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Global stabilization control of stochastic quantum systems

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Abstract The global stabilization control of arbitrary eigenstates for finite dimensional stochastic quantum systems with non-diagonal free Hamiltonian and non-regular measurement operator is studied in this paper. We propose a switching feedback control law, in which a constant control is used to steer the system state to a convergence domain, and another control law designed based on Lyapunov stability theorem, is used to attract the states in the convergence domain to the desired target state. The convergence to an arbitrary target eigenstate from any initial state is strictly proved. Moreover, numerical simulation experiments on a three-dimensional stochastic quantum system are implemented to demonstrate the effectiveness of the proposed control.

Keywords eigenstate, stochastic quantum systems, global stabilization, switching control, Lyapunov stability theorem

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1 Introduction

Although quantum mechanics has been established for more than 100 years, relative researches are still active in microscopic fields [1], and the state transfer control of quantum systems is one of the important aspects. In order to steer the system state to a desired target state, lots of control strategies have been developed [2–7], in which feedback control has higher accuracy and stronger robustness but involves the measurement of the system output [7]. The measurements in quantum systems, either projective measurement [8] or continuous measurement [9–12] will lead to a non-classical property. The system state will change inevitably in a probabilistic way, which is called quantum state reduction [13]. In the 1980s, Belavkinet et al. proposed the feedback control based on the measurement for quantum systems by combining continuous measurement and feedback control [10, 14, 15], that is, the quantum system is measured continuously and the system information can be obtained by the quantum filter, and one can use the state information in feedback control to change the system dynamics [13]. According to the quantum filter theory [10, 16–18], the feedback control based on measurement can be regarded as the state feedback control with the quantum filter [19]. The dynamical model of the quantum filter with

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continuous measurement follows a classical stochastic differential equation: stochastic master equation (SME) [10,11,19]. In order to obtain a deterministic manner by means of the control of stochastic quantum systems, van Handel et al. explored the use of stochastic Lyapunov techniques for the design of feedback controllers. They indicated that the global stabilization of the system state was difficult even for two-dimensional systems, and the design method of the proposed control was hard to be adopted widely due to the increase of computational complexity with higher dimensions [11]. For finite dimensional angular moment systems, Mirrahimi et al. proposed a switching control to achieve the global stabilization of eigenstates. The proof was done by the strict analysis on the sample paths of the quantum state [20]. Hereafter, Tsumura et al. proposed a continuous control to achieve the same task [1]. Recently, Ge et al. expanded finite dimensional angular moment systems to more general finite dimensional stochastic quantum systems. They designed a non-smooth state feedback control based on a non-smooth Lyapunov-like technique to solve the global stabilization problem of eigenstates. Moreover, the effectiveness of the feedback control strategies proposed based on SME was verified by the actual experiments [21–23]. The great challenge in the global stabilization of quantum SME states is: the system contains uncountable equilibriums, which include the eigenstates. The system state may converge to any one of the uncountable system equilibriums without control laws. Simultaneously, SME is a nonlinear stochastic differential equation, and the methods of designing control which are only suitable for linear systems are invalid at all. The adopted method of designing control needs to be suitable for nonlinear systems. The measurement operators of the stochastic quantum systems considered in the existing researches [3, 13, 20, 24–26] are all regular, which requires the eigenstates be all the system equilibriums in the setting of diagonal free Hamiltonian. We will loosen this restriction of the measurement operator to solve the problem of the stochastic systems with non-regular measurement operator and non-diagonal free Hamiltonian.

In this paper, we design a switching feedback control to make the quantum SME states converge to arbitrary target eigenstates. For deterministic systems, the Lyapunov stability theorem and LaSalle's invariance theorem are important tools in the analysis and design of control laws [20]. Similarly, these two theorems' stochastic counterparts also play essential roles in our work. To solve the problem of uncountable equilibrium points, the Lyapunov function proposed in this paper is used to distinguish the desired target state from other equilibrium points. Then, we design a switching feedback control which is composed of two parts: one is a constant control used to steer the state to a convergence domain from the outside of the convergence domain; the other one is the control designed based on Lyapunov stability theorem, which is used to steer the state in the convergence domain to the target state. The proposed control laws steer the system state to the desired target state rather than other equilibrium points from an arbitrary initial state in the state space. The convergence is analyzed and proved in detail.

The rest of this paper is organized as follows. The mathematical models of the controlled system and the control problem are described in Section 2. In Section 3, the switching control strategy which makes the target state globally stable is designed, and the main results of this paper are also given in this section. In Section 4, the proof of the main results is given. Furthermore, the numerical simulation experiments are performed on a three-dimensional system in Section 5. Finally, Section 6 concludes this paper.

