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Robust state estimation for uncertain linear systems with random parametric uncertainties

Huabo LIU[1,2*] & Tong ZHOU[1,3]

[1]*Department of Automation, Tsinghua University, Beijing* 100084*, China;*
[2]*College of Automation and Electrical Engineering, Qingdao University, Qingdao* 266071*, China;*
[3]*Tsinghua National Laboratory for Information Science and Technology (TNList), Tsinghua University, Beijing* 100084*, China*

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Abstract In this paper, we investigate state estimations of a dynamical system with random parametric uncertainties which may arbitrarily affect a plant state-space model. A robust estimator is derived based on expectation minimization of estimation errors. An analytic solution similar to that of the well-known Kalman filter is derived for this new robust estimator which can be realized recursively with a comparable computational complexity. Under some weak assumptions, it is proved that this estimator converges to a stable system, the covariance matrix of estimation errors is bounded, and the estimation is asymptotically unbiased. Numerical simulations show that the obtained robust filter has an estimation accuracy comparable to other robust estimators and can be applied in a wider range.

Keywords robustness, state estimation, recursive estimation, parametric uncertainty, regularized least-squares

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1 Introduction

State estimation has been extensively utilized in system control and signal processing. The Kalman filter is the optimal estimator under the criterion of mean-squares for linear systems with normal external disturbances and widely applied in numerous fields such as control, finance, communication and so on [1–3]. Moreover, some variants of Kalman filter are put forward and applied, such as extended Kalman filter and unscented Kalman filter (see [1, 4] and the references therein).

Since modelling errors are generally unavoidable on account of complex manufacture process and modelling inaccuracies, robust state estimators have been developed whose performances do not degrade appreciably when actual plant parameters deviate from their nominal ones in a reasonable way. The H_∞ robust filters are based on the 2-induced norm boundedness of an operator mapping the disturbances to the estimation errors [5–9]. Abundant results have been obtained about the extensive applications of H_∞ estimation, such as [9] for a general class of uncertain discrete-time stochastic neural networks and the references therein for more. The imperfection of H_∞ filters is that certain existence conditions should be verified at every estimation instant in their on-line realization. Set-valued estimation is to

* Corresponding author (email: liu-hb10@mails.tsinghua.edu.cn)

construct ellipsoids around state estimates based on the measurements and subject to norm-bounded noise disturbances [10–12]. The verification of existence conditions needs to be implemented, which is a limitation for recursive filtering. Guaranteed error variance designs are to ensure that the steady-state variance of the state estimation errors is upper bounded for all admissible uncertainties [13, 14]. The existence verification of a positive definite stabilizing solution of a discrete algebraic Riccati equation is required [15].

In [2] a framework based on regularized least-squares (RLS) is suggested for robust filter designs, which shares the similar form of the well-known Kalman filter operating on corrected parameters rather than given nominal ones. However, this robust filter focuses on a worst case analysis, which may be conservative under relatively "small" uncertainties. In this situation, a design methodology is proposed in [16] based on iteratively solving a tradeoff problem between nominal performance and robustness to the uncertainty in consideration of the uncertainty scales. In [17] an optimal filtering approach for uncertain discrete-time systems based on regularization and penalty function is proposed.

There is another paradigm in robust filter designs which is based on sensitivity penalization of estimation errors to plant parameter variations. In [18] it is adopted for single-input single-output systems in the frequency domain with transfer function representation and spectral factorization. In [3, 19] it is employed for multi-input multi-output time varying dynamical systems under state-space framework and the plant parameters are affected by modelling errors in a relatively arbitrary way. Based on nominal estimation performances and sensitivity penalization of estimation errors to parameter variations, an analytic expression of the robust state estimator has been derived [3, 19]. The estimator can be recursively implemented, and has a similar form and comparable computational complexity with the widely-applied Kalman filter. The possible limitation of sensitivity penalization based robust estimator is that the selection of the optimal design parameter γ is unclear.

In this paper we investigate the robust state estimation for time-varying linear systems with random parametric uncertainties. We propose a new robust state estimation method taking account of the random modelling errors based on the relationship between the Kalman filter and regularized least-squares. An analytic expression has been derived for this robust estimator, which can be recursively implemented and has a similar form and comparable computational complexity with the Kalman filter. It is proved that under some assumptions, as well as conditions like detectability and stabilizability, the estimator converges to a stable system, the covariance matrix of estimation errors is bounded, and the estimation is asymptotically unbiased (the expectation of estimation errors converges to zero at least exponentially). Some numerical simulations show that this robust estimator has comparable estimation performances and can be widely applied. Compared to the ones in [16] and [3, 19], the derived estimator does only require the off-line computations of some relevant matrices instead of the optimization of a design parameter, and the parametric uncertainties are permitted to affect the model arbitrarily.

The rest of this paper is organized as follows. In Section 2, a plant state-space model is given and the robust state estimator is derived. Some important properties such as convergence and boundedness are discussed in Section 3. Numerical simulation results are provided in Section 4. Finally, Section 5 concludes this paper. The appendices are included to give a derivation of the recursive estimation procedure and proofs of the theoretical results.

Notation: Given a column vector x and a positive-definite matrix W, $\|x\|$ and $\|x\|_W$ are defined to denote the Euclidean norm and its weighted version, namely, $\sqrt{x^{\mathrm{T}}x}$ and $\sqrt{x^{\mathrm{T}}Wx}$, respectively. $\mathrm{E}(*)$ represents the mathematical expectation of a random variable, vector or matrix and δ_{ij} the Kronecker delta function. $\mathrm{col}\{X_j\}$ denotes the vector/matrix stacked by X_j.

2 Plant dynamics description and robust state estimator design

Consider the following uncertain linear system,

$$\begin{cases} x_{i+1} = A_i(\varepsilon_i)\,x_i + B_i(\varepsilon_i)\,w_i, \\ y_i = C_i(\varepsilon_i)\,x_i + v_i, \end{cases} \qquad i \geqslant 0, \tag{1}$$

where x_i is the state, w_i is the process noise, y_i is the measurement, and v_i is the measurement error. x_0, w_i and v_i are uncorrelated random vectors with $\mathrm{E}(w_i) = 0,\ \mathrm{E}(v_i) = 0$ and $\mathrm{E}(\mathrm{col}(x_0 - \mathrm{E}(x_0), w_i, v_i)(*)^{\mathrm{T}}) = \mathrm{diag}\{\Pi_0, Q_i\delta_{ij}, R_i\delta_{ij}\}$, in which Π_0, Q_i and R_i are known positive definite matrices. Moreover, ε_i is composed of L real valued scalar bounded uncertainties $\varepsilon_{i,k},\ k = 1, \ldots, L$, which denotes parametric modelling errors at the ith sampled instant. It is assumed that the L uncertainties are independent of each other.

Remark 1. In System (1) the way in which the plant parameters are affected by modelling errors ε_i can be arbitrary, while in [2] and [16], it is required that system matrices depend linearly on a norm bounded uncertainty matrix, and in [3,19] the elements of system matrices are needed to be differentiable functions of ε_i. This characteristic makes System (1) capture possibly more of the real systems' behaviours than the ones in [2, 3, 16, 19].

From [2], the Kalman filter admits a deterministic interpretation as the solution to a RLS problem, as follows.

$$\hat{x}_{i+1|i+1} = A_i\left(0\right)\hat{x}_{i|i+1} + B_i\left(0\right)\hat{w}_{i|i+1}, \tag{2}$$
$$\begin{pmatrix} \hat{x}_{i|i+1} \\ \hat{w}_{i|i+1} \end{pmatrix} = \arg\min_{x_i, w_i}\left[\left\|x_i - \hat{x}_{i|i}\right\|^2_{P^{-1}_{i|i}} + \|w_i\|^2_{Q_i^{-1}} + \|y_{i+1} - C_{i+1}\left(0\right)x_{i+1}\|^2_{R^{-1}_{i+1}}\right],$$

where $\hat{x}_{i|l}$ stands for the optimal estimator of x_i based on measurements $y_j|_{j=0}^{l}$, and $P_{i|l}$ the corresponding estimation error covariance matrix. The cost function of the RLS problem is the regularized squares residual norm. The interpretation means that given an initial estimate $\hat{x}_{i|i}$ for x_i, one seeks to ameliorate it by incorporating the additional information provided by the new measurement y_{i+1}.

