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Ordered proposition fusion based on consistency and uncertainty measurements

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Abstract The fusion of ordered propositions is an important and widespread problem in artificial intelligence, but existing fusion methods have difficulty handling the fusion of ordered propositions. In this paper, we propose a solution based on consistency and uncertainty measurements. The main contributions of this paper are as follows. First, we propose the concept of convexity degree, mean, and center of basic support function to comprehensively describe the basic support function of ordered propositions. Second, we introduce entropy as a measure of uncertainty in the basic support function of ordered propositions. Third, we generalize the indeterminacy of the basic support function and propose a novel method to measure the consistency between two basic support functions. Finally, based on the above researches, we propose a novel algorithm for fusing ordered propositions. Theoretical analysis and experimental results demonstrate that the proposed method outperforms state-of-the-art methods.

Keywords uncertainty processing, ordered proposition, evidence theory, dempster rule of combination, information fusion

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1 Introduction

Fusing information from multiple sources can yield higher quality information and significantly enhance the effectiveness of decision making. In recent years, with the development of information technologies such as information retrieval, and computer networks, the volume of information has grown explosively. Thus, the significance of information fusion has gradually increased and it has become a critical technology for intelligent information processing and big data processing.

It is infeasible to provide a universal fusion method for all information fusion problems. Individual problems must be summarized and analyzed, and a specialized fusion algorithm must be used for each problem. The fusion of ordered propositions is an important and widespread problem. A set of ordered propositions describe same characteristics or features of a subject with a gradually increasing or decreasing intensity. For example, professors evaluate students on a scale of "Excellent, Good, Fair, Poor" and

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agricultural experts evaluate the fertility of cultivated land by "high, middle, low". A set of ordered propositions can be represented as a basic support function (similar to a belief function) whose elements denote the truth-value (belief value) of each proposition. Fusion of ordered propositions means combining multiple basic support functions from different sources into a single basic support function. The truth-value of a basic support function for a set of ordered propositions must satisfy the convex property, meaning the curve of the function must be convex and unimodal, because we cannot say that fertility is both high and low simultaneously.

The Dempster-Shafer theory [1,2] is a commonly used method for fusing propositions [3]. In this theory, the problem domain is denoted by a finite nonempty set Θ, which is also called the frame of discernment. A set of propositions are represented as a belief function and Dempster's rule of combination is used for combining belief functions. As an example, for the belief functions μ and ν, the fusion result ω is defined as $\omega(A) = \sum_{B\cap C=A} \lambda_1(B)\lambda_2(C)/(1-K)$, where $A, B, C \in 2^\Theta$, and $K = \sum_{B\cap C=\emptyset} \mu(B)\nu(C)$ denotes the conflict degree between μ and ν. Dempster-Shafer theory based approaches have been widely used in many fields including information fusion [4–11]. However, these approaches cannot handle the fusion of ordered propositions because they cannot ensure that the fusion result satisfies the convex property.

Previously, fusion algorithms based on centroid were proposed [12–14] to fuse the belief functions of ordered propositions and ensure satisfaction of the convex property. These types of algorithms work in two phases. First, they find the most probable proposition based on the centroid of the mass. Second, they reallocate the truth-value of each proposition to construct a new convex belief function as the fusion result. However, this approach has a few shortcomings. First, the consistency between two basic support functions is not considered, so it cannot adapt fusion strategies for different consistencies. Second, the centroid does not precisely reflect the location of the proposition that is most likely to be true. As a result, centroid-based algorithms sometimes obtain inaccurate results.

In order to overcome the shortcomings of the aforementioned methods, we propose a novel method for the fusion of ordered propositions based on consistency and uncertainty measurements. We provide a comprehensive analysis of the basic support functions of ordered propositions, introduce entropy as a measure of the uncertainty of the basic support function, provide theoretical analysis, and present a novel method for measuring the consistency between two basic support functions. Finally, based on these concepts, we propose a novel algorithm for fusing ordered propositions.

The remainder of this paper is organized as follows. Section 2 provides a brief discussion of ordered propositions and centroid-based algorithms. In Section 3, we explain the shortcomings of the centroid-based algorithms. In Section 4, we provide definitions and properties for basic support functions, such as convex degree and center. Section 5 discusses the proposed method for measuring the consistency and uncertainty of basic support functions. The proposed fusion algorithm is described in Section 6. Section 7 presents experiments, case studies, and comparisons for the proposed method. Finally, our conclusion and direction for future work are summarized in Section 8.

2 Preliminaries

In this section, we provide some background knowledge about ordered propositions [15].

We first present an example of "source-rock evaluation". The evaluation results for source-rocks can be represented by a quad-tuple. The first element of the quad-tuple denotes the truth-value of the proposition "the source-rock richness is high". The second element of the quad-tuple denotes the truth-value of the proposition "the source-rock richness is normal". The third element denotes the truth-value of the proposition "the source-rock richness is low". Finally, the fourth element of the quad-tuple denotes the truth-value of the proposition "the source-rock richness is zero (meaning it is not a source-rock)". The range of truth-values for each proposition is [0, 1]. The quad-tuple is a basic support function, similar to a belief function in the Dempster-Shafer Theory.

For instance, a resulting quad-tuple of (0.1, 0.6, 0.0, 0.0) means that the truth-value of the first proposition is 0.1, the truth-value of the second proposition is 0.6, the truth-value of the third proposition is 0.0, and the truth-value of the fourth proposition is 0.

Definition 1 (Ordered proposition). For a set of propositions $P_1, P_2, \ldots, P_n$, let $|P_i|$ represent the truth-value of P_i and $|P_m| = \max\{|P_1|, \ldots, |P_n|\}$. $P_1, P_2, \ldots, P_n$ are ordered propositions, if

(1) $\forall i = 1, 2, \ldots, n$, all subject items of P_i are S, and P_i has the form of P_i= "S is s_i";

(2) $\forall i = 1, 2, \ldots, n$, s_i describes the same characteristics or features of S;

(3) $\forall i = 1, 2, \ldots, m-1$, $|P_i| \leqslant |P_{i+1}|$; and $\forall i = m, m+1, \ldots, n-1$, $|P_i| \geqslant |P_{i+1}|$.

Using source-rock evaluation as an example, S=source-rock richness and s_i is $\{\text{high, normal, low, zero}\}$. Thus P_1 is "the source-rock richness is high", P_2 is "the source-rock richness is normal", P_3 is "the source-rock richness is low", and P_4 is "the source-rock richness is zero". The degree of the characteristics or features increases or decreases gradually. This gradient can be represented by "less than or equal to" relationships between propositions.

Definition 2 ('less than or equal to' relationship). For a set of ordered propositions $P = \{P_1, P_2, \ldots, P_n\}$, and $1 \leqslant i, j \leqslant n, P_i$ is less than or equal to P_j (denoted $P_i \leqslant P_j$) if and only if: $i \leqslant j$ and $(P, \leqslant)$ is a totally ordered set.

For ordered propositions, the basic support function is convex, meaning the curve of the function is unimodal, because we cannot say "the source-rock richness is both high and low simultaneously".

Definition 3 (Convex property). $|P_1|, |P_2|, \ldots, |P_n|$ have the convex property, if $\forall i \leqslant j \leqslant k$ satisfies $|P_j| \geqslant \min\{|P_i|, |P_k|\}$.

$|P_i|$ represents the truth-value of proposition $P_i (1 \leqslant i \leqslant n)$. Guan et al. [16] introduced the basic support function to describe the truth-value of a set of propositions and we refined the definition for ordered propositions.

Definition 4 (Basic support function of ordered propositions). For a set of ordered propositions $S = \{s_1, s_2, \ldots, s_n\}$, a function λ is called a basic support function of the ordered propositions if

(1) λ is defined on $\{\overline{S}\} \cup \{\{s_i\} | 1 \leqslant i \leqslant n\}$, where $\bar{S}$ indicates indeterminacy;

(2) $\lambda(s_i) \geqslant 0,\ 1 \leqslant i \leqslant n$;

(3) $\sum_{1 \leqslant i \leqslant n} \lambda(s_i) \leqslant 1$;

(4) $\lambda(\overline{S}) = 1 - \sum_{1 \leqslant i \leqslant n} \lambda(s_i)$.

Take aforementioned example of "source-rock evaluation", the evaluation result is a quadri-tuple (0.1, 0.6, 0.0, 0.0), this quadri-tuple implies a basic support function $\lambda : \lambda(s_1) = 0.1, \lambda(s_2) = 0.6, \lambda(s_3) = 0.0, \lambda(s_4) = 0.0, \lambda(\bar{S}) = 0.3$, in detail,

$\lambda(s_1) = 0.1$ means the truth-value of the 1st proposition "the source-rock richness is high" is 0.1;

$\lambda(s_2) = 0.6$ means the truth-value of the 2nd proposition "the source-rock richness is normal" is 0.6;

$\lambda(s_3) = 0.0$ means the truth-value of the 3rd proposition "the source-rock richness is low" is 0.0;

$\lambda(s_4) = 0.0$ means the truth-value of the 4th proposition "the source-rock richness is zero" is 0.0;

$\lambda(\bar{S}) = 0.3$ is indeterminate part of source-rock richness.

We now briefly describe the main idea of the centroid-based method. For two basic support functions λ_1 and λ_2, the authorities (or weights) of λ_1 and λ_2 are Ω_1 and Ω_2. Let ω be the fusion result of λ_1 and λ_2, then ω is defined as follows when using centroid-based method:

$$\omega(s_i) = \begin{cases} \sum\limits_{1 \leqslant k \leqslant i} \left\{\Omega_1 \times \lambda_1(s_k)[1 + \lambda_1(\overline{S})] + \Omega_2 \times \lambda_2(s_k)[1 + \lambda_2(\overline{S})]\right\} / (g - k + 1), & \text{if } i < g, \\ \sum\limits_{1 \leqslant k \leqslant g} \left\{\Omega_1 \times \lambda_1(s_k)[1 + \lambda_1(\overline{S})] + \Omega_2 \times \lambda_2(s_k)[1 + \lambda_2(\overline{S})]\right\} / (g - k + 1) \\ \quad + \sum\limits_{g+1 \leqslant k \leqslant n} \left\{\Omega_1 \times \lambda_1(s_k)[1 + \lambda_1(\overline{S})] + \Omega_2 \times \lambda_2(s_k)[1 + \lambda_2(\overline{S})]\right\} / (k - g + 1), & \text{if } i = g, \\ \sum\limits_{i \leqslant k \leqslant n} \left\{\Omega_1 \times \lambda_1(s_k)[1 + \lambda_1(\overline{S})] + \Omega_2 \times \lambda_2(s_k)[1 + \lambda_2(\overline{S})]\right\} / (k - g + 1), & \text{if } i > g, \end{cases}$$

where $g(\omega) = \begin{cases} \lceil \mathrm{gd} \rceil, \text{if } \mathrm{gd} - \lfloor \mathrm{gd} \rfloor \geqslant \Delta_2 \\ \lfloor \mathrm{gd} \rfloor, \text{if } \mathrm{gd} - \lfloor \mathrm{gd} \rfloor \leqslant \Delta_1 \end{cases}$ and $\mathrm{gd} = \sum_{i=1}^{n} i\{\Omega_1 \lambda_1(s_i)/[1 - \lambda_1(\overline{S})] + \Omega_2 \lambda_2(s_i)/[1 - \lambda_2(\overline{S})]\}$.