2 Description of the system model and control problem

The state of a finite dimensional quantum filter can be described by the density matrix, and the state ρ_t follows SME [11, 20],

$$\begin{aligned} \mathrm{d}\rho_t &= \left(-\mathrm{i}\left[H_0,\rho_t\right] + L\rho_t L^\dagger - \tfrac{1}{2}L^\dagger L\rho_t - \tfrac{1}{2}\rho_t L^\dagger L\right)\mathrm{d}t - \mathrm{i}\left[H_1,\rho_t\right]u_t\mathrm{d}t \\ &\quad + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \mathrm{tr}\left(L\rho_t + \rho_t L^\dagger\right)\rho_t\right)\mathrm{d}\omega_t \\ &= \alpha\left(\rho_t\right)\mathrm{d}t + \beta\left(\rho_t\right)\mathrm{d}t + \delta\left(\rho_t\right)\mathrm{d}\omega_t, \\ \mathrm{d}y_t &= \mathrm{d}\omega_t + \sqrt{\eta}\,\mathrm{tr}\left(L\rho_t + \rho_t L^\dagger\right)\mathrm{d}t, \end{aligned} \tag{1}$$

where, $\mathrm{i} = \sqrt{-1}$ is the imaginary; ρ_t is the density matrix of the system obtained by the filter, and $\rho_t \in \mathcal{S} = \left\{\rho_t \in \mathbb{C}^{n\times n} : \rho_t = {\rho_t}^\dagger \geqslant 0, \mathrm{tr}\,(\rho_t) = 1, \mathrm{tr}\left({\rho_t}^2\right) \leqslant 1\right\}$, in which $\mathbb{C}$ represents the set of complex numbers, ${\rho_t}^\dagger$ is the conjugate transpose of ρ_t, and $\mathcal{S}$ represents the state space; ω_t is the 1-dimensional Wiener process defined on the classical complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, in which Ω is a sample space, $\mathcal{F}$ is a σ-algebra and a set of subsets of Ω, $\mathcal{P}$ is a probability measure; $\mathrm{d}\omega_t$ is the Wiener increment with $\mathrm{E}\,(\mathrm{d}\omega_t) = 0$ and $\mathrm{E}((\mathrm{d}\omega_t)^2) = \mathrm{d}t$, where $\mathrm{E}\,(\cdot)$ represents the expected value of a random variable; y_t is the measurement record of the output; H_0 and H_1 are $n \times n$ Hermitian matrices, called the free Hamiltonian and control Hamiltonian, respectively; $L \in \mathbb{C}^{n\times n}$ is the Hermitian measurement operator which determines the way of the system interacting with the measurement apparatus; $\eta \in (0, 1]$ is the measurement efficiency; $u_t \in \mathbb{R}$ is the control input, where $\mathbb{R}$ represents the set of real numbers; $[H_0, \rho_t] = H_0\rho_t - \rho_t H_0$ represents the commutator of H_0 and ρ_t.

The control problem can be described as: to design control laws to globally asymptotically render the stochastic quantum system (1) from any initial state to the desired target state almost surely, i.e.,

$$\mathbb{P}\left(\lim_{t\to\infty} \rho_t = \rho_f\right) = 1, \quad \forall \rho_0 \in \mathcal{S}, \tag{2}$$

where, $\mathbb{P}\,(\cdot)$ represents the probability of events.

In order to interpret the control problem more clearly, we introduce the definition of "globally stable" first.

Definition 1. Let ρ_e be an equilibrium of (1), i.e., $\mathrm{d}\rho_t|_{\rho_t=\rho_e} = 0$, so if the equilibrium ρ_e satisfies $\lim_{\rho_0\to\rho_e}\mathbb{P}\,(\sup_{0<t<\infty} |\rho_t - \rho_e| \geqslant \epsilon) = 0, \forall \epsilon > 0$, then ρ_e is said to be stable in probability; if the equilibrium ρ_e is stable in probability and satisfies $\mathbb{P}\,(\lim_{t\to\infty}\rho_t = \rho_e) = 1$, $\forall \rho_0 \in \mathcal{S}_l$, $\mathcal{S}_l \subset \mathcal{S}$, then ρ_e is said to be locally stable. Moreover, ρ_e is said to be globally stable when $\mathcal{S}_l = \mathcal{S}$ [27].