We improve the cost function of the RLS problem considering the appreciable deterioration of estimation performances because of model uncertainties which are generally unavoidable. For notational simplicity, define matrices respectively as follows, $\Psi_i = R^{-1}_{i+1}, H_i(\varepsilon_i, \varepsilon_{i+1}) = C_{i+1}\left(\varepsilon_{i+1}\right)\left[A_i\left(\varepsilon_i\right) \quad B_i\left(\varepsilon_i\right)\right]$, $\beta_i(\varepsilon_i, \varepsilon_{i+1}) = y_{i+1} - C_{i+1}(\varepsilon_{i+1})A_i\left(\varepsilon_i\right)\hat{x}_{i|i}$, $\Phi_i = \mathrm{diag}\{P^{-1}_{i|i}, Q_i^{-1}\}$ and $\alpha_i = \mathrm{col}(x_i - \hat{x}_{i|i}, w_i)$. The new cost function of the RLS at every instant is suggested to be (3) in view of the effect of random modelling errors.

$$\begin{aligned} J(\alpha_i) &= \mathrm{E}\Big\{\left\|x_i - \hat{x}_{i|i}\right\|^2_{P^{-1}_{i|i}} + \|w_i\|^2_{Q_i^{-1}} + \|y_{i+1} - C_{i+1}\left(\varepsilon_{i+1}\right)x_{i+1}\|^2_{R^{-1}_{i+1}}\Big\} \\ &= \mathrm{E}\left\{\|\alpha_i\|^2_{\Phi_i} + \|H_i(\varepsilon_i, \varepsilon_{i+1})\alpha_i - \beta_i(\varepsilon_i, \varepsilon_{i+1})\|^2_{\Psi_i}\right\} \\ &= \|\alpha_i\|^2_{\Phi_i} + \mathrm{E}\left\{\|H_i(\varepsilon_i, \varepsilon_{i+1})\alpha_i - \beta_i(\varepsilon_i, \varepsilon_{i+1})\|^2_{\Psi_i}\right\}. \end{aligned} \tag{3}$$

From the cost function we can conclude that the expectation of estimation performances on account of the random modelling errors is taken into account and an estimate of x_{i+1} is calculated using the formula of (2). When there are no modelling errors, the estimator through minimizing the cost function of (3) collapses to the standard Kalman filter.

It is not difficult to know that the cost function $J(\alpha_i)$ is a strictly convex function from the matrices Φ_i and Ψ_i, which has a global unique minimum at $\partial J\left(\alpha_i\right)/\partial\alpha_i = 0$ expressed as $\alpha_{i\mathrm{opt}}$. It is determined by (4) as follows.

$$\begin{aligned} &\left(\Phi_i + \mathrm{E}\left\{\begin{bmatrix} A_i^{\mathrm{T}}\left(\varepsilon_i\right) \\ B_i^{\mathrm{T}}\left(\varepsilon_i\right) \end{bmatrix} C^{\mathrm{T}}_{i+1}\left(\varepsilon_{i+1}\right)R^{-1}_{i+1}C_{i+1}\left(\varepsilon_{i+1}\right)\begin{bmatrix} A_i\left(\varepsilon_i\right) & B_i\left(\varepsilon_i\right) \end{bmatrix}\right\}\right)\alpha_{i\mathrm{opt}} \\ = \ &\mathrm{E}\left\{\begin{bmatrix} A_i^{\mathrm{T}}\left(\varepsilon_i\right) \\ B_i^{\mathrm{T}}\left(\varepsilon_i\right) \end{bmatrix} C^{\mathrm{T}}_{i+1}\left(\varepsilon_{i+1}\right)\right\}R^{-1}_{i+1}y_{i+1} - \mathrm{E}\left\{\begin{bmatrix} A_i^{\mathrm{T}}\left(\varepsilon_i\right) \\ B_i^{\mathrm{T}}\left(\varepsilon_i\right) \end{bmatrix} C^{\mathrm{T}}_{i+1}\left(\varepsilon_{i+1}\right)R^{-1}_{i+1}C_{i+1}\left(\varepsilon_{i+1}\right)A_i\left(\varepsilon_i\right)\right\}\hat{x}_{i|i}. \end{aligned} \tag{4}$$

For notational simplicity, define matrices H_{i1}, H_{i2} and H_{i3} as follows. It is obvious that $H_{i3} = H_{i1}\left[I\ 0\right]^{\mathrm{T}}$.

$$
\begin{aligned}
&\mathrm{E}\left\{\begin{bmatrix} A_i^{\mathrm{T}}(\varepsilon_i) \\ B_i^{\mathrm{T}}(\varepsilon_i) \end{bmatrix} C_{i+1}^{\mathrm{T}}(\varepsilon_{i+1}) R_{i+1}^{-1} C_{i+1}(\varepsilon_{i+1}) \begin{bmatrix} A_i(\varepsilon_i) & B_i(\varepsilon_i) \end{bmatrix}\right\} = H_{i1}, \\
&\mathrm{E}\left\{\begin{bmatrix} A_i^{\mathrm{T}}(\varepsilon_i) \\ B_i^{\mathrm{T}}(\varepsilon_i) \end{bmatrix} C_{i+1}^{\mathrm{T}}(\varepsilon_{i+1})\right\} = H_{i2}, \\
&\mathrm{E}\left\{\begin{bmatrix} A_i^{\mathrm{T}}(\varepsilon_i) \\ B_i^{\mathrm{T}}(\varepsilon_i) \end{bmatrix} C_{i+1}^{\mathrm{T}}(\varepsilon_{i+1}) R_{i+1}^{-1} C_{i+1}(\varepsilon_{i+1}) A_i(\varepsilon_i)\right\} = H_{i3}.
\end{aligned} \tag{5}
$$

Remark 2. In the proposed robust state estimation method, the key point is that H_{i1} and H_{i2} can be computed off-line, while the plant parameters are affected by modelling errors in an arbitrary way. The statistic characteristics of ε_i are assumed to be known, and the calculation of H_{i1} and H_{i2} is solvable through direct algebraic manipulations if these functions are simple. Otherwise, H_{i1} and H_{i2} can be computed through stochastic simulations [20]. For example, when the statistical distributions of ε_i are known, 10000 realizations of ε_i can be obtained, by which H_{i1} and H_{i2} can be computed. On the other hand, in the process of system modelling, plenty of parameter matrices such as $A_i(\varepsilon_i), B_i(\varepsilon_i), C_i(\varepsilon_i)$ on modelling error realizations are obtained taking the existence of modelling errors into consideration. On the basis of these parameter matrices corresponding to modelling error realizations, H_{i1} and H_{i2} can also be calculated. This means that the derived robust state estimator may have wider applications than the ones in [2, 3, 16, 19].

According to the aforementioned explanations we can provide the following recursive procedure to compute the estimate of the plant state when there exist random parameter uncertainties. The derivative details are provided in Appendix A.

(1) Initialization. Designate $P_{0|0}$ and $\hat{x}_{0|0}$ as $P_{0|0} = (\Pi_0^{-1} + \mathrm{E}\{C_0^{\mathrm{T}}(\varepsilon_0)R_0^{-1}C_0(\varepsilon_0)\})^{-1}$ and $\hat{x}_{0|0} = P_{0|0}\mathrm{E}\{C_0^{\mathrm{T}}(\varepsilon_0)\}R_0^{-1}y_0$, respectively.