$g(\omega)$ is the index of the most probable proposition in ω. If $\Delta_1 < \mathrm{gd} - \lfloor \mathrm{gd} \rfloor < \Delta_2$, two intermediate fusion results are constructed. First, ω_1 is constructed based on $g = \lfloor \mathrm{gd} \rfloor$, and ω_2 is constructed based on $g = \lceil \mathrm{gd} \rceil$. Second, the final fusion result ω is constructed by combining ω_1 and ω_2: let $\delta = \mathrm{gd} - \lfloor \mathrm{gd} \rfloor$, then $\omega(s_k) = (1-\delta)\omega_1(s_k) + \delta\omega_2(s_k)$ for all k such that $1 \leqslant k \leqslant n$.

3 Shortcomings of centroid-based methods

We first present some typical counter-examples to illustrate the shortcomings of centroid-based methods for fusing ordered propositions.

Example 1. The basic support functions for fusion are $\lambda_1 = (0.65, 0.2, 0.1, 0.05)$ and $\lambda_2 = (0.7, 0.15, 0.1, 0.05)$, and the weights of them are $\Omega_1 = \Omega_2 = 0.5$.

λ_1 and λ_2 are so consistent that they both imply that the 1st proposition is most likely to be true. Take "source-rock evaluation" as an example. λ_1 and λ_2 are both imply that source-rock richness is high. Thus, the clarity of the fusion result should be increased, meaning the truth-value of the 1st proposition $\omega(s_1)$ should be greater than $\max\{0.65, 0.7\}$.

However, using a centroid-based method, the centroid point of the basic support functions is calculated as gd$=\sum_{1\leqslant i \leqslant n} 0.5 \times \{\lambda_1(s_i) + \lambda_2(s_i)\} \times i$=1.525. The final fusion result is constructed based on the centroid point and the resulting basic support function is $\omega = (0.561, 0.367, 0.057, 0.015)$. $\omega(s_1) = 0.561$ is less than both 0.65 and 0.7. This result is incorrect and unintuitive.

Example 2. The basic support functions for fusion are $\lambda_1 = (0.25, 0.25, 0.25, 0.25)$ and λ_2 =(0.25, 0.25, 0.25, 0.25), and their weights are $\Omega_1 = \Omega_2 = 0.5$.

The basic support functions are completely indeterminate. In the example of source-rock evaluation, λ_1 and λ_2 indicate nothing about source-rock richness. However, using the centroid-based method, gd $= 2.5$, and the fusion result is $\omega = (0.104, 0.396, 0.396, 0.104)$, which is completely unreasonable.

Example 3. The basic support functions for fusion are $\lambda_1 = (0.1, 0.2, 0.2, 0.5)$ and $\lambda_2 = (0.1, 0.15, 0.15, 0.6)$, and their weights are $\Omega_1 = \Omega_2 = 0.5$.

λ_1 and λ_2 are consistent and they all imply the 4th proposition is most probable to be true, but using the centroid-based method, gd $= 3.175$, and the fusion result is $\omega = (0.033, 0.121, 0.571, 0.275)$, it implies the 3rd proposition is most probable to be true, and this is obviously incorrect.

Example 4. The basic support functions for fusion are $\lambda_1 = (0.05, 0.1, 0.15, 0.7)$ and $\lambda_2 = (0.05, 0.1, 0.15, 0.7)$, and their weights are $\Omega_1 = \Omega_2 = 0.5$.

λ_1 and λ_2 are completely consistent because they are identical. λ_1 and λ_2 imply that the 4th proposition is most likely to be true. Thus, the clarity of the fusion result should be increased, meaning the truth-value of the 4th proposition $\omega(s_4)$ should be greater than 0.7. However, using the centroid-based method, gd $= 3.5$, and the fusion result is $\omega = (0.015, 0.056, 0.344, 0.585)$. Although this result agrees with the 4th proposition, $\omega(s_4)$ is less than 0.7, meaning the clarity of the fusion result is decreased.

In summary, the reasons for the aforementioned shortcomings of centroid-based methods include: (1) The consistency between two basic support functions is not considered, so these methods cannot adapt fusion strategies based on different consistencies. (2) The centroid cannot precisely reflect the location of the proposition that is most likely to be true. These problems sometimes cause centroid-based methods to yield incorrect results.

4 Some definitions and properties of basic support functions

For a set of ordered propositions $S = \{s_1, s_2, \ldots, s_n\}$, we present the following definitions to describe their properties.

Definition 5 (Determinate part and indeterminate part of basic support function)**.** For a basic support function λ, the Determinate Part $\lambda(S)$ and Indeterminate Part $\lambda(\bar{S})$ are defined as

$$\lambda(S) = \sum\nolimits_{i=1,\ldots,n} \lambda(s_i), \qquad \lambda(\overline{S}) = 1 - \lambda(S). \tag{1}$$

It can be observed from Definition 5 that the determinate part of a basic support function λ is the sum of the truth-values of λ. The greater the determinate part is, the lower indeterminate part is. This means that λ becomes more effective as the determinate part increases.

Definition 6 (Mean of a basic support function). The mean value of a basic support function λ is defined as

$$\overline{\lambda} = (1/n)\sum_{i=1}^{n}\lambda(s_i). \tag{2}$$

Definition 7 (Degree of convexity). The degree of convexity of a basic support function λ is defined as

$$\text{convex}(\lambda) = \max\{\lambda(s_1),\ \lambda(s_2),\ \ldots,\ \lambda(s_n)\} - \overline{\lambda}. \tag{3}$$

It can be observed from Definition 7 that the maximum of convex(λ) is $1-\bar{\lambda}$. Thus, we can normalize (3) to find a normalized convex(λ) as follows:

$$\text{NC}(\lambda) = (\max\{\lambda(s_1),\ \lambda(s_2),\ \ldots,\ \lambda(s_n)\} - \overline{\lambda})/(1-\overline{\lambda}). \tag{4}$$

The greater the degree of convexity is, the more clear and determinate the basic support function is. The degree of convexity reflects the degree of clarify of an opinion that the basic support function (evidence) describes.

5 Measures of consistency and uncertainty of basic support functions

In this section, we propose methods to measure the degree of consistency and uncertainty of basic support functions.

5.1 Consistency of basic support function

Definition 8 (Generalized indeterminacy). For basic support function $\lambda = (\lambda(s_1), \lambda(s_2), \ldots, \lambda(s_n))$, λ is a generalized indeterminate basic support function if $\lambda(s_1) = \lambda(s_2) = \cdots = \lambda(s_n)$ or $\lambda(s_1) \approx \lambda(s_2) \approx \cdots \approx \lambda(s_n) (0 \leqslant \lambda(s_k) \leqslant 1/n, k = 1, \ldots, n)$.

$\lambda(s_j) \approx \lambda(s_k)$ indicates that $\lambda(s_j)$ is approximately equal to $\lambda(s_k)$, meaning $\forall 1 \leqslant j, k \leqslant n, |\lambda(s_j) - \lambda(s_k)| \leqslant \epsilon$ ($\epsilon > 0$ is a small real number).

Definition 9 (Center of a basic support function). For a basic support function $\lambda = (\lambda(s_1), \lambda(s_2), \ldots, \lambda(s_n))$, the center of λ is defined as

$$\text{CI}(\lambda) = \begin{cases} \arg\max\limits_{i=1,\ldots,n} \lambda(s_i), & \text{NC}(\lambda) \geqslant \theta, \\ \dfrac{\sum_{i=1,\ldots,n \wedge \lambda(s_i) \geqslant \tau\cdot\bar{\lambda}} \lambda(s_i) \times i}{\sum_{i=1,\ldots,n \wedge \lambda(s_i) \geqslant \tau\cdot\bar{\lambda}} \lambda(s_i)}, & \text{otherwise}, \end{cases} \tag{5}$$

θ could be set to 0.55 based on our experience and experiments, $1 < \tau \leqslant 1.5$.

Definition 10 (Consistency between basic support functions). If CI(μ) and CI(ν) are the centers of the basic support functions μ and ν, then $\bar{\mu}$ and $\bar{\nu}$ are the mean value of μ and ν. The consistency between μ and ν is defined as

$$\Delta G(\mu, \nu) = |\text{CI}(\mu) - \text{CI}(\nu)|/(n-1). \tag{6}$$

If ΔG =1, then μ and ν are inconsistent. If ΔG =0, then μ and ν are consistent. Otherwise, if $0 < \Delta G < 1$, then μ and ν are considered to be partially consistent. The consistency between μ and ν can be divided into 3 degrees.

$0 \leqslant \Delta G \leqslant \delta_1$: The consistency between μ and ν is high.

$\delta_1 < \Delta G \leqslant \delta_2$: The consistency between μ and ν is medium.

$\delta_2 < \Delta G$: The consistency between μ and ν is poor.

Based on our experience and experiments, $\delta_1 = 1/6$ and $\delta_2 = 1/3$ are suitable for the proposed method.

5.2 Uncertainty measure of basic support functions based on entropy

Uncertainty measures can supply additional viewpoints for analyzing data and help us to define the substantive characteristics of datasets [17, 18]. We propose an entropy-based method to measure the uncertainty of a basic support function. We first use three basic support functions as an example: $\lambda_1 = (0, 1, 0, 0), \lambda_2 = (0.15, 0.7, 0.1, 0.05)$, and $\lambda_3 = (0.25, 0.25, 0.25, 0.25)$. The degrees of uncertainty of the basic support functions vary significantly. The degree of uncertainty of λ_1 is the lowest, because it is completely certain. The degree of uncertainty of λ_2 is medium. The degree of uncertainty of λ_3 is the highest because it is completely uncertain.

Shannon [19] defined entropy as a measure of the uncertainty of information. We extend that definition and introduce the concept of entropy for basic support functions.