According to Definition 1, the control problem can be denoted as global stabilization via state feedback control. In this paper, the measurement operator $L = \mathrm{diag}\,(l_1, l_2, \ldots, l_n)$ is non-regular, i.e., at least two diagonal elements of L are the same. Further, we set that $l_1 = l_2 \neq l_k$ for $k = 3, \ldots, n$ as well as $l_1 + l_2 \neq l_i + l_j$ for $i, j \neq 1, 2$. The free Hamiltonian $H_0 = \left[\begin{smallmatrix} H_{01} & 0 \\ 0 & D \end{smallmatrix}\right]$, where $H_{01} = \left[\begin{smallmatrix} h_{0-11} & h_{0-12} \\ h_{0-21} & h_{0-22} \end{smallmatrix}\right]$ with $h_{0-12} \neq 0$ and D represents a diagonal matrix. The control Hamiltonian $H_1 = \left[\begin{smallmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{smallmatrix}\right] = \{h_{ij}\}_{n\times n}$ is connected, i.e., $h_{i(i+1)} \neq 0$ for $i = 1, \ldots, n-1$. Moreover, $h_{i1} = h_{i2}$ or $h_{i1} = -h_{i2}$, $\forall i = 3, \ldots, n$ cannot hold simultaneously, and $h_{1i} \neq 0$ and $h_{2i} \neq 0$ for $i = 3, \ldots, n$. Then, we investigate the global stabilization of an arbitrary eigenstate including $\rho_f = \rho_i = \mathrm{diag}\{\underbrace{0\cdots0}_{i-1}10\cdots0\}$ for $i = 3, \ldots, n$ and $\rho_f = \left[\begin{smallmatrix} \rho_{f1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$, in which $\rho_{f1} = \left[\begin{smallmatrix} a & b \\ b^* & 1-a \end{smallmatrix}\right]$ with $b \neq 0$ and $a\,(1-a) = bb^*$, where b^* is the conjugate complex number of b. It is obvious that ρ_f satisfies $[H_0, \rho_f] = 0$, $\mathrm{tr}\,(\rho_f) = 1$ and $\mathrm{tr}\left({\rho_f}^2\right) = 1$. It is easy to prove that ρ_f is a controllable equilibrium. Because a can be selected as any real value, ρ_f is an arbitrary eigenstate.

3 Control design and main results

In this section, we design control laws to steer the system state to the given target eigenstate ρ_f from an arbitrary initial state, and derive the main results of this paper.

3.1 Control design

The idea of control design in this paper is from the Lyapunov stability theorem and LaSalle's invariance theorem for SME. The Lyapunov stability theorem is used to determine whether the target state is stable in probability, while the LaSalle's invariance theorem is used to determine whether ρ_t converges to the target state ρ_f. Namely, we need to construct a suitable Lyapunov function V, and design the control law $u\,(\rho_t)$ to ensure $\mathcal{L}V \leqslant 0$ and $V\,(\rho_f) = 0$, where $\mathcal{L}$ represents the infinitesimal generator [8, 28, 29], and thereby the target state ρ_f is stable in probability. Then, we analyze the largest invariant set and determine the convergence domain $\mathcal{Q}_{\mathrm{cd}}$, so that the target state is locally stable in $\mathcal{Q}_{\mathrm{cd}}$ under $u\,(\rho_t)$. In order to make the target state globally stable, we need to design another control: a constant control, to ensure the system state enters $\mathcal{Q}_{\rm cd}$ in a finite time when the system state is not in $\mathcal{Q}_{\rm cd}$. Then, the control $u(\rho_t)$ and the constant control compose the control u_t, under which the target state is globally stable.

Now, we focus on the target eigenstates with the form $\rho_f = \left[\begin{smallmatrix} \rho_{f1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. We design the control law $u(\rho_t)$ with the Lyapunov method. Here, the Lyapunov function is constructed as

$$V(\rho_t) = V_1(\rho_t) + c\mathrm{tr}^2(P\rho_t), \tag{3}$$

where, $V_1(\rho_t) = 1 - \mathrm{tr}(\rho_t\rho_f)$; $P = \left[\begin{smallmatrix} P_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a Hermitian matrix and satisfies $\mathrm{tr}(P\rho_f) = 0$, in which $P_1 = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{smallmatrix}\right]$ with $p_{11} \neq 0$. According to $\mathrm{tr}(P\rho_f) = 0$, $\Re(p_{12}b^*) = \frac{1}{2}((a-1)p_{11}x - ap_{11})$ with $x = p_{22}/p_{11} \neq \pm 1$, in which $\Re(\cdot)$ represents the real part of a complex number.