(2) Parameter modification. Denote

$$
H_{i1} - \begin{bmatrix} A_i^{\mathrm{T}}(0) \\ B_i^{\mathrm{T}}(0) \end{bmatrix} C_{i+1}^{\mathrm{T}}(0) R_{i+1}^{-1} C_{i+1}(0) \begin{bmatrix} A_i(0) & B_i(0) \end{bmatrix}
$$

as $G_i = \begin{bmatrix} G_{i11} & G_{i12} \\ G_{i12}^{\mathrm{T}} & G_{i22} \end{bmatrix}$ and define matrices $\hat{A}_i(0), \hat{B}_i(0), \hat{P}_{i|i}$ and $\hat{Q}_i$ respectively as follows.

$$
\begin{aligned}
&\hat{P}_{i|i}^{-1} = P_{i|i}^{-1} + G_{i11}, \\
&\hat{Q}_i^{-1} = Q_i^{-1} + G_{i22} - G_{i12}^{\mathrm{T}}\hat{P}_{i|i}G_{i12}, \\
&\hat{A}_i(0) = \left(A_i(0) - \hat{B}_i(0)\hat{Q}_i G_{i12}^{\mathrm{T}}\right)\left(I - \hat{P}_{i|i}G_{i11}\right), \\
&\hat{B}_i(0) = B_i(0) - A_i(0)\hat{P}_{i|i}G_{i12}.
\end{aligned} \tag{6}
$$

(3) State estimate updating. Calculate $\hat{x}_{i+1|i+1}$ and $P_{i+1|i+1}$ respectively as

$$
\hat{x}_{i+1|i+1} = \hat{A}_i(0)\hat{x}_{i|i} + P_{i+1|i+1}\begin{pmatrix} P_{i+1|i}^{-1}\left(A_i(0)\hat{P}_{i|i}\begin{bmatrix} I & 0 \end{bmatrix} + \hat{B}_i(0)\hat{Q}_i\begin{bmatrix} -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} & I \end{bmatrix}\right) H_{i2}R_{i+1}^{-1}y_{i+1} \\ -C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)\hat{A}_i(0)\hat{x}_{i|i} \end{pmatrix}.
$$

$$
\begin{aligned}
&P_{i+1|i} = A_i(0)\hat{P}_{i|i}A_i^{\mathrm{T}}(0) + \hat{B}_i(0)\hat{Q}_i\hat{B}_i^{\mathrm{T}}(0), \\
&R_{e,i+1} = R_{i+1} + C_{i+1}(0)P_{i+1|i}C_{i+1}^{\mathrm{T}}(0), \\
&P_{i+1|i+1} = P_{i+1|i} - P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{e,i+1}^{-1}C_{i+1}(0)P_{i+1|i}.
\end{aligned} \tag{7}
$$

Based on the above results, the form of this estimation procedure is consistent with the time and measurement update form of the robust estimator derived in [2, 16] and has a similar form with the one in [3, 19]. The robust estimator is accommodated with modified parameters and estimation error covariance matrix rather than nominal ones taking account of estimation degradation owing to plant parameter variations. At the same time, it is not difficult to derive a prediction form and an information form for the robust state estimator.

3 Some properties of the estimator

In this section, some important asymptotic properties of the derived state estimator are investigated. Assume $\varepsilon_{i,k}$ is normalized in magnitude to be contractive and the set $\mathcal{E}$ is composed of these modelling errors, that is, $\mathcal{E} = \{\varepsilon | |\varepsilon_{i,k}| \leqslant 1,\ k = 1, \ldots, L\}$. Moreover, we adopt two assumptions for the asymptotic behaviours analysis of the robust state estimator.

(A1) The nominal model parameters are time invariant and only ε_i is time varying.

(A2) The uncertain linear system of (1) is exponentially stable in the sense of Lyapunov. Moreover, the relevant matrices are all bounded for $i > 0$ and $\varepsilon_i \in \mathcal{E}$.

In the above assumptions, the first one is necessary in investigating the steady-state behaviours of the estimator and the second one is to guarantee the unbiasedness of the estimator and the boundedeness of the estimation errors. A convergence property of $P_{i|i-1}$ can be established and its proof is deferred to Appendix B.

For symbol simplicity, the relevant matrices are abbreviated as A, B, C, R, Q, H_1, H_2 and G respectively in the following discussions. Moreover, define some matrices as follows:

$$U = \left(Q^{-1} + G_{22} - G_{12}^{\mathrm{T}} G_{11}^{-1} G_{12}\right)^{-1},\ D = BU^{\frac{1}{2}},$$

$$J = \begin{bmatrix} 0 & U^{\frac{1}{2}} G_{12}^{\mathrm{T}} G_{11}^{-\frac{1}{2}} \end{bmatrix},\ W = \begin{bmatrix} I & 0 \\ 0 & I + G_{11}^{-\frac{1}{2}} G_{12} U G_{12}^{\mathrm{T}} G_{11}^{-\frac{1}{2}} \end{bmatrix},\ F = \begin{bmatrix} R^{-\frac{1}{2}} C \\ G_{11}^{\frac{1}{2}} \end{bmatrix}.$$

Theorem 1. Assume that Condition (A1) is satisfied, (A, F) is detectable and

$$(A - BUG_{12}^{\mathrm{T}}(I + G_{11}^{-1} G_{12} U G_{12}^{\mathrm{T}})^{-1}, D(I + U^{1/2} G_{12}^{\mathrm{T}} G_{11}^{-1} G_{12} U^{1/2})^{-1/2})$$

is stabilizable. Then, for arbitrary $\Pi_0 > 0$, $P_{i|i-1}$ converges exponentially to a unique positive semi-definite matrix P, while A_{pi} converges to a constant stable matrix A_p. Here, $A_{pi} = A - (AP_{i|i-1}F^{\mathrm{T}} + DJ) \times (W + FP_{i|i-1}F^{\mathrm{T}})^{-1}F$ and $A_p = A - (APF^{\mathrm{T}} + DJ)(W + FPF^{\mathrm{T}})^{-1}F$.

We know that the convergence of $P_{i|i-1}$ is equivalent to that of $P_{i|i}$ from the relation between $P_{i|i-1}$ and $P_{i|i}$, and the conditions of Theorem 1 are also the ones for the convergence of $P_{i|i}$.

Eq. (7) can be rewritten as follows,

$$\hat{x}_{i+1|i+1} = A_{fi}\hat{x}_{i|i} + P_{i+1|i+1}P_{i+1|i}^{-1}B_{fi}y_{i+1}, \tag{8}$$

where $A_{fi} = [I - P_{i+1|i+1}C^{\mathrm{T}}R^{-1}C]\hat{A}_i, B_{fi} = (A\hat{P}_{i|i}[\,I\ 0\,] + \hat{B}_i\hat{Q}_i[\,-G_{12}^{\mathrm{T}}\hat{P}_{i|i}\ I\,])H_2R^{-1}$.

Based on (8) and Theorem 1, we present the asymptotic stability of the robust state estimator in Theorem 2. Its proof is deferred to Appendix C.

Theorem 2. Assume that all the conditions of Theorem 1 are satisfied, and the derived robust state estimator converges to a time-invariant stable system.

After the convergence of the estimation procedure and the stability of the steady state estimator are established, we investigate the unbiasedness of the estimate and the boundedness of the covariance matrix of the estimation errors under the condition that System (1) is exponentially stable.