Definition 11 (Entropy of a basic support function)**.** Suppose Θ is a space containing all n-tuple basic support functions other than $(\lambda(s_1) = 0, \lambda(s_2) = 0, \ldots, \lambda(s_n) = 0), n \geqslant 2$, and let $x \ln(1/x) = 0$ if $x = 0$. For a basic support function $\lambda = (\lambda(s_1), \lambda(s_2), \ldots, \lambda(s_n)) \in \Theta$, the entropy of λ is defined as

$$H(\lambda) = \sum_{i=1}^{n} \lambda(s_i) \ln \left(\frac{1}{\lambda(s_i)}\right). \tag{7}$$

Eq. (7) shows that $H(\lambda)$ reaches the maximum value of ln$n$ if $\lambda(s_1) = \lambda(s_2) = \cdots = \lambda(s_n) = 1/n$. Additionally, $H(\lambda)$ reaches the minimum value of 0 if $\exists 1 \leqslant k \leqslant n, \lambda(s_k) = 1$.

For a basic support function with a unique maximum truth-value, entropy measures how much uncertainty exists in the basic support function. The greater the entropy, the greater the degree of uncertainty. For example, take two basic support functions $\lambda_1 = (0.005, 0.99, 0.005, 0, 0)$ and λ_2=(0.0049995, 0.990001, 0.0049995, 0, 0), then $H(\lambda_1) = 0.062933$ and $H(\lambda_2) = 0.062928$. The maximum truth-value of λ_2 is 0. 990001, which is 0.000001 greater than maximum truth-value of λ_1, which is 0.99. This means that the degree of the uncertainty of λ_1 is higher. Thus, $H(\lambda_1)$ is greater than $H(\lambda_2)$. We present Theorem 1 to provide theoretical support for this conclusion.

Theorem 1. Suppose that Θ is a function space containing all n-tuple basic support functions other than $(0, \ldots, 0)$, $n \geqslant 2, \lambda = (x_1, x_2, \ldots, x_n)$ is a basic support function with a unique maximum in Θ. Let $x\ln(1/x) = 0$ if $x = 0$. Suppose $x_k = \max\{x_1, x_2, \ldots, x_n\}$ is the maximum truth-value in λ. Take another basic support function $\lambda' = (x_1 - \Delta_1, \ldots, x_{k-1} - \Delta_{k-1}, x_k + \Delta_k, x_{k+1} - \Delta_{k+1}, \ldots, x_n - \Delta_n) \in \Theta$, and let $0 < \Delta_k, x_k < 1; \Delta_1, \Delta_2, \ldots, \Delta_{k-1}, \Delta_{k+1}, \ldots, \Delta_n \geqslant 0, \Delta_k = \Delta_1 + \Delta_2 + \cdots + \Delta_{k-1} + \Delta_{k+1} + \cdots + \Delta_n$. Then, $H(\lambda) > H(\lambda')$, meaning λ and λ' both satisfy the following equation:

$$\left(\sum_{j=1,\ldots,n} x_j \ln \frac{1}{x_j}\right) - \left((x_k + \Delta_k) \ln \frac{1}{x_k + \Delta_k} + \sum_{j=1,\ldots,n \,\wedge\, j \neq k} (x_j - \Delta_j) \ln \frac{1}{x_j - \Delta_j}\right) > 0.$$

Proof. By mathematical induction.

(1) Firstly we prove the theorem is true if $n = 2$.

Suppose $\lambda = (x_1, x_2) \in \Theta$ is a basic support function with single maximum, and $x_2 > x_1 \geqslant \Delta > 0, x_2 + \Delta \leqslant 1, x_1 - \Delta \geqslant 0, \lambda' = (x_1 - \Delta,\ x_2 + \Delta)$.

We know $x\ln(1/x)$ is monotonically decreasing on interval $[1/e, 1]$, and monotonically increasing over interval $[0, 1/e]$, it gets maximum at $x = 1/e$. The gradient is $(x \ln(1/x))' = -\ln x - 1$.

If $x_1 \leqslant 1/e \leqslant x_2$, then we have $x_2 \ln(1/x_2) > (x_2 + \Delta) \ln(1/(x_2 + \Delta))$ because of the monotonic decreasing of $x \ln(1/x)$, and we also have $x_1 \ln(1/x_1) > (x_1 - \Delta) \ln(1/(x_1 - \Delta))$ because of the monotonic increasing of $xln(1/x)$. So we obtain $x_1 \ln(1/x_1) + x_2 \ln(1/x_2) > (x_1 - \Delta) \ln(1/(x_1 - \Delta)) + (x_2 + \Delta) \ln(1/(x_2 + \Delta))$.

If $1/e \leqslant x_1 \leqslant x_2$, then for the gradient at x_1 and x_2, we have $0 \geqslant (x_1 \ln(1/x_1))' = -\ln x_1 - 1 > -\ln x_2 - 1 = (x_2 \ln(1/x_2))'$, thus $(x_1 - \Delta) \ln(1/(x_1 - \Delta)) - x_1 \ln(1/x_1) < x_2 \ln(1/x_2) - (x_2 + \Delta) \ln(1/(x_2 + \Delta))$. So we obtain $x_1 \ln(1/x_1) + x_2 \ln(1/x_2) > (x_1 - \Delta) \ln(1/(x_1 - \Delta)) + (x_2 + \Delta) \ln(1/(x_2 + \Delta))$.

If $x_1 \leqslant x_2 \leqslant 1/e$, then for the gradient at x_1 and x_2, we have $(x_1 \ln(1/x_1))' = -\ln x_1 - 1 > -\ln x_2 - 1 = (x_2 \ln(1/x_2))' \geqslant 0$, thus $(x_2 + \Delta)\ln(1/(x_2 + \Delta)) - x_2 \ln(1/x_2) < x_1 \ln(1/x_1) - (x_1 - \Delta)\ln(1/(x_1 - \Delta))$. So we obtain $x_1 \ln(1/x_1) + x_2 \ln(1/x_2) > (x_1 - \Delta)\ln(1/(x_1 - \Delta)) + (x_2 + \Delta)\ln(1/(x_2 + \Delta))$.

That means the entropy of λ' is less than λ, because λ' is more determinate.

(2) Assume Theorem 1 is ture for k $(2 \leqslant k < n)$, we then prove that Theorem 1 is also ture for $k = n$.

Because $\lambda = (x_1, x_2, \ldots, x_n)$ is convex, it can be derived that $\min\{x_1, \ldots, x_n\} = \{x_1, x_n\}$. We might as well assume $x_n = \min\{x_1, x_2, \ldots, x_n\}, x_k = \max\{x_1, x_2, \ldots, x_n\}, 1 < k < n$, then basic support function $(x_1, x_2, \ldots, x_{n-1})$ satisfies Theorem 1 based on the inductive assumption.

(2.1) Prove that $x_n \ln(1/x_n) - (x_n - \Delta_n)\ln(1/(x_n - \Delta_n)) \geqslant 0$. We prove $x_n < 1/e$ by contradiction at first.

If $x_n \geqslant 1/e$, then $\forall n \geqslant 3$, $\sum_{j=1,\ldots,n} x_j \geqslant n \sum x_n \geqslant n \sum 1/e \geqslant 3/e$. Because $e < 3$, it can be derived that $\sum_{j=1,\ldots,n} x_j > 1$. So $x_n < 1/e$ is proved.

If $x = 1/e$, then $x\ln(1/x)$ gets maximum $1/e$.

If $0 < \Delta < x < 1/e$, then we get $x \ln(1/x) > (x - \Delta)\ln(1/(x - \Delta))$.

If $1/e < x \leqslant 1$, then we get $x \ln(1/x) > (x + \Delta)\ln(1/(x + \Delta))$.

If $0 < x < 1/e$, then $x_n \ln(1/x_n) > (x_n - \Delta_n)\ln(1/(x_n - \Delta_n)) > 0$.

If $x_n = 0$, then $\Delta_n = 0$, so $x_n \ln(1/x_n) - (x_n - \Delta_n)\ln(1/(x_n - \Delta_n)) = 0$.

(2.2) Based on the inductive assumption, we have

$$\left(\sum_{j=1,\ldots,n-1} x_j \ln(1/x_j)\right) - \left((x_k + \Delta_k)\ln(1/(x_k + \Delta_k)) + \sum_{j=1,\ldots,n \,\wedge\, j \neq k} (x_j - \Delta_j)\ln(1/(x_j - \Delta_j))\right) > 0.$$

From (2.1) we know that $x_n \ln(1/x_n) - (x_n - \Delta_n)\ln(1/(x_n - \Delta_n)) \geqslant 0$.

So it can be derived that

$$\left(\sum_{j=1,\ldots,n} x_j \ln(1/x_j)\right) - \left((x_k + \Delta_k)\ln(1/(x_k + \Delta_k)) + \sum_{j=1,\ldots,n \wedge\, j \neq k} (x_j - \Delta_j)\ln(1/(x_j - \Delta_j))\right) > 0.$$

Theorem 1 illustrates that if a basic support function has a unique maximum truth-value, then the less entropy the basic support function has, the more determinate opinion this basic support function represents.

However, let us consider basic support functions $\lambda_1 = (0.5, 0.5, 0.0, 0.0)$ and $\lambda_2 = (0.15, 0.7, 0.1, 0.05)$ for source-rock richness. λ_1 cannot identify whether source-rock richness is high or normal, and λ_2 implies that source-rock richness is normal, meaning λ_2 is more determinate than λ_1. However, $H(\lambda_1) = 0.693$ and $H(\lambda_2) = 0.914$, meaning the entropy of λ_2 is larger than that of λ_1. Similarly, $\lambda_3 = (0.25, 0.5, 0.15, 0.1)$ is more determinate than $\lambda_4 = (0.4, 0.4, 0.15, 0.05)$ and $\lambda_5 = (1/3, 1/3, 1/3, 0)$, but $H(\lambda_3) = 1.208$ is greater than $H(\lambda_4) = 1.167$ and $H(\lambda_5) = 1.099$. Furthermore, λ_4 is more determinate than λ_5, but $H(\lambda_4)$ is greater than $H(\lambda_5)$. These examples indicate that if a basic support function has multiple maximum truth-values, simple entropy does not accurately measure uncertainty. In these situations, we extend the entropy of basic support functions by using Definition 12.

Definition 12 (Extended entropy of basic support functions)**.** Suppose that Θ is a space containing all n-tuple basic support functions except $(\lambda(s_1) = 0, \lambda(s_2) = 0, \ldots, \lambda(s_n) = 0), n \geqslant 2$, and let $x \ln(1/x) = 0$ if $x = 0$. For a basic support function $\lambda = (\lambda(s_1), \lambda(s_2), \ldots, \lambda(s_n)) \in \Theta$, let $\lambda(s_k) = \max\{\lambda(s_1), \lambda(s_2), \ldots, \lambda(s_n)\}, 1 \leqslant k \leqslant n$. If $\forall \beta\lambda(s_k) \leqslant \lambda(s_j) \leqslant \lambda(s_k), \beta \geqslant 0.9$, then $\lambda(s_j)$ is called quasi-maximum. The extended entropy of λ is defined as

$$E(\lambda) = \begin{cases} H(\lambda), & \text{if } \lambda \text{ has single maximum and no quasi-maximum,} \\ H(\lambda) + (\ln n - H(\lambda))(n'/n)^{\alpha}, & \text{if } \lambda \text{ is convex} \wedge 2 \leqslant n' \leqslant n, \\ H(\lambda) + (\ln n - H(\lambda))(|k-j|/n)^{\alpha}, & \text{if } \lambda \text{ is not convex} \wedge 2 \leqslant |k-j| < n, \end{cases} \tag{8}$$

where $\alpha = 0.1$, n' is the total number of maxima and quasi-maxima in λ, k' and j' are the indices of top two truth-values in λ that maximize $|k' - j'|$.