It should be noticed that the matrix P in V plays an important role in solving the problem of uncountable system equilibrium points. We can reduce and know determinately the states in the invariant set by designing P, which is one of our tasks in this paper. The control u_t is also realized by designing P. The parameter $c \in \mathbb{R}$ and $c \geqslant 0$, which is used to make sure that only the target state is contained in the convergence domain $\mathcal{Q}_{\rm cd}$ for the states in invariant set.

We perform a straightforward computation and obtain the infinitesimal operator of ρ_t acting on $V(\rho_t)$,

$$\mathcal{L}V(\rho_t) = G(\rho_t) + u_t g(\rho_t), \tag{4}$$

where, $G(\rho_t) = 2c\mathrm{tr}(P\rho_t)\mathrm{tr}(P\alpha) + c\mathrm{tr}^2(P\delta)$, $g(\rho_t) = \mathrm{tr}(-\mathrm{i}H_1[\rho_t, 2c\mathrm{tr}(P\rho_t)P - \rho_f])$.

Design the control law as

$$u_t = u(\rho_t) = -\frac{G(\rho_t)}{g(\rho_t)} - k_v g(\rho_t), \tag{5}$$

in which $k_v \in \mathbb{R}$ and $k_v > 0$, and

$$\mathcal{L}V(\rho_t) = -k_v g^2(\rho_t) \leqslant 0. \tag{6}$$

According to LaSalle's invariance theorem, for any initial state, the system state ρ_t converges to the invariant set $\mathcal{R}$ as

$$\mathcal{R} = \{\rho_{\mathcal{R}} : \mathcal{L}V(\rho_{\mathcal{R}}) = 0\} = \{\rho_{\mathcal{R}} : g(\rho_{\mathcal{R}}) = 0\}, \tag{7}$$

in probability under $u(\rho_t)$. Notice that the states $\rho_{\mathcal{R}}$ in $\mathcal{R}$ are included in the equilibrium points, so $\rho_{\mathcal{R}}$ has the form $\rho_{\mathcal{R}} = \left[\begin{smallmatrix} \rho_{\mathcal{R}1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $\rho_{\mathcal{R}1} = \left[\begin{smallmatrix} \rho_{\mathcal{R}11} & \rho_{\mathcal{R}12} \\ \rho_{\mathcal{R}21} & \rho_{\mathcal{R}22} \end{smallmatrix}\right]$, or $\rho_{\mathcal{R}}$ is an eigenstate.

For the target state ρ_f, it is obvious that $g(\rho_f) = 0$, so $\rho_f \in \mathcal{R}$. Based on Lyapunov stability theorem and LaSalle's invariance theorem for SME, if $\mathcal{R}$ only contains ρ_f, $u(\rho_t)$ will make ρ_t converge to ρ_f from any initial state, and ρ_f is globally stable. However, $\mathcal{R}$ may contain other states besides ρ_f, and the number of the states in invariant set $\mathcal{R}$ is even uncountable. In this case, ρ_t may converge to any one state in $\mathcal{R}$ rather than ρ_f under $u(\rho_t)$. Namely, $u(\rho_t)$ can only make the target state locally stable rather than globally stable. Thus, we need to analyze the states in $\mathcal{R}$, and change $u(\rho_t)$ to make the target state globally stable.

3.2 Analysis of invariant set and main results

We analyze the state $\rho_{\mathcal{R}}$ in $\mathcal{R}$ first. Because $\rho_{\mathcal{R}}$ is included in equilibrium points, $\rho_{\mathcal{R}}$ satisfies the condition $[H_0, \rho_{\mathcal{R}}] = 0$. Based on (7), $g(\rho_{\mathcal{R}}) = 0$. According to $[H_0, \rho_{\mathcal{R}}] = 0$, one can get a general form of $\rho_{\mathcal{R}}$ as $\rho_{\mathcal{R}} = \left[\begin{smallmatrix} \rho_{\mathcal{R}1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where $\rho_{\mathcal{R}1} = \left[\begin{smallmatrix} \rho_{\mathcal{R}11} & \rho_{\mathcal{R}12} \\ \rho_{\mathcal{R}21} & 1-\rho_{\mathcal{R}11} \end{smallmatrix}\right]$ and $\rho_{\mathcal{R}12} = \frac{2\rho_{\mathcal{R}11}-1}{2a-1}b$ $(a \neq 0.5)$ while $\rho_{\mathcal{R}11} = \rho_{\mathcal{R}22} = 0.5$ and $\Im(h_{0-12}\rho_{\mathcal{R}21}) = 0$ $(a = 0.5)$, in which $\Im(\cdot)$ represents the imaginary of a complex number. Since $[H_0, \rho_f] = 0$, we have $h_{0-22} = Xh_{0-11}$ or $h_{0-12} = \frac{1-X}{2a-1}b$ when $a \neq 0.5$, and $h_{0-22} = h_{0-11}$ or $\Im(h_{0-12}b^*) = 0$ when $a = 0.5$. Then, there are three types of states which make $g(\rho_{\mathcal{R}}) = 0$:

(1) $\rho_{\mathcal{R}} \in \mathcal{R}_1 = \{\rho_{\mathcal{R}} : \mathrm{tr}(H_1[\rho_{\mathcal{R}}, \rho_f]) = 0$ and $\mathrm{tr}(P\rho_{\mathcal{R}}) \cdot \mathrm{tr}(H_1[\rho_{\mathcal{R}}, P]) = 0\}$, i.e.,

$$\Im(h_{12}\rho_{\mathcal{R}21}) \cdot (2a-1) + \Im(h_{21}b) \cdot (2\rho_{\mathcal{R}11} - 1) = 0, \tag{8}$$

$$\Im\left(h_{12}\rho_{\mathcal{R}21}\right)\cdot\left(p_{11}-p_{22}\right)+\Im\left(h_{21}p_{12}\right)\cdot\left(2\rho_{\mathcal{R}11}-1\right)=0. \tag{9}$$

Because $\operatorname{tr}\left(P\rho_{\mathcal{R}}\right)\neq 0$ for $\rho_{\mathcal{R}}\neq\rho_f$, $\operatorname{tr}\left(P\rho_{\mathcal{R}}\right)$ in (9) is omitted.

(2) $\rho_{\mathcal{R}}\in\mathcal{R}_2=\{\rho_{\mathcal{R}}:\operatorname{tr}\left(H_1\left[\rho_{\mathcal{R}},\rho_f\right]\right)\neq 0$ and $2c\operatorname{tr}\left(P\rho_{\mathcal{R}}\right)\operatorname{tr}\left(H_1\left[\rho_{\mathcal{R}},P\right]\right)=\operatorname{tr}\left(H_1\left[\rho_{\mathcal{R}},\rho_f\right]\right)\}$.

(3) $\rho_{\mathcal{R}}\in\mathcal{R}_3=\{\rho_i:i=3,\ldots,n$ and $g\left(\rho_i\right)=0\}$.

Then, $\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3\cup\rho_f$.

For different target states, we divide ρ_f into four cases according to $\Im\left(b\right)$ and a, and analyze the states which make $g\left(\rho_{\mathcal{R}}\right)=0$, respectively.

Case 1. $\Im\left(b\right)=0$ and $a\neq 0.5$. In this case, $\rho_{\mathcal{R}21}=\frac{2\rho_{\mathcal{R}11}-1}{2a-1}b\in\mathrm{R}$, so Eq. (8) can be written as

$$\rho_{\mathcal{R}21}\Im\left(h_{12}\right)\cdot\left(2a-1\right)=b\Im\left(h_{12}\right)\cdot\left(2\rho_{\mathcal{R}11}-1\right), \tag{10}$$

and Eq. (9) can be written as

$$\rho_{\mathcal{R}21}\Im\left(h_{12}\right)\cdot\left(p_{11}-p_{22}\right)+\Im\left(h_{21}p_{12}\right)\cdot\left(2\rho_{\mathcal{R}11}-1\right)=0. \tag{11}$$

When $\Im\left(h_{12}\right)=0$ and $\rho_{R11}\neq 0.5$, Eq. (11) becomes $h_{21}\Im\left(p_{12}\right)\cdot\left(2\rho_{\mathcal{R}11}-1\right)=0$. Letting $\Im\left(p_{12}\right)\neq 0$, then Eq. (11) does not hold. When $\Im\left(h_{12}\right)\neq 0$ and $\rho_{\mathcal{R}11}\neq 0.5$, Eq. (11) becomes $\rho_{\mathcal{R}21}=-\frac{2\rho_{\mathcal{R}11}-1}{p_{11}-p_{22}}\cdot\frac{\Im(h_{21}p_{12})}{\Im(h_{12})}$. Let $\frac{\Im(h_{21}p_{12})}{\Im(h_{12})}\neq\frac{p_{22}-p_{11}}{2a-1}b$, then Eq. (11) does not hold, either. Moreover, when $\rho_{\mathcal{R}11}=0.5$ and $\rho_{\mathcal{R}21}=0$, the state is denoted as ρ_{I_2}, i.e., $\rho_{I_2}=\left[\begin{smallmatrix}\rho_{I_21} & 0\\ 0 & 0\end{smallmatrix}\right]$ with $\rho_{I_21}=\left[\begin{smallmatrix}0.5 & 0\\ 0 & 0.5\end{smallmatrix}\right]$. For ρ_{I_2}, Eq. (11) necessarily holds. Thus $\rho_{I_2}\in\mathcal{R}_1$ in this case.