For simple denotation, define $\bar{x}_i, \hat{\bar{x}}_{i|i}$ and $\tilde{\bar{x}}_{i|i}$ respectively as $\bar{x}_i = [I + \Omega_i(0)]x_i, \hat{\bar{x}}_{i|i} = [I + \Omega_i(0)]\hat{x}_{i|i}$ and $\tilde{\bar{x}}_{i|i} = \bar{x}_i - \hat{\bar{x}}_{i|i}$, where $\Omega_i(\varepsilon_i) = P_{i|i-1}C_i^{\mathrm{T}}(0)R_i^{-1}C_i(\varepsilon_i)$. It is obvious that $\tilde{\bar{x}}_{i|i} = [I + \Omega_i(0)](x_i - \hat{x}_{i|i})$. Define matrix $\bar{A}_i(\varepsilon_i)$ as $(I + \Omega_{i+1}(0))\, A_i(\varepsilon_i)\, (I + \Omega_i(0))^{-1}$. Then it can be directly proved from (8) and the proof of Theorem 2 that

$$\begin{bmatrix} \tilde{\bar{x}}_{i+1|i+1} \\ \hat{\bar{x}}_{i+1|i+1} \end{bmatrix} = \tilde{A}_i(\varepsilon_i, \varepsilon_{i+1}) \begin{bmatrix} \tilde{\bar{x}}_{i|i} \\ \hat{\bar{x}}_{i|i} \end{bmatrix} + \tilde{B}_i(\varepsilon_i, \varepsilon_{i+1}) \begin{bmatrix} w_i \\ v_{i+1} \end{bmatrix}, \tag{9}$$

where

$$\tilde{A}_i(\varepsilon_i,\varepsilon_{i+1})=\left[\begin{array}{c|c} (I+\Omega_{i+1}(0)-B_{fi}C_{i+1}(\varepsilon_{i+1})) & (I+\Omega_{i+1}(0)-B_{fi}C_{i+1}(\varepsilon_{i+1})) \\ \times(I+\Omega_{i+1}(0))^{-1}\bar{A}_i(\varepsilon_i) & \times(I+\Omega_{i+1}(0))^{-1}\bar{A}_i(\varepsilon_i)-A_{pi} \\ \hline B_{fi}C_{i+1}(\varepsilon_{i+1})(I+\Omega_{i+1}(0))^{-1}\bar{A}_i(\varepsilon_i) & (B_{fi}C_{i+1}(\varepsilon_{i+1}))(I+\Omega_{i+1}(0))^{-1}\bar{A}_i(\varepsilon_i)+A_{pi} \end{array}\right],$$

$$\tilde{B}_i(\varepsilon_i,\varepsilon_{i+1})=\left[\begin{array}{c|c} ((I+\Omega_{i+1}(0))-B_{fi}C_{i+1}(\varepsilon_{i+1}))B_i(\varepsilon_i) & -B_{fi} \\ \hline B_{fi}C_{i+1}(\varepsilon_{i+1})B_i(\varepsilon_i) & B_{fi} \end{array}\right].$$

Based on these relations and the stability of matrix $A_i(\varepsilon_i)$, a condition is obtained for the boundedness of estimation errors of the robust filter, as well as its asymptotic unbiasedness. Noting that the form of (9) is similar to the one of Eq. (16) in [21], the proof can be accomplished by the same line and ignored here.

Theorem 3. Suppose that Assumptions (A1) and (A2) are simultaneously satisfied, (A,F) is detectable and $(A-BUG_{12}^{\mathrm{T}}(I+G_{11}^{-1}G_{12}UG_{12}^{\mathrm{T}})^{-1}, D(I+U^{1/2}G_{12}^{\mathrm{T}}G_{11}^{-1}G_{12}U^{1/2})^{-1/2})$ is stabilizable. Then, the robust state estimator is asymptotically unbiased, and at every sampled instant i, its estimation errors have a finite covariance matrix.

4 Numerical simulations

In this section, we compare the performances of the derived state estimator by some examples with those of the Kalman filters based on actual parameters and nominal parameters respectively, as well as the ones in [2, 3, 16, 19]. In these simulations, 10^3 time-domain input-output data pairs are generated for plant state estimation, in which all the initial states are set to zero, while disturbances w_i and v_i are produced following normal distributions.

Besides, 5×10^2 simulations are performed for each set of numerical experiment settings to compute the ensemble-average estimation error variance at every sampled instant. The size of the ensemble-average is approximated by the averaged value of the square of the Euclidean distance from the actual plant state to its estimate, that is, $\mathrm{E}\|x_i-\hat{x}_{i|i}\|^2\approx\frac{1}{500}\sum_{j=1}^{500}\|x_i-\hat{x}_{i|i}^{(j)}\|^2$.

This example is borrowed from [1] and is also used in [2, 16, 19], in which it is assumed that

$$\begin{aligned} &A_i(\varepsilon_i)=\begin{bmatrix}0.9802 & 0.0196+0.099\varepsilon_i\\ 0.0000 & 0.9802\end{bmatrix},\ B_i(\varepsilon_i)=\begin{bmatrix}1.0000 & 0.0000\\ 0.0000 & 1.0000\end{bmatrix},\\ &C_i(\varepsilon_i)=\begin{bmatrix}1.0000 & -1.0000\end{bmatrix},\ R_i=1.0000,\\ &Q_i=\begin{bmatrix}1.9608 & 0.0195\\ 0.0195 & 1.9605\end{bmatrix},\ \Pi_0=\begin{bmatrix}1.0000 & 0.0000\\ 0.0000 & 1.0000\end{bmatrix}. \end{aligned}$$

In Figure 1(a), the modelling error ε_i is generated according to a uniform distribution in $[-1,1]$ and is kept unchanged for each experiment. In Figure 1(c) the modelling error is generated according to the same distribution in every sampled instant. We can see that the derived estimator has nearly the same performances to the other robust estimators. The design parameters λ [2], α [16] and γ [19] are set respectively as recommended values by the authors.

In Figure 2, the (1,2) entry of $A_i(\varepsilon_i)$ related to modelling error is set to be $0.0196+0.99\varepsilon_i$, like the one in [16], which means that the uncertainty is "large" compared to the original case. In the situation of a fixed uncertainty, the performance of the Kalman filter with nominal parameter degrades significantly, the derived robust filter obtains similar performance compared to the robust filters in [16, 19] and only 1dB performance decreases compared with the filter in [2]. On the other hand, in the time-varying case, the filter is comparable to the ones in [2, 16, 19].

In Figure 3, the (1,2) entry of $A_i(\varepsilon_i)$ is modified to be $0.3912+0.099\varepsilon_i$, which means that the uncertainty is relatively "small" owing to the enlargement of nominal parameter. It shows that the filter in [2] is