The 2nd line in (8) means that λ has a higher extended entropy when it has a greater number of maxima and quasi-maxima. The 3rd line in (8) means that λ has a higher extended entropy if its extreme points are further apart. These conditions both imply that λ is more indeterminate.

Take the aforementioned example of a basic support function $\lambda_1 = (0.5, 0.5, 0.0, 0.0)$ with $H(\lambda_1) \approx 0.693, \ln(4) - H(\lambda_1) \approx 0.693, n' = 2, (n'/n)^\alpha = 0.5^{0.1}$. Then, $E(\lambda_1) \approx 1.34$, which is greater than the entropy of λ_2 and λ_3, which is a reasonable result. $H(\lambda_4) \approx 1.167$, $\ln(4) - H(\lambda_4) \approx 0.219, \alpha = 0.1$, and $(n'/n)^\alpha = 0.5^{0.1}$. Then, $E(\lambda_4) \approx 1.372; H(\lambda_5) \approx 1.099$, $\ln(4) - H(\lambda_5) \approx 0.288$, $\alpha = 0.1$, $(n'/n)^\alpha = 0.75^{0.1}$. Thus, $E(\lambda_5) \approx 1.378$ and $E(\lambda_4) < E(\lambda_5)$ is reasonable. As a final example, consider $\lambda = (0.395, 0.07, 0.07, 0.07, 0.395), H(\lambda) \approx 1.292, (|k' - j'|/n)^\alpha = 0.8^{0.1}$. Then, $E(\lambda) = H(\lambda) + (\ln 5 - H(\lambda)) \times 0.8^{0.1} \approx 1.602$.

6 The proposed method for fusing the basic support functions of ordered propositions

6.1 The general algorithm

Let μ and ν be the n-dimensional basic support functions for fusion. The weights of μ and ν are Ω_μ and Ω_ν, Δ is the set of all indeterminate basic support functions, and γ is the indeterminate basic support function$(1/n, \ldots, 1/n)$. Additionally, the basic support function resulting from fusion is ω. The proposed fusion algorithm is outlined in Algorithm 1.

Algorithm 1 Fusion of basic support functions of ordered propositions

Input: Basic support functions μ and ν, the weights Ω_μ and Ω_ν;
Output: Fusion result ω;
/* One of the basic support functions is indeterminate */
if $\mu \in \Delta$ **and** $\nu \in \Delta$ **then**
  $\omega \leftarrow \gamma$;
  return
else if $\mu \in \Delta$ **and** $\nu \notin \Delta$ **then**
  $\omega \leftarrow \nu$;
  return
else if $\mu \notin \Delta$ **and** $\nu \in \Delta$ **then**
  $\omega \leftarrow \mu$;
  return
end if
/* Calculate the initial fusion result */
for $i = 1$ to n **do**
  $\omega'(s_i) \leftarrow \Omega_\mu \cdot \mu(s_i)(1 + \mu(\overline{S})) + \Omega_\nu \cdot \nu(s_i)(1 + \nu(\overline{S}))$;
end for
/* Calculate the center and consistency */
Calculate $\mathrm{CI}(\omega')$ with (5);
Calculate $\Delta G(\mu, \nu)$ with (6);
/* Fusion */
if $\mathrm{CI}(\omega')$ is Integer **then**
  Execute the fusion strategy described in Subsection 6.2 for the situation that $\mathrm{CI}(\omega')$ is integer;
else
  Execute the fusion strategy described in Subsection 6.3 for the situation that $\mathrm{CI}(\omega')$ is non-integer;
end if

6.2 Fusion strategy when $\mathrm{CI}(\omega')$ is an integer

The aim of this section is to construct the final fusion result ω from the initial fusion result ω' when $\mathrm{CI}(\omega')$ is an integer. Typically, if μ and ν are consistent, the clarity or determinacy of the fusion result should increase, meaning that the entropy of the final fusion result should decrease. Conversely, if μ and ν are inconsistent, the entropy of the final fusion result should be higher than that of the initial fusion result. Thus, we use different strategies based on the degree of consistency between μ and ν.

Table 1 Process of generating ω from ω' by (9) with $\varsigma = 0.2$

	Truth-value obtained by $\omega(s_1)$	Truth-value obtained by $\omega(s_2)$	Truth-value obtained by $\omega(s_3)$	Truth-value obtained by $\omega(s_4)$	Truth-value obtained by $\omega(s_5)$
Truth-value distributed from $\omega'(s_1)$ to $\omega(s_1),\ldots,\omega(s_5)$	0.01389	0.01667	0.01944	0	0
Truth-value distributed from $\omega'(s_2)$ to $\omega(s_1),\ldots,\omega(s_5)$	0	0.06818	0.08182	0	0
Truth-value distributed from $\omega'(s_3)$ to $\omega(s_1),\ldots,\omega(s_5)$	0	0	0.6	0	0
Truth-value distributed from $\omega'(s_4)$ to $\omega(s_1),\ldots,\omega(s_5)$	0	0	0.08182	0.06818	0
Truth-value distributed from $\omega'(s_5)$ to $\omega(s_1),\ldots,\omega(s_5)$	0	0	0.01944	0.01667	0.01389
Total truth-value of $\omega(s_1),\ldots,\omega(s_5)$	0.01389	0.08485	0.80252	0.08485	0.01389

(1) High consistency. If $0 \leqslant \Delta G(\mu,\nu) \leqslant \delta_1$, this indicates that μ is very consistent with ν. Thus, each truth-value in ω' should be strongly correlated to the center of ω', increasing the clarity of the fusion result. Therefore, we obtain the final fusion result ω by using an arithmetic sequence with an initial term of 1 and a common difference of 0.2 to positively regulate ω' as follows:

$$\omega(s_i) = \begin{cases} \displaystyle\sum_{k=1}^{i} \frac{\omega'(s_k)\,[1+\varsigma(i-k)]}{\sum_{j=0}^{\mathrm{CI}(\omega')-k}(1+j\varsigma)}, & \text{if } i < \mathrm{CI}(\omega'), \\ \displaystyle\sum_{1\leqslant k\leqslant \mathrm{CI}(\omega')} \frac{\omega'(s_k)\,[1+\varsigma(i-k)]}{\sum_{j=0}^{\mathrm{CI}(\omega')-k}(1+j\varsigma)} + \sum_{k=\mathrm{CI}(\omega')+1}^{n} \frac{\omega'(s_k)\,[1+\varsigma(k-\mathrm{CI}(\omega'))]}{\sum_{j=0}^{k-\mathrm{CI}(\omega')}(1+j\varsigma)}, & \text{if } i = \mathrm{CI}(\omega'), \\ \displaystyle\sum_{k=i}^{n} \frac{\omega'(s_k)\,[1+\varsigma(k-i)]}{\sum_{j=0}^{k-\mathrm{CI}(\omega')}(1+j\varsigma)}, & \text{if } i > \mathrm{CI}(\omega'), \end{cases} \tag{9}$$

where $\varsigma = 0.2$ is the common difference.

We use Example 5 to explain (9).

Example 5. The basic support functions for fusion are $\mu = (0.05, 0.15, 0.6, 0.15, 0.05)$ and $\nu = (0.05, 0.15, 0.6, 0.15, 0.05)$. The initial fusion result $\omega' = (0.05, 0.15, 0.6, 0.15, 0.05)$, $\mathrm{CI}(\omega') = 3$ is an integer. The process of constructing the final fusion result ω from ω' by using (9) with $\varsigma = 0.2$ is illustrated in Table 1.

The arithmetic sequence with an initial term of 1 and a common difference of 0.2 is 1, 1.2, 1.4, ... In the 1st row of Table 1, $\omega(s_1)$ obtains 1/(1+1.2+1.4) of truth-value from $\omega(s_1)'$, that is 0.05×1/3.6=0.01389. $\omega(s_2)$ obtains 1.2/(1+1.2+1.4) from $\omega(s_1)'$, that is 0.05×1.2/3.6=0.01667. $\omega(s_3)$ obtains 1.4/(1+1.2+1.4) from $\omega(s_1')$, that is 0.05×1.4/3.6=0.01944. That also means the truth-value of $\omega(s_1)'$ is divided into 3 parts: 0.01389, 0.01667, 0.01944, and 0.01389 is distributed to $\omega(s_1)$, 0.01667 is distributed to $\omega(s_2)$, 0.01944 is distributed to $\omega(s_3)$. For the 2nd row of Table 1, $\omega(s_2)'$ distributes 1/(1+1.2) of its truth-value to $\omega(s_2)$, that is 0.15×1/2.2=0.06818. And $\omega(s_2)'$ distributes 1.2/(1+1.2) of its truth-value to $\omega(s_3)$, that is 0.15×1.2/2.2=0.08182. For the 3rd row of Table 1, $\omega(s_3)'$ distributes 1/1 of its truth-value to $\omega(s_3)$, that is 0.6. Finally, we obtain $\omega = (0.01389, 0.08485, 0.80252, 0.08485, 0.01389)$.

(2) Medium consistency. If $\delta_1 < \Delta G(\mu,\nu) \leqslant \delta_2$, this indicates that the consistency between μ and ν is medium, so each truth-value in ω' can normally converge to the center of ω'. Thus, we use an arithmetic sequence with an initial term of 1 and a common difference of 0.1 to regulate the initial fusion result ω' positively, meaning we use (9) with $\varsigma = 0.1$ to generate the final fusion result ω.