Case 2. $\Im\left(b\right)=0$ and $a\neq 0.5$. In this case, when $\rho_{\mathcal{R}11}\neq 0.5$ and $\Im\left(\rho_{\mathcal{R}12}\right)=\frac{2\rho_{\mathcal{R}11}-1}{2a-1}\Im\left(b\right)\neq 0$, Eq. (8) becomes $\left(2\rho_{\mathcal{R}11}-1\right)\Im\left(h_{12}b^*+h_{21}b\right)=0$. Due to

$$\Im\left(h_{12}b^*+h_{21}b\right)=0,$$

Eq. (8) holds and Eq. (9) becomes $\frac{p_{11}-p_{22}}{2a-1}\Im\left(h_{12}b^*\right)+\Im\left(h_{21}p_{12}\right)=0$. Let $\frac{p_{11}-p_{22}}{2a-1}\neq-\frac{\Im(h_{21}p_{12})}{\Im(h_{12}b^*)}$ by designing P_1, and Eq. (9) does not hold. On the other hand, when $\rho_{\mathcal{R}11}=0.5$ and $\Im\left(\rho_{\mathcal{R}12}\right)=0$, Eqs. (8) and (9) hold due to $\Im\left(h_{12}\rho_{\mathcal{R}21}\right)=0$ and $\left(2\rho_{\mathcal{R}11}-1\right)=0$. Thus, ρ_{I_2} also belongs to $\mathcal{R}_1$ in this case.

Case 3. $\Im\left(b\right)=0$ and $a=0.5$. In this case, $\Im\left(h_{0-12}\right)=0$, $\Im\left(\rho_{\mathcal{R}21}\right)=0$, and $\rho_{\mathcal{R}11}=0.5$. When $\Re\left(\rho_{\mathcal{R}21}\right)\neq 0$, Eq. (8) becomes $\Im\left(h_{21}b\right)\cdot\left(2\rho_{\mathcal{R}11}-1\right)=0$, which means that Eq. (8) holds. Simultaneously, Eq. (9) becomes $\rho_{\mathcal{R}21}\Im\left(h_{12}\right)\cdot\left(p_{11}-p_{22}\right)=0$. Letting $\Im\left(h_{12}\right)\neq 0$ and $p_{11}\neq p_{22}$, Eq. (9) does not hold. On the other hand, when $\Re\left(\rho_{\mathcal{R}21}\right)=0$, $\rho_{\mathcal{R}21}=0$, i.e., $\rho_{\mathcal{R}}=\rho_{I_2}$, it is obvious that Eqs. (8) and (9) hold, and thereby $\rho_{I_2}\in\mathcal{R}_1$ in this case.

Case 4. $\Im\left(b\right)\neq 0$ and $a=0.5$. In this case, $\Im\left(h_{0-12}\right)\neq 0$. When $\Im\left(\rho_{\mathcal{R}21}\right)\neq 0$, Eq. (8) becomes $\Im\left(h_{21}b\right)\cdot\left(2\rho_{\mathcal{R}11}-1\right)=0$, which means (8) holds. Simultaneously, Eq. (9) becomes

$$\Im\left(h_{12}\rho_{\mathcal{R}21}\right)\cdot\left(p_{11}-p_{22}\right)=0. \tag{12}$$

When $\Im\left(h_{12}\rho_{\mathcal{R}21}\right)\neq 0$, i.e., $h_{12}\neq\rho_{\mathcal{R}12}$, and $p_{11}\neq p_{22}$, Eq. (12) does not hold. When $h_{12}=\rho_{\mathcal{R}12}$, Eq. (12) holds, i.e., $\rho_{\mathcal{R}}\in\mathcal{R}_1$ in which $\rho_{\mathcal{R}1}=\left[\begin{smallmatrix}0.5 & h_{12}\\ h_{21} & 0.5\end{smallmatrix}\right]$. This state is denoted as ρ_h. When $\Im\left(\rho_{\mathcal{R}21}\right)=0$ and $\Re\left(\rho_{\mathcal{R}21}\right)=0$, we have $\rho_{\mathcal{R}}=\rho_{I_2}\in\mathcal{R}_1$. Moreover, when $\Im\left(\rho_{\mathcal{R}21}\right)=0$ and $\Re\left(\rho_{\mathcal{R}21}\right)\neq 0$, the state has the general form of $\rho_{\mathcal{R}}$.