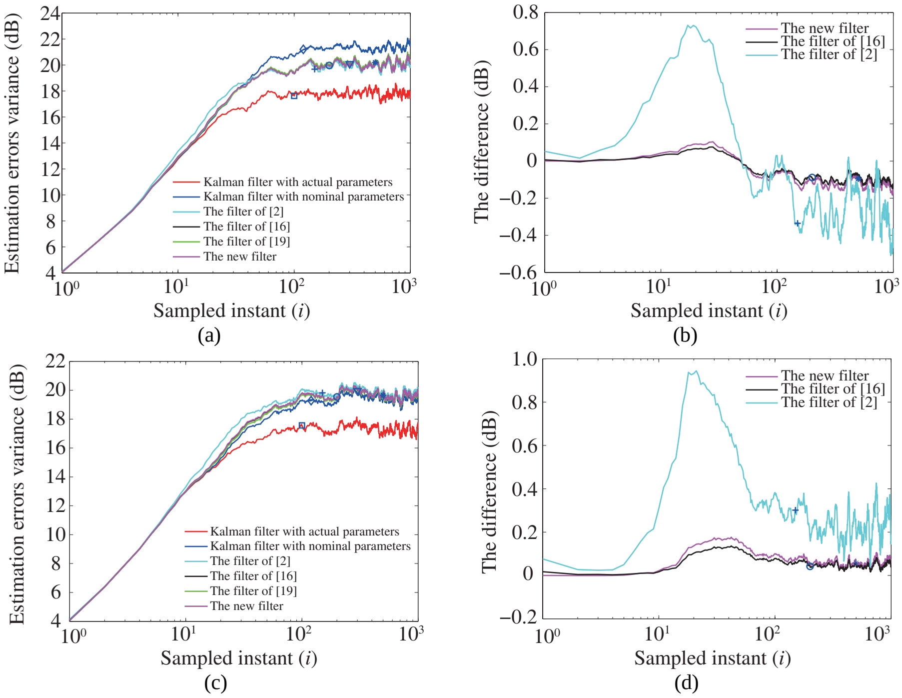

Figure 1 Estimation error variance with model uncertainty. −□−: Kalman filter with actual parameters; −◇−: Kalman filter with nominal parameters; −▽−: Filter of [19]; −∘−: Filter of [16]; −+−: Filter of [2]; −∗−: the new filter. For clarity, the performance differences of the filters are provided between the corresponding methods and the benchmark, where the benchmark is the performance curve based on the approach in [19]. The same line styles and markers are used in the following figures to represent the curves obtained from the same numerical simulation settings. (a) Estimation error variance with a fixed uncertainty; (b) the detailed difference with a fixed uncertainty; (c) estimation error variance with a time-varying uncertainty; (d) the detailed difference with a time-varying uncertainty.

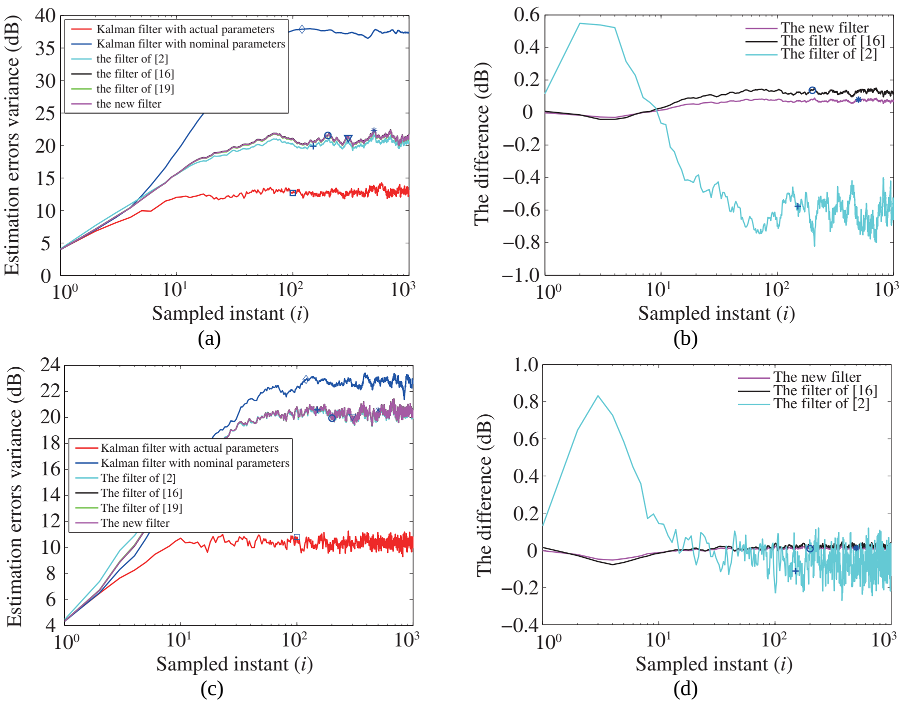

Figure 2 Estimation error variance under a large uncertainty. −□−: Kalman filter with actual parameters; −◇−: Kalman filter with nominal parameters; −▽−: Filter of [19]; −∘−: Filter of [16]; −+−: Filter of [2]; −∗−: the new filter. (a) Estimation error variance with a fixed uncertainty; (b) the detailed difference with a fixed uncertainty; (c) estimation error variance with a time-varying uncertainty; (d) the detailed difference with a time-varying uncertainty.

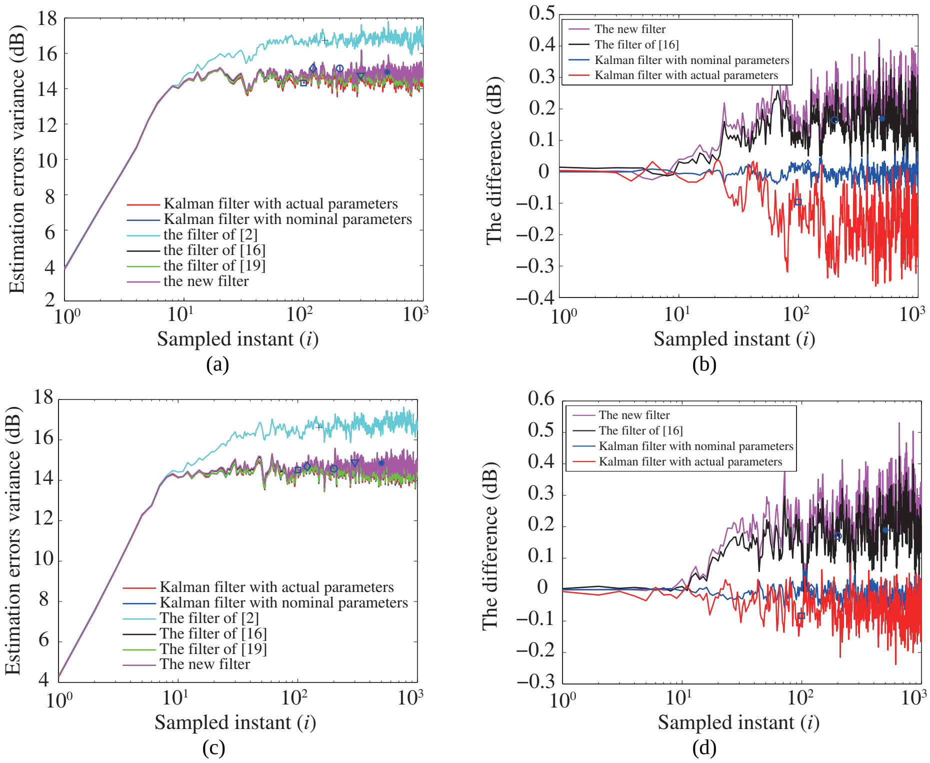

Figure 3 Estimation error variance under a relatively small uncertainty. −□−: Kalman filter with actual parameters; −◇−: Kalman filter with nominal parameters; −▽−: Filter of [19]; − ∘ −: Filter of [16]; − + −: Filter of [2]; − ∗ −: the new filter. (a) Estimation error variance with a fixed uncertainty; (b) the detailed difference with a fixed uncertainty; (c) estimation error variance with a time-varying uncertainty; (d) the detailed difference with a time-varying uncertainty.

conservative in the situations with both a fixed uncertainty and a time-varying uncertainty, and the new filter has comparable performances with other filters and does not undergo the conservativeness.

Figures 1–3 show that the derived robust state estimator is not sensitive to the relative magnitude of the uncertainties and exhibits estimation performances comparable with those of the robust filters in [16, 19]. Therefore, on the premise that some relevant matrices are computed off-line, there is no need to solve parameter selections or optimization problems on the optimal designs for the new state estimator compared to the existing robust state estimators. This fact means that it is a convenient and valid estimation method in applications.