(3) Poor consistency. If $\Delta G(\mu,\nu) > \delta_2$, this indicates that the consistency between μ and ν is poor. Thus we regulate ω' through two steps to generate ω: positive regulation and negative regulation. First, the positive regulation is performed. This involves generating ω by regulating ω' positively based on an

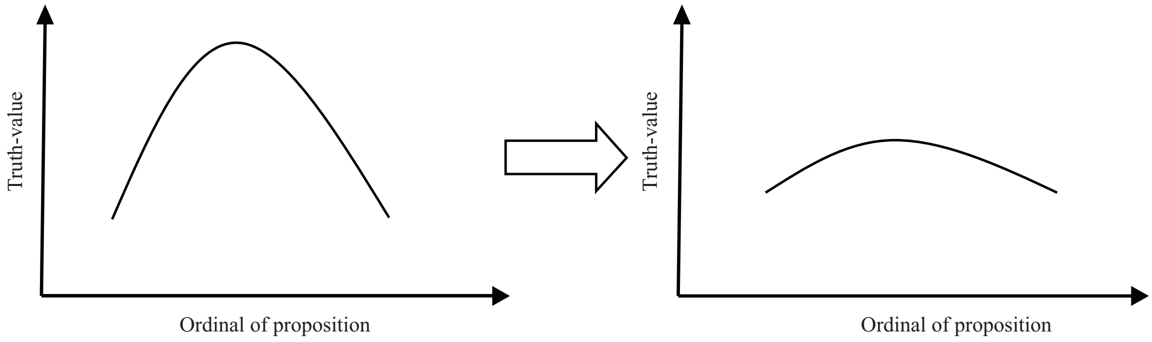

Figure 1 Explanation of negative regulation for basic support function.

arithmetic sequence with an initial term of 1 and a common difference of 0, meaning using 9 with $\varsigma = 0$. Because μ and ν are poorly consistent, the final fusion result is expected to more indeterminate. Second, we perform negative regulation. The process for negative regulation is to continuously squeeze the curve of the truth-values in ω vertically to make the curve smoother, until its entropy approximately equals the entropy of ω', meaning $E(\omega') \approx E(\omega)$. Theorem 1 implies that positive regulation for basic support functions causes entropy to decrease and negative regulation causes entropy to increase. The principle of negative regulation is illustrated in Figure 1.

The procedure for negative regulation is outlined in Algorithm 2.

Algorithm 2 Negative regulation of basic support function

Input: Initial fusion result ω' and basic support function ω after positive regulation;
Output: Fusion result ω;
$\sigma \leftarrow 1$;
while $|E(\omega) - E(\omega')| \leqslant \epsilon$ **do**
    $I \leftarrow$ index of maximum truth-value of ω;
    $k \leftarrow 1$;
    for $k = I$ to $n-1$ **do**
        if $\omega(s_k) > \omega(s_{k+1})$ **then**
            $\delta \leftarrow (\omega(s_k) - \omega(s_{k+1}))\sigma$;
            $\rho \leftarrow \omega(s_k) + \omega(s_{k+1})$;
            $\omega(s_{k+1}) \leftarrow \omega(s_{k+1}) + \delta\omega(s_{k+1})/\rho$;
            $\omega(s_k) \leftarrow \omega(s_k) - \delta + \delta\omega(s_k)/\rho$;
        end if
    end for
    for $k = I; k > 1; k--$ **do**
        if $\omega(s_k) > \omega(s_{k-1})$ **then**
            $\delta \leftarrow (\omega(s_k) - \omega(s_{k-1}))\sigma$;
            $\rho \leftarrow \omega(s_k) + \omega(s_{k-1})$;
            $\omega(s_{k-1}) \leftarrow \omega(s_{k-1}) + \delta\omega(s_{k-1})/\rho$;
            $\omega(s_k) \leftarrow \omega(s_k) - \delta + \delta\omega(s_k)/\rho$;
        end if
    end for
    if $E(\omega) < E(\omega') - \epsilon$ **then**
        $\sigma \leftarrow 1$;
    else if $E(\omega) > E(\omega') - \epsilon$ **then**
        $\sigma \leftarrow \sigma/2$;
    end if
end while

6.3 Fusion strategy when $\mathrm{CI}(\omega')$ is a non-integer

The aim of this section is to construct the final fusion result ω from the initial fusion result ω' when $\mathrm{CI}(\omega')$ is not an integer. Similarly to Subsection 6.2, we perform different strategies based on the degree of consistency between μ and ν.

(1) High consistency. If $0 \leqslant \Delta G(\mu,\nu) \leqslant \delta_1$, this means that the consistency between μ and ν is high. In order to construct the final fusion result ω, we use an arithmetic sequence with an initial term of 1 and a common difference(ς) of 0.2 to regulate the initial fusion result ω' positively as follows:

$$\omega(s_i)=\begin{cases}\displaystyle\sum_{k=1}^{i}\frac{\omega'(s_k)\left[1+\varsigma(i-k)\right]}{\left(\left(\sum_{j=0}^{\lceil \mathrm{CI}(\omega')\rceil-k}\left[1+j\varsigma\right]\right)-\varsigma\right)}, & \text{if } i<\lfloor \mathrm{CI}(\omega')\rfloor,\\ \omega'\left(s_{\lfloor \mathrm{CI}(\omega')\rfloor}\right)+\Gamma\left(\lceil \mathrm{CI}(\omega')\rceil-\mathrm{CI}(\omega')\right), & \text{if } i=\lfloor \mathrm{CI}(\omega')\rfloor \wedge \omega'(s_{\lfloor \mathrm{CI}(\omega')\rfloor})\neq\omega'(s_{\lceil \mathrm{CI}(\omega')\rceil}),\\ \omega'(s_{\lceil \mathrm{CI}(\omega')\rceil})+\Gamma\left(\mathrm{CI}(\omega')-\lfloor \mathrm{CI}(\omega')\rfloor\right), & \text{if } i=\lceil \mathrm{CI}(\omega')\rceil \wedge \omega'(s_{\lfloor \mathrm{CI}(\omega')\rfloor})\neq\omega'(s_{\lceil \mathrm{CI}(\omega')\rceil}),\\ \omega'\left(s_{\lfloor \mathrm{CI}(\omega')\rfloor}\right)+\Gamma/2, & \text{if } i=\lfloor \mathrm{CI}(\omega')\rfloor \wedge \omega'(s_{\lfloor \mathrm{CI}(\omega')\rfloor})=\omega'(s_{\lceil \mathrm{CI}(\omega')\rceil}),\\ \omega'(s_{\lceil \mathrm{CI}(\omega')\rceil})+\Gamma/2, & \text{if } i=\lceil \mathrm{CI}(\omega')\rceil \wedge \omega'(s_{\lfloor \mathrm{CI}(\omega')\rfloor})=\omega'(s_{\lceil \mathrm{CI}(\omega')\rceil}),\\ \displaystyle\sum_{k=i}^{n}\frac{\omega'(s_k)\left[1+\varsigma(k-i)\right]}{\left(\left(\sum_{j=0}^{k-\lfloor \mathrm{CI}(\omega')\rfloor}\left[1+j\varsigma\right]\right)-\varsigma\right)}, & \text{if } i>\lceil \mathrm{CI}(\omega')\rceil,\end{cases} \tag{10}$$

where

$$\Gamma=\Gamma_1+\Gamma_2,$$

$$\Gamma_1=\sum_{k=1}^{\lfloor \mathrm{CI}(\omega')\rfloor-1}\frac{\omega'(s_k)\left[1+\varsigma(\lfloor \mathrm{CI}(\omega')\rfloor-k)\right]}{\left(\sum_{j=1}^{\lceil \mathrm{CI}(\omega')\rceil-k+1}\left[1+(j-1)\varsigma\right]\right)-\varsigma}+\sum_{k=\lceil \mathrm{CI}(\omega')\rceil+1}^{n}\frac{\omega'(s_k)\left[1+\varsigma(k-\lceil \mathrm{CI}(\omega')\rceil)\right]}{\left(\sum_{j=1}^{k-\lfloor \mathrm{CI}(\omega')\rfloor+1}\left[1+\varsigma(j-1)\right]\right)-\varsigma},$$

$$\Gamma_2=\sum_{k=1}^{\lfloor \mathrm{CI}(\omega')\rfloor-1}\frac{\omega'(s_k)\left[1+\varsigma(\lceil \mathrm{CI}(\omega')\rceil-k)-\varsigma\right]}{\left(\sum_{j=1}^{\lceil \mathrm{CI}(\omega')\rceil-k+1}\left[1+(j-1)\varsigma\right]\right)-\varsigma}+\sum_{k=\lceil \mathrm{CI}(\omega')\rceil+1}^{n}\frac{\omega'(s_k)\left[1+\varsigma(k-\lceil \mathrm{CI}(\omega')\rceil)\right]}{\left(\sum_{j=1}^{k-\lfloor \mathrm{CI}(\omega')\rfloor+1}\left[1+\varsigma(j-1)\right]\right)-\varsigma}.$$

(2) Medium consistency. If $\delta_1 < \Delta G(\mu,\nu) \leqslant \delta_2$, we use an arithmetic sequence with an initial term of 1 and a common difference of 0.1 to regulate the initial fusion result ω' positively using (10) with $\varsigma = 0.1$.

(3) Poor consistency. If $\Delta G(\mu,\nu) > \delta_2$, this means that the consistency between μ and ν is poor. Similarly to Subsection 6.2, we first regulate ω' positively using (10) with $\varsigma = 0$, and then perform negative regulation using Algorithm 2 to obtain the final fusion result ω.

6.4 Theoretical analysis of proposed algorithm

We provide theoretical support for the legitimacy and effectiveness of the proposed algorithm in Theorems 2–5.

Theorem 2. If μ and ν are basic support functions for ordered propositions and ω is the fusion result of μ and ν when using Algorithm 1, then ω is a basic support function.

Proof. (1) If Algorithm 1 terminates at Lines 4, 7, or 10 of the algorithm, obviously ω is a valid basic support function.

(2) For initial fusion result ω', we have $\sum_{i=1}^{n}\omega'(s_i) = \Omega_\mu\sum_{i=1}^{n}\mu(s_i)(1+\mu(\bar{S})) + \Omega_\nu\sum_{i=1}^{n}\nu(s_i)(1+\nu(\bar{S}))$. Due to $\mu(\bar{S}) = 1-\sum_{i=1}^{n}\mu(s_i)$, which implies $\sum_{i=1}^{n}\omega'(s_i) = \Omega_\mu\sum_{i=1}^{n}\mu(s_i)(2-\sum_{i=1}^{n}\mu(s_i)) + \Omega_\nu\sum_{i=1}^{n}\nu(s_i)(2-\sum_{i=1}^{n}\nu(s_i))$.

We know the function $f(x) = x(2-x) = -x^2+2x \leqslant 1$ at interval $x\in[0,1]$, and $0 \leqslant \sum_{i=1}^{n}\mu(s_i)$, $\sum_{i=1}^{n}\nu(s_i) \leqslant 1$, so we obtain $\sum_{i=1}^{n}\omega'(s_i) \leqslant \Omega_\mu+\Omega_\nu \leqslant 1$.