According to Cases 1–4, $\rho_{\mathcal{R}}\in\mathcal{R}_1$ only when $\rho_{\mathcal{R}}=\rho_{I_2}$ as long as P_1 in P satisfies certain conditions, otherwise $\rho_{\mathcal{R}}\notin\mathcal{R}_1$. Because Eq. (8) holds, we know that $\rho_R\notin R$. With $\mathcal{R}_3$, the invariant set $\mathcal{R}=\{\rho_f,\rho_{I_2}$ and $\rho_i,i=3,\ldots,n\}$ for Cases 1–3. According to the analyses in Cases 4, $\rho_{\mathcal{R}}\in\mathcal{R}_1$ only when $\rho_{\mathcal{R}}=\rho_{I_2}$ or ρ_h as long as P_1 satisfies certain conditions, otherwise $\rho_{\mathcal{R}}\notin\mathcal{R}_1$. Similarly, because Eq. (8) must hold, there is no state $\rho_{\mathcal{R}}\in\mathcal{R}_2$. Combined with $\mathcal{R}_3$, the invariant set $\mathcal{R}=\{\rho_f,\rho_h,\rho_{I_2}$ and $\rho_i,i=3,\ldots,n\}$ for Case 4. Based on the analyses for $\mathcal{R}$, the process of reducing the states in invariant set contains two steps: (1) reducing the system equilibriums to $\rho_{\mathcal{R}}$ based on $[H_0,\rho_f]=0$ and $[H_0,\rho_{\mathcal{R}}]=0$; (2) reducing the state from $\rho_{\mathcal{R}}$ to ρ_{I_2} or ρ_h by designing P.

Thus, a main result of this paper, which gives a solution of the control problem, is described in the Theorem 1.

Theorem 1. For system (1) with the same settings in Section 2, the system state ρ_t converges to the target eigenstate $\rho_f = \left[\begin{smallmatrix} \rho_{f1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $\rho_{f1} = \left[\begin{smallmatrix} a & b \\ b^* & 1-a \end{smallmatrix}\right]$ from an arbitrary initial state under the switching control

$$u_t = \begin{cases} u_c = 1, & \begin{array}{c}\text{if } \rho_t \in \mathcal{Q}_{\geqslant M+Q/4} \text{ or } \rho_t \in \Phi \text{ when } \rho_t \text{ last} \\ \text{enters } \Phi \text{ through the boundary } \mathcal{Q}_{M+Q/4},\end{array} \\ u(\rho_t), & \begin{array}{c}\text{if } \rho_t \in \mathcal{Q}_{\leqslant M} \text{ or } \rho_t \in \Phi \text{ when } \rho_t \text{ last enters} \\ \Phi \text{ through the boundary } \mathcal{Q}_M,\end{array} \end{cases} \tag{13}$$