5 Conclusion

This paper investigates robust state estimations based on a minimization of expectation of estimation errors subject to norm-bounded and random parametric uncertainties. The derived estimator takes a form similar to that of the Kalman filter and can be easily implemented in a recursive manner with a comparable computational burden. Moreover, there are no restrictions on the ways in which uncertainties affect plant parameters, and model uncertainties can take an arbitrary structure. Numerical simulations show that the robust estimator of this paper overcomes the conservativeness of [2] and has comparable performances under several circumstances on the magnitude of uncertainties with the improved robust state estimators in [16, 19].

It is interesting to generalize the derived robust state estimator to uncertain linear systems with delayed and missing measurements. On the other hand, it still remains challenging to give a quantitative estimate for the size of tolerable modelling errors.

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Conflict of interest The authors declare that they have no conflict of interest.

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Appendix A Derivation of the estimation procedure

To estimate the initial state x_0, the cost function $J(\alpha_0)$ is set as follows,

$$J_0 = \left[\mathrm{E}\left\{ \|x_0\|^2_{\Pi_0^{-1}} + \|y_0 - C_0(\varepsilon_0)\, x_0\|^2_{R_0^{-1}} \right\} \right].$$

We obtain the following estimate of the initial state $\hat{x}_{0|0} = P_{0|0}\mathrm{E}\left\{C_0^{\mathrm{T}}(\varepsilon_0)\right\} R_0^{-1} y_0$, where

$$P_{0|0} = \left(\Pi_0^{-1} + \mathrm{E}\left\{ C_0^{\mathrm{T}}(\varepsilon_0)\, R_0^{-1} C_0(\varepsilon_0) \right\} \right)^{-1}.$$

From the definition of H_{i1}, it is normally true that $G_i \geqslant 0$. Denote $\alpha_{i\mathrm{opt}}, C_{i+1}(0)[A_i(0) \quad \hat{B}_i(0)]$ and $\hat{x}_{i|i+1} + \hat{P}_{i|i} G_{i12} \hat{w}_{i|i+1}$ as $\mathrm{col}(\hat{x}_{i|i+1} - \hat{x}_{i|i}, \hat{w}_{i|i+1}), \hat{H}_i$ and $\tilde{x}_{i|i+1}$.

From the following algebraic relation,

$$\begin{aligned}\left(\begin{bmatrix} P_{i|i}^{-1} & \\ & Q_i^{-1} \end{bmatrix} + G_i \right) &= \left(\begin{bmatrix} P_{i|i}^{-1} & \\ & Q_i^{-1} \end{bmatrix} + \begin{bmatrix} G_{i11} & G_{i12} \\ G_{i12}^{\mathrm{T}} & G_{i22} \end{bmatrix} \right) = \begin{bmatrix} I & 0 \\ G_{i12}^{\mathrm{T}} \hat{P}_{i|i} & I \end{bmatrix} \\ &\quad \times \begin{bmatrix} \hat{P}_{i|i}^{-1} & 0 \\ 0 & Q_i^{-1} + G_{i22} - G_{i12}^{\mathrm{T}} \hat{P}_{i|i} G_{i12} \end{bmatrix} \begin{bmatrix} I & \hat{P}_{i|i} G_{i12} \\ 0 & I \end{bmatrix},\end{aligned} \tag{A1}$$

substituting (A1) into (4), and multiplying $[I \quad 0; G_{i12}^{\mathrm{T}}\hat{P}_{i|i} \quad I]^{-1}$ from the left sides of (4), we can get that

$$\left(\begin{bmatrix} \hat{P}_{i|i}^{-1} & 0 \\ 0 & \hat{Q}_i^{-1} \end{bmatrix} + \hat{H}_i^{\mathrm{T}}\Psi_i\hat{H}_i\right)\begin{pmatrix} \tilde{x}_{i|i+1} - \hat{x}_{i|i} \\ w_{i|i+1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} & I \end{pmatrix}\left(H_{i2}R_{i+1}^{-1}y_{i+1} - H_{i3}\hat{x}_{i|i}\right).$$

Defining variable $\tilde{x}_{i+1|i+1} = A_i(0)\tilde{x}_{i|i+1} + \hat{B}_i(0)\hat{w}_{i|i+1}$, we can get the following two expressions, for which the direct computation of matrix inverse is avoided.

$$\begin{aligned} \tilde{x}_{i|i+1} &= \hat{x}_{i|i} + \hat{P}_{i|i}\begin{bmatrix} I & 0 \end{bmatrix}\left(H_{i2}R_{i+1}^{-1}y_{i+1} - H_{i3}\hat{x}_{i|i}\right) - \hat{P}_{i|i}A_i^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)\left(\tilde{x}_{i+1|i+1} - A_i(0)\hat{x}_{i|i}\right), \\ w_{i|i+1} &= \hat{Q}_i\begin{bmatrix} -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} & I \end{bmatrix}\left(H_{i2}R_{i+1}^{-1}y_{i+1} - H_{i3}\hat{x}_{i|i}\right) - \hat{Q}_i\hat{B}_i^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)\left(\tilde{x}_{i+1|i+1} - A_i(0)\hat{x}_{i|i}\right). \end{aligned} \tag{A2}$$

Therefore,

$$\begin{aligned} \tilde{x}_{i+1|i+1} &= A_i(0)\tilde{x}_{i|i+1} + \hat{B}_i(0)w_{i|i+1} \\ &= \begin{pmatrix} A_i(0)\hat{P}_{i|i}\begin{bmatrix} I & 0 \end{bmatrix}H_{i2}R_{i+1}^{-1}y_{i+1} - A_i(0)\hat{P}_{i|i}A_i^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)\tilde{x}_{i+1|i+1} \\ +A_i(0)\left(I - \hat{P}_{i|i}\begin{bmatrix} I & 0 \end{bmatrix}H_{i3} + \hat{P}_{i|i}A_i^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)A_i(0)\right)\hat{x}_{i|i} \end{pmatrix} \\ &\quad + \begin{pmatrix} \hat{B}_i(0)\hat{Q}_i\begin{bmatrix} -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} & I \end{bmatrix}H_{i2}R_{i+1}^{-1}y_{i+1} - \hat{B}_i(0)\hat{Q}_i\hat{B}_i^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)\tilde{x}_{i+1|i+1} \\ +\hat{B}_i(0)\left(\hat{Q}_i\hat{B}_i^{\mathrm{T}}(0)C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)A_i(0) - \hat{Q}_i\begin{bmatrix} -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} & I \end{bmatrix}H_{i3}\right)\hat{x}_{i|i} \end{pmatrix} \\ &= A_i(0)\hat{P}_{i|i}\begin{bmatrix} I & 0 \end{bmatrix}H_{i2}R_{i+1}^{-1}y_{i+1} + \hat{B}_i(0)\hat{Q}_i\begin{bmatrix} -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} & I \end{bmatrix}H_{i2}R_{i+1}^{-1}y_{i+1} \\ &\quad -P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)\tilde{x}_{i+1|i+1} \\ &\quad + \begin{pmatrix} A_i(0) - A_i(0)\hat{P}_{i|i}\begin{bmatrix} I & 0 \end{bmatrix}H_{i3} - \hat{B}_i(0)\hat{Q}_i\begin{bmatrix} -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} & I \end{bmatrix}H_{i3} \\ +P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)A_i(0) \end{pmatrix}\hat{x}_{i|i} \\ &= \left(A_i(0)\hat{P}_{i|i}\begin{bmatrix} I & 0 \end{bmatrix} + \hat{B}_i(0)\hat{Q}_i\begin{bmatrix} -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} & I \end{bmatrix}\right)H_{i2}R_{i+1}^{-1}y_{i+1} \\ &\quad -P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)\tilde{x}_{i+1|i+1} + \hat{A}_i(0)\hat{x}_{i|i}. \end{aligned}$$