(3) If μ and ν are highly consistent or moderately consistent, Algorithm 1 uses ω' to construct the

fusion result ω by an arithmetic sequence with common difference of ς. For $1 \leqslant i < \mathrm{CI}(\omega')$, we have

$$\sum_{i=1}^{\mathrm{CI}(\omega')-1} \omega(s_i) = \sum_{i=1}^{\mathrm{CI}(\omega')-1} \sum_{k=1}^{i} \frac{\omega'(s_k)(1+\varsigma(i-k))}{\sum\limits_{j=0}^{\mathrm{CI}(\omega')-k}(1+j\varsigma)} = \sum_{i=1}^{\mathrm{CI}(\omega')-1} \sum_{k=1}^{i} \frac{\omega'(s_k)(1+\varsigma(i-k))}{(\mathrm{CI}(\omega')-k+1)(1+\varsigma(\mathrm{CI}(\omega')-k)/2)}$$
$$= \sum_{i=1}^{\mathrm{CI}(\omega')-1} \left(\frac{\omega'(s_1)(1+(i-1)\varsigma)}{\mathrm{CI}(\omega')(1+\varsigma(\mathrm{CI}(\omega')-1)/2)} + \frac{\omega'(s_2)(1+(i-2)\varsigma)}{(\mathrm{CI}(\omega')-1)(1+\varsigma(\mathrm{CI}(\omega')-2)/2)} + \cdots + \frac{\omega'(s_i)}{(\mathrm{CI}(\omega')-i+1)(1+\varsigma(\mathrm{CI}(\omega')-i)/2)} \right).$$

Similarly, for $\mathrm{CI}(\omega') < i \leqslant n$, we have

$$\sum_{i=\mathrm{CI}(\omega')+1}^{n} \omega(s_i) = \sum_{i=\mathrm{CI}(\omega')+1}^{n} \sum_{k=i}^{n} \frac{\omega'(s_k)(1+\varsigma(k-i))}{\sum_{j=0}^{k-\mathrm{CI}(\omega')}(1+j\varsigma)}$$
$$= \sum_{i=\mathrm{CI}(\omega')+1}^{n} \sum_{k=i}^{n} \frac{\omega'(s_k)(1+\varsigma(k-i))}{(k-\mathrm{CI}(\omega')+1)(1+\varsigma(k-\mathrm{CI}(\omega'))/2)}$$
$$= \sum_{i=\mathrm{CI}(\omega')+1}^{n} \left(\frac{\omega'(s_n)(1+(n-i)\varsigma)}{(n-\mathrm{CI}(\omega')+1)(1+\varsigma(n-\mathrm{CI}(\omega'))/2)} + \cdots + \frac{\omega'(s_i)}{(i-\mathrm{CI}(\omega')+1)(1+\varsigma(i-\mathrm{CI}(\omega'))/2)} \right).$$

So for $1 \leqslant i \leqslant n$, the sum of $\omega(s_i)$ can be derived as follows:

$$\sum_{i=1}^{n} \omega(s_k) = \frac{\omega'(s_1)(\mathrm{CI}(\omega')+\varsigma(1+2+\cdots+\mathrm{CI}(\omega')-1))}{\mathrm{CI}(\omega')(1+\varsigma(\mathrm{CI}(\omega')-1)/2)} + \cdots + \frac{\omega'(s_n)(1+(n-i)\varsigma)}{(n-\mathrm{CI}(\omega')+1)(1+\varsigma(n-\mathrm{CI}(\omega'))/2)}$$
$$= \frac{\omega'(s_1)(\mathrm{CI}(\omega')+\varsigma\mathrm{CI}(\omega')(\mathrm{CI}(\omega')-1)/2)}{\mathrm{CI}(\omega')(1+\varsigma(\mathrm{CI}(\omega')-1)/2)} + \cdots + \frac{\omega'(s_n)(n-\mathrm{CI}(\omega')+1+\varsigma(n-\mathrm{CI}(\omega'))(n-\mathrm{CI}(\omega')+1)/2)}{(n-\mathrm{CI}(\omega')+1)(1+\varsigma(n-\mathrm{CI}(\omega'))/2)}$$
$$= \frac{\omega'(s_1)\mathrm{CI}(\omega')(1+\varsigma(\mathrm{CI}(\omega')-1)/2)}{\mathrm{CI}(\omega')(1+\varsigma(\mathrm{CI}(\omega')-1)/2)} + \cdots + \frac{\omega'(s_n)(n-\mathrm{CI}(\omega')+1)(1+\varsigma(n-\mathrm{CI}(\omega'))/2)}{(n-\mathrm{CI}(\omega')+1)(1+\varsigma(n-\mathrm{CI}(\omega'))/2)}$$
$$= \omega'(s_1) + \cdots + \omega'(s_n) \leqslant 1.$$

(4) If μ and ν are in poor consistency, the algorithm firstly conducts positive regulation to ω', we know the result of positive regulation, ω, is a basic support function according to (3). Secondly, the negative regulation is conducted to ω by Algorithm 2, in each iteration of this process, for $I \leqslant k \leqslant n$, $\omega(s_k)$ decreases by $\delta - \delta\omega_k/\rho$, and $\omega(s_{k+1})$ increases by $\delta\omega(s_{k+1})\rho$, due to $\rho = \omega_k + \omega(s_{k+1})$, which implies $\omega_k/\rho + \omega(s_{k+1})\rho = 1$, i.e., $\omega(s_{k+1})\rho = 1 - \omega_k/\rho$. It can be derived that $\delta\omega(s_{k+1})\rho = \delta - \delta\omega_k/\rho$, i.e., the decrease of $\omega(s_k)$ equals the increase of $\omega(s_{k+1})$. Likewise, the decrease of $\omega(s_k)$ equals the increase of $\omega(s_{k-1})$ for $1 \leqslant k \leqslant I$. So the sum of $\omega(s_k)$ of the result of negative regulation is less than or equal to 1.

Synthesizing (1)–(4), we obtain $\sum_{i=1}^{n} \omega(s_i) \leqslant 1$, and obviously $\omega(s_i) \geqslant 0$ for all i such that $1 \leqslant i \leqslant n$.

Therefore, the fusion result ω is a valid basic support function.

Lemma 1. Suppose μ and ν are basic support functions for ordered propositions and let ω be the fusion result of μ and ν when using Algorithm 1. Let m be the index of the maximum truth-value in ω. Then, ω is monotonically increasing over the interval $[1, m]$ and is monotonically decreasing over the interval $[m, n]$.

Proof. Let ω' be the initial fusion result in Algorithm 1.

(1) If μ and ν are in high or medium consistence, ω will be generated by conducting positive regulation to ω', i.e., using an arithmetic sequence with common difference of ς.

For all i, j such that $1 \leqslant i \leqslant j \leqslant m$, based on (3) of the proof of Theorem 2, we have

$$\omega(s_j) - \omega(s_i) \geqslant \frac{\omega'(s_1)(j-i)\varsigma}{m(1+\varsigma(m-1)/2)} + \frac{\omega'(s_2)(j-i)\varsigma}{(m-1)(1+\varsigma(m-2)/2)} + \cdots + \frac{\omega'(s_i)(j-i)\varsigma}{(m-i+1)(1+\varsigma(m-i)/2)} + \cdots + \frac{\omega'(s_j)}{(m-j+1)(1+\varsigma(m-j)/2)}.$$

Due to $m \geqslant j, m \geqslant i$, and $j \geqslant i$, each item of above equation is greater than or equal to 0, thus $\omega(s_j) - \omega(s_i) \geqslant 0$.

Similarly, for all i, j such that $m \leqslant i \leqslant j \leqslant n$, we have

$$\omega(s_i) - \omega(s_j) \geqslant \frac{\omega'(s_n)(j-i)\varsigma}{(n-m+1)(1+\varsigma(n-m)/2)} + \cdots + \frac{\omega'(s_j)(j-i)\varsigma}{(j-m+1)(1+\varsigma(j-m)/2)} + \cdots + \frac{\omega'(s_i)}{(i-m+1)(1+\varsigma(i-m)/2)} \geqslant 0.$$

(2) If μ and ν are in poor consistency, the algorithm generates ω firstly by conducting positive regulation to ω', from (1) we know the result of positive regulation is a basic support function. Secondly, the negative regulation is conducted to ω by Algorithm 2.

For $m \leqslant k \leqslant n$, in each iteration of negative regulation, $\omega(s_k)$ and $\omega(s_{k+1})$ are updated, let $\hat{\omega}(s_k)$ and $\hat{\omega}(s_{k+1})$ be the value after update, i.e., $\hat{\omega}(s_{k+1}) = \omega(s_{k+1}) + \delta\omega(s_{k+1})/\rho$ and $\hat{\omega}(s_k) = \omega(s_k) - \delta + \delta\omega(s_k)/\rho$; we have

$$\begin{aligned}
\hat{\omega}(s_k) - \hat{\omega}(s_{k+1}) &= \omega(s_k) - \delta + \delta\omega(s_k)/\rho - \omega(s_{k+1}) - \delta\omega(s_{k+1})/\rho \\
&= \omega(s_k) - \omega(s_{k+1}) - (\omega(s_k) - \omega(s_{k+1}))\sigma + \delta(\omega(s_k) - \omega(s_{k+1}))/\rho \\
&= (\omega(s_k) - \omega(s_{k+1}))(1 - \sigma + \delta/\rho).
\end{aligned}$$

Due to $\omega(s_k) \geqslant \omega(s_{k+1})$ and $\rho \leqslant 1$, it implies $\hat{\omega}(s_k) - \hat{\omega}(s_{k+1}) \geqslant 0$. Likewise, for $1 \leqslant k \leqslant m$, in each iteration of negative regulation, $\hat{\omega}(s_k) - \hat{\omega}(s_{k-1}) \geqslant 0$. That means during negative regulation, ω keeps its monotonic increasing on interval $[1, m]$ and monotonic decreasing on interval $[m, n]$.

Finally, synthesizing (1)–(2), ω is monotonically increasing on the interval $[1, m]$, and monotonically decreasing on the interval $[m, n]$.

Theorem 3. If the basic support function ω is the result of Algorithm 1, then ω satisfies the convex property.

Proof. Let m be the index of maximum truth-value of ω, from Lemma 1, for all i, j, k such that $1 \leqslant i \leqslant j \leqslant k \leqslant n$:

If $1 \leqslant j \leqslant m$, then $\omega(s_j) \geqslant \omega(s_i) \geqslant \min\{\omega(s_i), \omega(s_k)\}$, because ω is increasing on this interval.

If $m \leqslant j \leqslant n$, then $\omega(s_j) \geqslant \omega(s_k) \geqslant \min\{\omega(s_i), \omega(s_k)\}$, because ω is decreasing on this interval. Therefore, according to Definition 3, ω satisfies convex property.

Theorems 2 and 3 ensure the legitimacy of the proposed algorithm.

