}}\boldsymbol{l}_{i,\,3p}(k)]I_{3p}$, we see that

$$\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))=\boldsymbol{l}_{i1}(k)^{\mathrm{T}}\boldsymbol{l}_{i1}(k)+\cdots+\boldsymbol{l}_{i,\,3p}(k)^{\mathrm{T}}\boldsymbol{l}_{i,\,3p}(k).$$

Since $\boldsymbol{l}_{i1}(k),\ldots,\boldsymbol{l}_{i,\,3p}(k)$ are vectors made up of basis functions, $\boldsymbol{l}_{i1}(k)^{\mathrm{T}}\boldsymbol{l}_{i1}(k)+\cdots+\boldsymbol{l}_{i,\,3p}(k)^{\mathrm{T}}\boldsymbol{l}_{i,\,3p}(k)>0$. Thus $\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))>0$. From (B2) it follows that

$$\begin{aligned}\frac{\Delta(k)^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}&<\frac{||\boldsymbol{\phi}_{k-1}\|^2\varepsilon^2+2\|\boldsymbol{\phi}_{k-1}\|\cdot\varepsilon\varepsilon_0+\varepsilon_0^2}{\lambda_{\min}(L_k(k)^{\mathrm{T}}L_k(k))\|\boldsymbol{\phi}_{k-1}\|^2}\\&\leqslant\frac{\varepsilon^2}{\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))}+\frac{2\varepsilon\varepsilon_0}{\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))\sqrt{c}}+\frac{\varepsilon_0^2}{\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))c}\\&=\frac{[\varepsilon\sqrt{c}+\varepsilon_0]^2}{\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))c}.\end{aligned} \tag{B4}$$

Provided that $P_0(x),\ldots,P_q(x)$ are continuous functions, then $P_0(z_{ijk}),\ldots,P_q(z_{ijk})$ are bounded on $[0,1]$. So, $L_i(k)$ is bounded. Therefore, $\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))$ is also bounded, and its infimum $\bar{\lambda}_{\min}(L_i(k)^{\mathrm{T}}L_i(k))$ is well-defined.

Since $\bar{\lambda}_{\min}(L_i(k)^{\mathrm{T}}L_i(k))>0$, from (B3) and (B4) we see that Eq. (24) holds. Using a method similar to that followed in the proof of (3.6.8)[2)], we obtain

$$\lim_{k\to\infty}\frac{a_i(k-1)[e_i(k)^2-4\Delta(k)^2]}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}=0. \tag{B5}$$

Thus, there exists a constant $M>0$ such that for any $k\geqslant1$,

$$\frac{|a_i(k-1)[e_i(k)^2-4\Delta(k)^2]|}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}\leqslant M. \tag{B6}$$

As $a_i(k-1)=0$ or 1, $a_i(k-1)^2=a_i(k-1)$. From (B6) and (24), it is clear that

$$\frac{a_i(k-1)^2e_i(k)^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}=\frac{a_i(k-1)(e_i(k)^2-4\Delta(k)^2)}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}+\frac{4a_i(k-1)\Delta(k)^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}\leqslant M+4s_{0i}^2.$$

This establishes (25).

From (B5) it follows that for any $\in>0$, there exists integer $k_0\geqslant1$ such that if $k>k_0$, we have

$$\frac{|a_i(k-1)[e_i(k)^2-4\Delta(k)^2]|}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}<\in.$$

From the previous result, and the method followed in the proof of (25), formula (26) can also be proved, provided that M in (B6) is replaced by $\in$. Especially, by taking $\in=5s_{0i}^2$ in $\overline{M}_{ei}$, we have $\overline{M}_{ei}=3s_{0i}$, namely, Eq. (27) holds.

(3) Next, we prove (28).

Applying (12) and (22), and following a method similar to that of the proof of (3.6.13)[2)], it is easy to show that

$$\|\hat{\boldsymbol{\theta}}_i(k)-\hat{\boldsymbol{\theta}}_i(k-1)\|^2=\frac{a_i(k-1)\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)\,e_i(k)^2}{[c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)]^2}\leqslant\frac{a_i(k-1)e_i(k)^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}.$$

From this and (26), we have $\lim_{k\to\infty}\sup\|\hat{\boldsymbol{\theta}}_i(k)-\hat{\boldsymbol{\theta}}_i(k-1)\|\leqslant\overline{M}_{ei}$, namely, formula (28) holds.

(4) We take one among the Legendre, Chebyshev, Laguerre, and Hermite polynomials as the basis function. In this case, noting that $P_0(z_{ijk})=1$, then we have

$$\lambda_{\min}(L_i(k)^{\mathrm{T}}L_i(k))=\boldsymbol{l}_{i1}(k)^{\mathrm{T}}\boldsymbol{l}_{i1}(k)+\cdots+\boldsymbol{l}_{i,\,3p}(k)^{\mathrm{T}}\boldsymbol{l}_{i,\,3p}(k)>1.$$

From (B3) and (B4), it can be easily seen that

$$\frac{\Delta(k)^2}{c+\boldsymbol{\varphi}_i(k-1)^{\mathrm{T}}\boldsymbol{\varphi}_i(k-1)}\leqslant\frac{[\varepsilon\sqrt{c}+\varepsilon_0]^2}{c}.$$

Hence, we can take $s_{0i}=s_0=\frac{\varepsilon\sqrt{c}+\varepsilon_0}{\sqrt{c}}$ in (24). Because Eqs. (25) and (26) are all deduced by using (24), we can also take $s_{0i}=s_0=\frac{\varepsilon\sqrt{c}+\varepsilon_0}{\sqrt{c}}$ in (25) and (26). This completes the proof of Lemma 6.