Then,

$$\left(I + P_{i+1|i}C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)\right)\tilde{x}_{i+1|i+1} = \left(A_i(0)\hat{P}_{i|i}\begin{bmatrix} I & 0 \end{bmatrix} + \hat{B}_i(0)\hat{Q}_i\begin{bmatrix} -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} & I \end{bmatrix}\right)H_{i2}R_{i+1}^{-1}y_{i+1} + \hat{A}_i(0)\hat{x}_{i|i}.$$

Moreover we can get $[I+P_{i+1|i}C_{i+1}^{\mathrm{T}}R_{i+1}^{-1}C_{i+1}]^{-1}P_{i+1|i} = P_{i+1|i+1}$ based on the matrix inversion lemma, then $\tilde{x}_{i+1|i+1} = \hat{A}_i(0)\hat{x}_{i|i} + P_{i+1|i+1}(P_{i+1|i}^{-1}(A_i(0)\hat{P}_{i|i}[\, I \ 0\,] + \hat{B}_i(0)\hat{Q}_i[\, -G_{i12}^{\mathrm{T}}\hat{P}_{i|i} \ I\,])H_{i2}R_{i+1}^{-1}y_{i+1} - C_{i+1}^{\mathrm{T}}(0)R_{i+1}^{-1}C_{i+1}(0)\hat{A}_i(0)\hat{x}_{i|i})$. Note that Eq. (A2) has similar forms with those of [2,3,19], which implies that we can reasonably designate $\hat{x}_{i+1|i+1}$ as $\tilde{x}_{i+1|i+1}$. This completes the derivations.

Appendix B Proof of Theorem 1

To prove the conclusions of Theorems 1 and 2, the following results are utilized repeatedly.

Lemma 1. For arbitrary matrices A, B, C, D with compatible dimensions, assume that the inverses of all the involved matrices do exist. Then we have

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix}\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}, \\ A(I + BA)^{-1} &= (I + AB)^{-1}A, \\ (A + BCD)^{-1} &= A^{-1} - A^{-1}B\left(DA^{-1}B + C^{-1}\right)^{-1}DA^{-1}. \end{aligned} \tag{B1}$$

From (6) and Lemma A1, the following equalities can be straightforwardly achieved.

$$\begin{aligned} P_{i|i} &= \left(P_{i|i-1}^{-1} + C^{\mathrm{T}}R^{-1}C\right)^{-1} = P_{i|i-1} - P_{i|i-1}C^{\mathrm{T}}\left(R + CP_{i|i-1}C^{\mathrm{T}}\right)^{-1}CP_{i|i-1}, \\ \hat{P}_{i|i} &= \left(P_{i|i}^{-1} + G_{11}\right)^{-1} = P_{i|i} - P_{i|i}G_{11}^{\frac{1}{2}}\left(I + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}, \\ \hat{Q}_i &= \left(Q^{-1} + G_{22} - G_{12}^{\mathrm{T}}\left(P_{i|i}^{-1} + G_{11}\right)^{-1}G_{12}\right)^{-1} \\ &= \left(Q^{-1} + G_{22} - G_{12}^{\mathrm{T}}G_{11}^{-1}G_{12} + G_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}\right)^{-1}. \end{aligned} \tag{B2}$$

Lemma 2. The following equalities are established.

$$B\hat{Q}_iG_{12}^{\mathrm{T}}\hat{P}_{i|i}A^{\mathrm{T}} = BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I + G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}, \tag{B3}$$

$$
\begin{aligned}
&AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&\quad= AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&\quad\quad - AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}\hat{Q}_iG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}. \qquad \text{(B4)}
\end{aligned}
$$

Proof of Lemma A2. Combining Lemma A1 and (B2), we get that

$$
\begin{aligned}
B\hat{Q}_iG_{12}^{\mathrm{T}}\hat{P}_{i|i}A^{\mathrm{T}} &= BU\left(I+G_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}U\right)^{-1}G_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&= BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)\left(I+\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\right)\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&= BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}.
\end{aligned}
$$

Then the proof of (B3) is accomplished.

Next note that,

$$
\begin{aligned}
&AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&= AP_{i|i}G_{11}^{\frac{1}{2}}\left(\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}-\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}\right.\\
&\quad\times\left.\left(U^{-1}+G_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}\right)G_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}\right)G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&= AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&\quad - AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}\hat{Q}_iG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}.
\end{aligned}
$$

We get (B4). This completes the proof of the lemma.

Proof of Theorem 1. Now we give the proof of Theorem 1. According to the computation formula of $P_{i+1|i}$ and (B2), we have that,

$$
\begin{aligned}
P_{i+1|i} &= A\hat{P}_{i|i}A^{\mathrm{T}}+\hat{B}_i\hat{Q}_i\hat{B}_i^{\mathrm{T}}\\
&= A\left(P_{i|i}-P_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}\right)A^{\mathrm{T}}+\left(B-A\hat{P}_{i|i}G_{12}\right)\hat{Q}_i\left(B-A\hat{P}_{i|i}G_{12}\right)^{\mathrm{T}}\\
&= AP_{i|i}A^{\mathrm{T}}+B\hat{Q}_iB^{\mathrm{T}}-B\hat{Q}_iG_{12}^{\mathrm{T}}\hat{P}_{i|i}A^{\mathrm{T}}-A\hat{P}_{i|i}G_{12}\hat{Q}_iB^{\mathrm{T}}-AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&\quad +AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}\hat{Q}_iG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}.
\end{aligned}
$$

Substitute (B3) and (B4) into $P_{i+1|i}$ and combine with (B2), and we have,

$$
\begin{aligned}
P_{i+1|i} &= AP_{i|i}A^{\mathrm{T}}-AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&\quad -BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&\quad -AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}UB^{\mathrm{T}}+B\hat{Q}_iB^{\mathrm{T}}\\
&= AP_{i|i-1}A^{\mathrm{T}}-AP_{i|i-1}C^{\mathrm{T}}R^{-1/2}\left(I+R^{-1/2}CP_{i|i-1}C^{\mathrm{T}}R^{-1/2}\right)^{-1}R^{-1/2}CP_{i|i-1}A^{\mathrm{T}}\\
&\quad -AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&\quad -BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}}\\
&\quad -AP_{i|i}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}UB^{\mathrm{T}}\\
&\quad +BUB^{\mathrm{T}}-BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{-\frac{1}{2}}G_{12}UB^{\mathrm{T}}
\end{aligned}
$$

$$
\begin{aligned}
&= AP_{i|i-1}A^{\mathrm{T}} + BUB^{\mathrm{T}} - AP_{i|i-1}C^{\mathrm{T}}R^{-1/2}\left(I + R^{-1/2}CP_{i|i-1}C^{\mathrm{T}}R^{-1/2}\right)^{-1}R^{-1/2}CP_{i|i-1}A^{\mathrm{T}} \\
&\quad - \left(AP_{i|i}G_{11}^{\frac{1}{2}} + BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\right)\left(I + G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}\left(G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}} + G_{11}^{-\frac{1}{2}}G_{12}UB^{\mathrm{T}}\right) \\
&= AP_{i|i-1}A^{\mathrm{T}} + BUB^{\mathrm{T}} - \left[\, AP_{i|i-1}C^{\mathrm{T}}R^{-1/2} \;\; AP_{i|i}G_{11}^{\frac{1}{2}} + BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} \,\right] \\
&\quad \times \begin{bmatrix} \left(I + R^{-1/2}CP_{i|i-1}C^{\mathrm{T}}R^{-1/2}\right)^{-1} & \\ & \left(I + G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1} \end{bmatrix} \\
&\quad \times \begin{bmatrix} R^{-1/2}CP_{i|i-1}A^{\mathrm{T}} \\ G_{11}^{\frac{1}{2}}P_{i|i}A^{\mathrm{T}} + G_{11}^{-\frac{1}{2}}G_{12}U^{\mathrm{T}}B^{\mathrm{T}} \end{bmatrix}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\left[\, AP_{i|i-1}C^{\mathrm{T}}R^{-1/2} \;\; AP_{i|i}G_{11}^{\frac{1}{2}} + BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} \,\right] \\
&\quad = \left(AP_{i|i-1}F^{\mathrm{T}} + DJ\right)\begin{bmatrix} I & -\left(I + R^{-1/2}CP_{i|i-1}C^{\mathrm{T}}R^{-1/2}\right)^{-1}R^{-1/2}CP_{i|i-1}G_{11}^{\frac{1}{2}} \\ 0 & I \end{bmatrix} \\
&\quad = \left(AP_{i|i-1}F^{\mathrm{T}} + DJ\right)\begin{bmatrix} I & \left(I + R^{-1/2}CP_{i|i-1}C^{\mathrm{T}}R^{-1/2}\right)^{-1}R^{-1/2}CP_{i|i-1}G_{11}^{\frac{1}{2}} \\ 0 & I \end{bmatrix}^{-1},
\end{aligned}
$$