Theorem 4. In Algorithm 1, the negative regulation to basic support function ω will increase the entropy of ω.

Proof. From Lemma 1 we know the input basic support function ω for negative regulation satisfies convex property, let m be the index of maximum truth-value of ω, for $m \leqslant k \leqslant n$, from (4) of proof of Theorem 2, we know in each iteration $\omega(s_k)$ and $\omega(s_{k+1})$ become $\omega(s_k) - \Delta$ and $\omega(s_{k+1}) + \Delta, \Delta \geqslant 0$. Based on (1) of proof of Theorem 1, we obtain that the entropy increases. Likewise, for $1 \leqslant k \leqslant m$, in each iteration, $\omega(s_k)$ and $\omega(s_{k-1})$ become $\omega(s_k) - \Delta$ and $\omega(s_{k-1}) + \Delta, \Delta \geqslant 0$, and the entropy increases, too.

If μ and ν have high or medium consistency, Algorithm 1 will generate a more determinate result by using positive regulation. If μ and ν have poor consistency, Theorem 4 ensures that the algorithm generates a convex basic support function whose entropy is is as close to the initial fusion result (not convex) as possible, meaning the final fusion result has both the convex property and a large degree of uncertainty, which is reasonable. Thus, the rationality and precision of the algorithm is confirmed.

Table 2 Comparisons of uncertainty measures to basic support function

No.	Basic support function	Nonspecificity/AM	AU	Jousselme distance	Extended entropy
λ_1	(0.2, 0.2, 0.2, 0.2, 0.2)	0	1.609	0.632	1.609
λ_2	0.3, 0.15, 0.1, 0.15, 0.3)	0	1.522	0.563	1.608
λ_3	(0.45, 0.05, 0, 0.05,0.45)	0	1.018	0.505	1.596
λ_4	(0.5, 0, 0, 0, 0.5)	0	0.693	0.5	1.589
λ_5	(0.33, 0.33, 0.33, 0.01, 0)	0	1.144	0.577	1.586
λ_6	(0.39, 0.39, 0.15, 0.07, 0)	0	1.205	0.525	1.574
λ_7	(0.45, 0.45, 0.05, 0.05 ,0)	0	1.018	0.505	1.558
λ_8	(0.5, 0.5, 0, 0, 0)	0	0.693	0.5	1.529
λ_9	(0.23, 0.5, 0.17, 0.1, 0)	0	1.216	0.413	1.216
λ_{10}	(0.15, 0.7, 0.1, 0.05, 0)	0	0.914	0.25	0.914
λ_{11}	(0.05, 0.85, 0.05, 0.05, 0)	0	0.588	0.122	0.588
λ_{12}	(0.005, 0.99, 0.005, 0, 0)	0	0.063	0.009	0.063
λ_{13}	(0, 1, 0, 0 ,0)	0	0	0	0

Theorem 5 (Commutativity)**.** If the fusion result of the basic support functions μ and ν when using Algorithm 1 is ω_1, and the fusion result of ν and μ is ω_2, then $\omega_1 = \omega_2$.

Proof. (1) If Algorithm 1 terminates at Lines 4, 7, or 10, obviously the result is same for μ, ν and ν, μ. (2) At Line 14, due to $\omega'(s_i) = \Omega_\mu \cdot \mu(s_i)(1+\mu(\overline{S}))+\Omega_\nu \cdot \nu(s_i)(1+\nu(\overline{S})) = \Omega_\nu \cdot \nu(s_i)(1+\nu(\overline{S}))+\Omega_\mu \cdot \mu(s_i)(1+\mu(\overline{S}))$, so $\omega'(s_i)$ is same for μ, ν and ν, μ. (3) At Line 18, from (6), we obtain $\Delta G(\mu,\nu) = \Delta G(\nu,\mu)$. (4) From Line 17 to Line 23, the fusion result is generated from $\omega'(s_i)$, so the final fusion result is same for μ, ν and ν, μ. Finally, synthesizing (1)–(4), the proposed fusion method satisfies commutativity.

7 Case study and comparison

7.1 Comparison of uncertainty measures

We first compare the proposed extended entropy measure with several most commonly used and state-of-the-art uncertainty measures [20–24]: the nonspecificity measure, AU measure, ambiguity measure, and Jousselme distance-based measure. We select various typical basic support functions to test the measures. The results are displayed in Table 2.

In Table 2, the degree of uncertainty of the basic support functions decreases gradually, i.e., λ_i is more uncertain than $\lambda_{i+1} (1 \leqslant i \leqslant 12)$. It can be seen in Table 2 that the nonspecificity measure and ambiguity measure (AM) are ineffective for the basic support functions. If the basic support function has a unique maximum truth-value or is completely indeterminate, the AU measure, distance based measure, and extended entropy can measure its uncertainty effectively. The greater the value is, the higher the degree of uncertainty is. In particular, if a basic support function is completely indeterminate, such as λ_1, the value of the measure is maximized, and if a basic support function is completely determinate, such as λ_{13}, the value of the measure is minimized.

However, if a basic support function has multiple maximum truth-values or is not convex, the AU measure and Jousselme distance cannot measure the uncertainty accurately, but the extended entropy can. For instance, λ_6 is more determinate than λ_5, but AU(λ_6) >AU(λ_5). λ_9 is more determinate than λ_3 through λ_8, but AU(λ_9) is greater than AU(λ_3) through AU(λ_8). λ_{10} is more determinate than λ_4 and λ_8, but AU(λ_{10}) is greater than AU(λ_4) and AU(λ_8), which is unreasonable. However, the results for Jousselme distance and extended entropy are reasonable.

On the other hand, λ_8 is more determinate than λ_4. For example, if s_i is $\{\text{high}, \text{a little high}, \text{medium}, \text{a little low}, \text{low}\}$, λ_4 indicates that one cannot decide whether the object is high or low, which is close to completely indeterminate. λ_8 indicates that one knows the object is fairly high, which is more determinate. However, AU(λ_4) = AU(λ_8), and JD(λ_4) = JD(λ_8), meaning the AU measure and Jousselme distance cannot measure the uncertainty accurately in such kind of cases. λ_3 and λ_7 also fall into this category.

Table 3 Comparison of the algorithms for fusing $\mu = (0.1, 0.6, 0.3)$ and $\nu = (0.6, 0.3, 0.1)$

Fusion algorithm	Fusion result	Centroid/Center of basic support functions	Extended entropy
Centroid-based algorithm	(0.245, 0.66, 0.095)	1.926	0.842
Proposed algorithm	(0.3426, 0.4555, 0.2019)	1.629	1.048

Table 4 Comparison of the algorithms for fusing $\mu = (0.1, 0.2, 0.2, 0.5)$ and $\nu = (0.1, 0.15, 0.15, 0.6)$

Fusion algorithm	Fusion result	Centroid/Center of basic support functions	Extended entropy
Centroid-based algorithm	(0.033, 0.121, 0.571, 0.275)	3.325	1.044
Proposed algorithm	(0.019, 0.072, 0.217, 0.692)	4	0.851

Additionally, λ_5, λ_6, and λ_7 are more determinate than λ_4, but $\mathrm{JD}(\lambda_5)$, $\mathrm{JD}(\lambda_6)$, and $\mathrm{JD}(\lambda_7)$ are all greater than $\mathrm{JD}(\lambda_4)$. The result is the same for the AU measure. Similarly, λ_5 and λ_6 are more determinate than λ_3, but $\mathrm{JD}(\lambda_5)$ and $\mathrm{JD}(\lambda_6)$ are both greater than $\mathrm{JD}(\lambda_3)$. The AU measure again produces the same results. In these types of cases, the results of the AU measure and Jousselme distance are unreasonable. However, the proposed extended entropy can measure the uncertainty accurately.

We now provide some theoretical analysis and explanations for the aforementioned measures. In the nonspecificity measure and ambiguity measure, the cardinality of each subset of frame of discernment is 1 for basic support functions. Thus, according to the definitions of the measures, the value equals 0 for all basic support functions. The AU measure and Jousselme distance-based measure only consider the truth-value of each proposition and do not consider the distribution of truth-values in the basic support function. Thus, for basic support functions which have the same truth-values (λ_4 and λ_8), the measures return the same value. However, the extended entropy considers not only the truth-value, but also the distribution of truth-values in each proposition of the basic support function. Therefore, more reasonable results can be obtained by using the proposed measure.

7.2 Comparison with centroid-based algorithm

We use five typical cases as examples, present the procedure of the proposed fusion algorithm, and compare the results with traditional centroid-based algorithm [12–14].

(1) The basic support functions are $\mu = (0.25, 0.25, 0.25, 0.25)$ and $\nu = (0.25, 0.25, 0.25, 0.25)$, and their weights are $\Omega_1 = \Omega_2 = 0.5$. The centroid-based algorithm obtained gd $= 2.5$ and a fusion result of $\omega = (0.104166, 0.395833, 0.395833, 0.104166)$. However, the belief shifted from indeterminacy to determinacy, meaning the result is not reasonable. For the proposed method, the fusion result is (0.25, 0.25, 0.25, 0.25), which it is reasonable.

(2) The basic support functions are $\mu = (0.1, 0.6, 0.3)$ and $\nu = (0.6, 0.3, 0.1)$, and $\Omega_1 = \Omega_2 = 0.5$. Using the proposed algorithm, the initial fusion result is $\omega' = (0.35, 0.45, 0.2)$, and $\mathrm{CI}(\omega') = 1.62, E(\omega') = 1.04865$. The results of the centroid-based algorithm and the proposed algorithm are listed in Table 3.

Because μ is not well consistent with ν, the result ω is expected to be indeterminate, and $E(\omega)$ should not differ greatly from $E(\omega')$. Therefore, the result of the proposed algorithm is more reasonable.

(3) The basic support functions are $\mu = (0.1, 0.2, 0.2, 0.5)$ and $\nu = (0.1, 0.15, 0.15, 0.6)$, and $\Omega_1 = \Omega_2 = 0.5$. Using the proposed algorithm, the initial fusion result is $\omega' = (0.1, 0.175, 0.175, 0.55)$, $\mathrm{CI}(\omega') = 3.77$, and $E(\omega') = 1.169$. Additionally, $\mathrm{CI}(\mu) = 3.70, \mathrm{CI}(\nu) = 3.83$, and $\Delta G(\mu, \nu) = 0.043$. The fusion results are displayed in Table 4. μ and ν are consistent well, and they both imply that the 4th proposition is most likely to be true, so the resulting basic support function should reach its maximum at index 4 and the extended entropy of the result should be increased. The results of the proposed algorithm meet these requirements. However, the centroid-based algorithm generates a result indicating that the 3rd proposition is most likely to be true and the extended entropy is lower, which is not reasonable. Thus, the proposed algorithm obviously outperforms the centroid-based algorithm in this case.