therefore,

$$
\begin{aligned}
P_{i+1|i} &= AP_{i|i-1}A^{\mathrm{T}} + BUB^{\mathrm{T}} - \left(AP_{i|i-1}F^{\mathrm{T}} + DJ\right) \\
&\quad \times \begin{bmatrix} I + R^{-1/2}CP_{i|i-1}C^{\mathrm{T}}R^{-1/2} & R^{-1/2}CP_{i|i-1}G_{11}^{\frac{1}{2}} \\ G_{11}^{\frac{1}{2}}P_{i|i-1}C^{\mathrm{T}}R^{-1/2} & I + G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}} \end{bmatrix}^{-1}\left(AP_{i|i-1}F^{\mathrm{T}} + DJ\right)^{\mathrm{T}} \\
&= AP_{i|i-1}A^{\mathrm{T}} + DD^{\mathrm{T}} - \left(AP_{i|i-1}F^{\mathrm{T}} + DJ\right)\left(W + FP_{i|i-1}F^{\mathrm{T}}\right)^{-1}\left(AP_{i|i-1}F^{\mathrm{T}} + DJ\right)^{\mathrm{T}}. \qquad \text{(B5)}
\end{aligned}
$$

Notice that the last term of (B5) is a standard discrete Riccati recursion, and we can complete the proof of Theorem 1 by the same arguments as those asymptotic properties analysis of the Kalman filter [1].

This completes the proof.

Appendix C Proof of Theorem 2

In order to prove Theorem 2, the following results are needed, whose proofs are similar to the ones of Lemma A2 and are ignored here.

Lemma 3. The following equalities are established.

$$
\begin{aligned}
&AP_{i|i}G_{11}^{\frac{1}{2}}\left(I + G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}} \\
&\quad = A\left(P_{i|i}^{-1} + G_{11}\right)^{-1}G_{11} - A\left(P_{i|i}^{-1} + G_{11}\right)^{-1}G_{12}\hat{Q}_iG_{12}^{\mathrm{T}}\left(I + P_{i|i}G_{11}\right)^{-1}, \qquad \text{(C1)} \\
&BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I + G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}} = B\hat{Q}_iG_{12}^{\mathrm{T}}\left(I + P_{i|i}G_{11}\right)^{-1}.
\end{aligned}
$$

Proof of Theorem 2. From the definition of A_{pi} and the relations of Lemma A2, the following expressions are obtained.

$$
\begin{aligned}
A_{pi} &= A - \left(AP_{i|i-1}F^{\mathrm{T}} + DJ\right)\left(W + FP_{i|i-1}F^{\mathrm{T}}\right)^{-1}F \\
&= A - \left(AP_{i|i-1}\left[\, C^{\mathrm{T}}R^{-\frac{1}{2}} \;\; G_{11}^{\frac{1}{2}} \,\right] + \left[\, 0 \;\; BQG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} \,\right]\right) \\
&\quad \times \left(\begin{bmatrix} I & 0 \\ 0 & I + G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} R^{-\frac{1}{2}}C \\ G_{11}^{\frac{1}{2}} \end{bmatrix}P_{i|i-1}\left[\, C^{\mathrm{T}}R^{-\frac{1}{2}} \;\; G_{11}^{\frac{1}{2}} \,\right]\right)^{-1}\begin{bmatrix} R^{-\frac{1}{2}}C \\ G_{11}^{\frac{1}{2}} \end{bmatrix} \\
&= A - AP_{i|i-1}C^{\mathrm{T}}R^{-\frac{1}{2}}\left(I + R^{-\frac{1}{2}}CP_{i|i-1}C^{\mathrm{T}}R^{-\frac{1}{2}}\right)^{-1}R^{-\frac{1}{2}}C \\
&\quad - \left(AP_{i|i-1}G_{11}^{\frac{1}{2}} + BQG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\right)\left(I + G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}\left(I + P_{i|i-1}C^{\mathrm{T}}R^{-1}C\right)^{-1} \\
&= A\left(I + P_{i|i-1}C^{\mathrm{T}}R^{-1}C\right)^{-1} \\
&\quad - \left(AP_{i|i-1}G_{11}^{\frac{1}{2}} + BQG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\right)\left(I + G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}} + G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}\left(I + P_{i|i-1}C^{\mathrm{T}}R^{-1}C\right)^{-1}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&A-\left(AP_{i|i-1}G_{11}^{\frac{1}{2}}+BQG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\right)\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}\\
&=A-AP_{i|i-1}G_{11}^{\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}\\
&\quad-BUG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}\left(I+G_{11}^{-\frac{1}{2}}G_{12}UG_{12}^{\mathrm{T}}G_{11}^{-\frac{1}{2}}+G_{11}^{\frac{1}{2}}P_{i|i}G_{11}^{\frac{1}{2}}\right)^{-1}G_{11}^{\frac{1}{2}}\\
&=A-A\hat{P}_{i|i}G_{11}+A\hat{P}_{i|i}G_{12}\hat{Q}_iG_{12}^{\mathrm{T}}\left(I-\hat{P}_{i|i}G_{11}\right)-B\hat{Q}_iG_{12}^{\mathrm{T}}\left(I-\hat{P}_{i|i}G_{11}\right)\\
&=A\left(I-\hat{P}_{i|i}G_{11}\right)+\left(A\hat{P}_{i|i}G_{12}-B\right)\hat{Q}_iG_{12}^{\mathrm{T}}\left(I-\hat{P}_{i|i}G_{11}\right)\\
&=\left(A-\hat{B}_i\hat{Q}_iG_{12}^{\mathrm{T}}\right)\left(I-\hat{P}_{i|i}G_{11}\right)\\
&=\hat{A}_i(0).
\end{aligned}
$$

It is easy to get that $A_{pi}=\hat{A}_i(0)(I+P_{i|i-1}C^{\mathrm{T}}R^{-1}C)^{-1}$. From the definition, we know that $A_{fi}=(I+P_{i+1|i}C^{\mathrm{T}}R^{-1}\times C)^{-1}\hat{A}_i(0)$, therefore the following relation is established: $A_{fi}=(I+P_{i+1|i}C^{\mathrm{T}}R^{-1}C)^{-1}A_{pi}(I+P_{i|i-1}C^{\mathrm{T}}R^{-1}C)$.

From Theorem 1 we know that $P_{i|i-1}$ converges to a constant matrix under some conditions. As nominal system matrices are assumed to be time invariant, this convergence means that $\lim_{i\to\infty}(P_{i+1|i}C^{\mathrm{T}}R^{-1}C-P_{i|i-1}C^{\mathrm{T}}R^{-1}C)=0$. Therefore, it can be declared from the above equation that with the increment of the time index i, the set of eigenvalues of A_{fi} converges to that of A_{pi}, while the latter converges to a stable and constant matrix. This completes the proof.