(4) The basic support functions are $\mu = (0.05, 0.1, 0.15, 0.7)$ and $\nu = (0.05, 0.1, 0.15, 0.7)$, $\Omega_1 = \Omega_2 = 0.5$. Using the proposed algorithm, the initial fusion result is $\omega' = (0.05, 0.1, 0.15, 0.7)$ and the degree of convexity of ω' is $\mathrm{NC}(\omega') = 0.6 > 0.55$, so $\mathrm{CI}(\omega') = 4$ and $E(\omega') = 0.914$. The fusion results are shown in

Table 5 Comparison of the algorithms for fusing $\mu = (0.05, 0.1, 0.15, 0.7)$ and $\nu = (0.05, 0.1, 0.15, 0.7)$

Fusion algorithm	Fusion result	Centroid/Center of basic support functions	Extended entropy
Centroid-based algorithm	(0.015, 0.056, 0.344, 0.585)	3.63	0.904
Proposed algorithm	(0.010, 0.039, 0.115, 0.836)	4	0.570

Table 6 Comparison of the algorithms for fusing $\mu = (0.6, 0.2, 0.15, 0.05)$ and $\nu = (0.05, 0.1, 0.15, 0.7)$

Fusion algorithm	Fusion result	Centroid/Center of basic support functions	Extended entropy
Centroid-based algorithm	(0.108, 0.338, 0.429, 0.125)	2.571	1.230
Proposed algorithm	(0.142, 0.306, 0.388, 0.164)	2.598	1.303

Table 5. Because $\mu = \nu$, meaning μ and ν are completely consistent, the entropy of the final fusion result should be decreased, and the maximum truth-value should be emphasized. The results of the proposed algorithm are obviously more reasonable than the results of the centroid-based algorithm.

(5) The basic support functions are $\mu = (0.6, 0.2, 0.15, 0.05)$ and $\nu = (0.05, 0.1, 0.15, 0.7)$, $\Omega_1 = \Omega_2 = 0.5$. Using the proposed method, the initial fusion result is $\omega' = (0.325, 0.15, 0.15, 0.375)$, and $\mathrm{CI}(\omega') = 2.598$, $E(\omega') = 1.302$. In addition, $\mathrm{CI}(\mu) = 1.194$, $\mathrm{CI}(\nu) = 3.969$, $\Delta G(\mu, \nu) = 0.925$. The fusion results are shown in Table 6. μ and ν have poor consistency and both algorithms generate results indicating that the 3rd proposition is most likely to be true, which is reasonable. Because μ and ν are conflicting, it is reasonable that the degree of uncertainty in the result is high and close to the initial fusion result. The extended entropy of the result of the proposed algorithm is higher than that of the centroid-based algorithm and is closer to the entropy of the initial fusion result. Thus, the proposed algorithm generates more reasonable results than the centroid-based algorithm.

7.3 Comparison with related work

In this section, we compare the proposed algorithm with related work especially based on Dempster-Shafer theory.

In the classical Dempster-Shafer theory [1,2], two evidences μ and ν are combined by Dempster's rule of combination. For cases (1)–(4) in Subsection 7.2, the fusion results generated by Dempster-Shafer theory are (0.25, 0.25, 0.25, 0.25), (0.222, 0.667, 0.111), (0.027, 0.081, 0.081,0.811), and (0.005, 0.019, 0.043, 0.933). These results are consistent with the results of proposed algorithm. However, for case (5), the Dempster-Shafer theory generates a result of (0.279, 0.186, 0.209, 0.326), which does not satisfy convex property, meaning it is not a valid basic support function for ordered propositions, and is inferior to the proposed algorithm. We then provide a theoretical explanation by Theorem 6.

Theorem 6. Suppose that μ and ν are basic support functions for ordered propositions, ω is the fusion result of μ and ν by Dempster-Shafer theory. Let $m = \arg\max_i \mu(s_i)\nu(s_i)$ and $f(i) = \mu(s_i)\nu(s_i)$. If the function f is monotonically increasing on the interval [1, m], and monotonically decreasing on the interval [m, n] then ω is a basic support function for ordered propositions, meaning ω satisfies the convex property. Otherwise, ω does not satisfy the convex property.

Proof. For ordered propositions, if set $A \notin \{\overline{S}\} \cup \{\{s_i\}|1 \leqslant i \leqslant n\}$, then $\mu(A) = 0$, so $\omega(s_i) = \mu(s_i)\nu(s_i)/(1-K)$, K is normalization constant and is the same for all $\omega(s_i), (1 \leqslant i \leqslant n)$, that is to say ω has the same monotonicity with f, according to Definition 3 and based on the proof of Theorem 3, the theorem is proved.

Corollary 1. Suppose that μ and ν are basic support functions for ordered propositions, ω is the fusion result of μ and ν by Dempster-Shafer theory. Let $m_\mu = \arg\max_i \mu(s_i)$ and $m_\nu = \arg\max_i \nu(s_i)$. If $m_\mu = m_\nu$, then ω satisfies the convex property. Otherwise, it is not guaranteed that ω always satisfies the convex property.

Proof. Let $m = m_\mu = m_\nu$. On interval [1, m], we have

$$\mu(s_1) \leqslant \mu(s_2) \leqslant \cdots \leqslant \mu(s_m), \text{ and } \nu(s_1) \leqslant \nu(s_2) \leqslant \cdots \leqslant \nu(s_m),$$

which implies

$$\mu(s_1)\nu(s_1) \leqslant \mu(s_2)\nu(s_2) \leqslant \cdots \leqslant \mu(s_m)\nu(s_m).$$

Similarly, on interval $[m, n]$, we have

$$\mu(s_m)\nu(s_m) \geqslant \mu(s_{m+1})\nu(s_{m+1}) \geqslant \cdots \geqslant \mu(s_n)\nu(s_n),$$

according to Theorem 6, we obtain ω satisfies convex property.

If $m_\mu \neq m_\nu$, we provide a counter example such that Case (5) in Subsection 7.2, the result of Dempster-Shafer theory does not satisfy convex property.

Therefore, when μ and ν are substantially consistent, the Dempster-Shafer theory is equivalent to the proposed algorithm. However, when μ and ν are conflicting, Dempster-Shafer theory cannot guarantee reasonable results, especially when K=1, e.g., $\mu = (0.9, 0.1, 0, 0)$ and $\nu = (0, 0, 0.1, 0.9)$, Dempster-Shafer theory cannot be applied in this case because the denominator of Dempster's rule of combination is 0. Based on Theorems 2 and 3, we know that the proposed algorithm can generate reasonable result in such cases.

Murphy [4] proposed an approach that first averages the belief functions of the evidences, and then combines the two average belief functions by Dempster's rule of combination to generate the final fusion results. Han et al. [5] proposed a weighted evidence combination approach based on the variance of evidence. Song et al. [6] defined a weighted coefficient based on the credibility and falsity of evidence, and used the weighted coefficient to average the belief functions, which are then combined by Dempster's rule of combination. These three methods are equivalent when combining two basic support functions of ordered proposition. For cases (1)–(4) of Subsection 7.2, the results of the three methods are (0.25, 0.25, 0.25, 0.25), (0.336, 0.555, 0.109), (0.027, 0.082, 0.082, 0.809), (0.005, 0.019, 0.043, 0.933), which are reasonable. However, for case (5), the three methods generate a result of (0.363, 0.077, 0.077, 0.483), which does not satisfy the convex property.

Yang et al. [7] proposed a combination approach based on multi-criteria ranking, they evaluated and ranked the evidences according to the criteria of evidence precision, credibility, and conflict. They then selected the top ranked evidence as the final fusion result. When combing conflicting evidences, although the result satisfies the convex property, it only considers the opinion of one evidence, and ignores the opinion of the other evidences, e.g., the result for case (5) is (0.05, 0.1, 0.15, 0.7), which ignores the opinion of μ. Thus it is not considered a reasonable result for ordered propositions.

Florea et al. [8] proposed a robust combination rule in which the weights are a function of the conflict between evidences. Smarandache and Dezert [9] proposed a series of proportional conflict redistribution (PCR) combination rules, PCR5 is the most efficient version of their proposed rules, PCR5 is the most efficient version of their proposed rules, the fusion result ω of μ and ν is defined as

$$\omega(A) = \sum\nolimits_{A_1 \cap A_2 = A} \mu(A_1)\nu(A_2) + \sum\nolimits_{A \cap B = \emptyset} \left[\frac{\mu(A)^2 \nu(B)}{\mu(A) + \nu(B)} + \frac{\nu(A)^2 \mu(B)}{\nu(A) + \mu(B)} \right].$$

Fu et al. [10] proposed a combination rule based on proportional generalized conflict redistribution called PGCR-A, in their rule, the fusion result ω of μ and ν is defined as

$$\omega(A) = \sum\nolimits_{A_1 \cap A_2 = A} \frac{|A_1 \cap A_2|}{|A_1||A_2|} \mu(A_1)\nu(A_2) + \frac{1}{2}[\mu(A) + \nu(A)]\mathrm{GC},$$

where GC is generalized conflict and defined as sum of potential conflict and Shafer's conflicting. The results generated for case (5) by these three methods are (0.033, 0.022, 0.025, 0.038), (0.351, 0.094, 0.092, 0.463), and (0.32, 0.154, 0.156, 0.37). None of the results satisfy convex property, which is unreasonable for ordered propositions.

In summary, when μ and ν are substantially consistent, most DS-based methods are comparable to the proposed algorithm. However, when μ and ν are poorly consistent, the DS-based methods cannot ensure the convex property in the fusion results. The proposed algorithm can provide more reasonable results in such cases.

8 Conclusion and future work

In order to overcome the shortcomings of previous method for fusing ordered propositions, we proposed a novel method based on consistency and uncertainty measurements. We adopted basic support functions to represent the truth-values of each ordered proposition, and then proposed the concept of using the convexity degree, mean, and center of basic support functions to comprehensively describe the basic support functions of ordered propositions. We introduced entropy as a measure of the uncertainty of basic support functions for ordered propositions. Additionally, we generalized the indeterminacy of basic support functions and proposed a novel method to measure the consistency between two basic support functions. Finally, we proposed a novel algorithm to fuse the basic support functions of ordered propositions. Our theoretical analysis and experimental results demonstrate the effectiveness of the proposed method. In future, we will perform research on the properties of associativity and idempotency in the proposed method. The uncertainty measurement proposed in this paper has the potential to be applied to other types of propositions or arguments. If we want to do this, we must find a way to handle situations that the cardinality of a focal element is greater than 1, meaning the proposition contains multiple characteristics or features of an object. We will also need to adjust the formulation of the measures based on the specific types of propositions or arguments. This is an interesting topic that we concentrate on in future work.

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