in which $Q = \min\{V(\rho_t)\} - V(\rho_{I_n})$ with $\rho_t \in \mathcal{R}$ and $\rho_t \neq \rho_f$; $\rho_{I_n} = \frac{1}{n} I_n$, where I_n is the identity matrix; $M = V(\rho_{I_n}) + \frac{Q}{2} > 0$; $\mathcal{Q}_\lambda = \{\rho_t : V(\rho_t) = \lambda\}$ with $\lambda = M$ and $\lambda = M + Q/4$; $\mathcal{Q}_{>M} = \{\rho_t : V(\rho_t) > M\}$ and $\mathcal{Q}_{<M+Q/4} = \{\rho_t : V(\rho_t) < M + Q/4\}$; $\Phi = \mathcal{Q}_{<M+Q/4} \cap \mathcal{Q}_{>M} = \{\rho_t : M < V(\rho_t) < M + Q/4\}$. $u(\rho_t) = -\frac{G(\rho_t)}{g(\rho_t)} - k_v g(\rho_t)$, where $G(\rho_t) = 2c\mathrm{tr}(P\rho_t) \cdot \mathrm{tr}(P\alpha) + c\mathrm{tr}^2(P\delta)$ and $g(\rho_t) = \mathrm{tr}(-\mathrm{i}H_1[\rho_t, 2c\mathrm{tr}(P\rho_t)P - \rho_f])$, in which $P = \left[\begin{smallmatrix} P_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with $P_1 = \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{smallmatrix}\right]$. The conditions for designing P_1 are: (1) If $\Im(b) = 0$ and $a \neq 0.5$, $\frac{\Im(h_{21}p_{12})}{\Im(h_{12})} \neq \frac{p_{22}-p_{11}}{2a-1} b$; (2) If $\Im(b) \neq 0$ and $a \neq 0.5$, $\frac{p_{11}-p_{22}}{2a-1} \neq -\frac{\Im(h_{21}p_{12})}{\Im(h_{12}b^*)}$, otherwise, $p_{11} \neq p_{22}$. Moreover, the conditions for determining c are: if $\Im(b) \neq 0$ and $a = 0.5$, c is a constant which satisfies $\max\{(\frac{n}{n+2} - \frac{8n\Re(h_{12}b^*)}{4+n^2})N, \frac{2}{n+2}N\} < c < N$ with $N = \frac{n}{\mathrm{tr}^2(P)}$, otherwise, $\frac{2}{n+2}N < c < N$.

In Theorem 1, Q and M are used to divide the state space $\mathcal{S}$ into three parts: $\mathcal{Q}_{\geqslant M+Q/4}$, $\mathcal{Q}_{\leqslant M}$ and Φ. For different target states, $\mathcal{R} = \{\rho_f, \rho_{I_2}$ and $\rho_i, i = 3, \ldots, n\}$ or $\{\rho_f, \rho_h, \rho_{I_2}$ and $\rho_i, i = 3, \ldots, n\}$. The convergence domain $\mathcal{Q}_{\mathrm{cd}}$ is $\mathcal{Q}_{\leqslant M}$ in Theorem 1, and we will give the proof of Theorem 1 in Section 4 in detail.

4 Proof of main results

We denote by $\mathcal{M}_1$ and $\mathcal{M}_u$ two state domains in which the constant control $u_t = u_c$ and the control $u_t = u(\rho_t)$ are applied, respectively. For Cases 1–3, $\mathcal{R} = \{\rho_f, \rho_{I_2}$ and $\rho_i, i = 3, \ldots, n\}$ and

$$\min\{V(\rho_{I_2}), V(\rho_i) \text{ for } i = 3, \ldots, n\} = V(\rho_{I_2}).$$

The diagram of the proof is shown in Figure 1, in which $\mathcal{Q}_{\geqslant M+Q/4} \subset \mathcal{M}_1$ and $\mathcal{Q}_{\leqslant M} \subset \mathcal{M}_u$. Moreover, if ρ_t enters Φ through the boundary $\mathcal{Q}_M$, then $\Phi \subset \mathcal{M}_u$; otherwise $\Phi \subset \mathcal{M}_1$. Theorem 1 is proved through the following three steps:

Step 1. The state in $\mathcal{M}_1$ is steered to $\mathcal{M}_u$ in a finite time almost surely.

Step 2. The switching number of the system state between $\mathcal{M}_1$ and $\mathcal{M}_u$ is finite, and the system state stays in $\mathcal{M}_u$ finally.

Step 3. The states in $\mathcal{M}_u$ permanently converge to the target state ρ_f almost surely.

Step 1 can be finished under the constant control u_c, i.e., for any initial state $\rho_0 \in \mathcal{Q}_{>M}$, the state exits $\mathcal{Q}_{>M}$ and evolves to $\mathcal{M}_u$ in a finite time with probability 1. The proof of Step 1 contains two parts: (1) The system state ρ_t converges to a determined state which is included in $\mathcal{M}_u$ under the control $u_t = 1$ from $\mathcal{M}_1$. This conclusion will be given in detail by Lemma 1. (2) The transfer of the system state from $\mathcal{M}_1$ to $\mathcal{M}_u$ under the control $u_t = 1$ is completed in a finite time. This conclusion will be given in detail by Lemma 2.

Lemma 1. Under the control $u_t = 1$, $\lim_{t\to\infty} \mathrm{E}(\rho_t) = \rho_{I_n}$.

Proof. Let $\bar{\rho}_t = \mathrm{E}(\rho_t)$, and then $\bar{\rho}_t$ under the control $u_t = 1$ follows the equation:

$$\frac{\mathrm{d}\bar{\rho}_t}{\mathrm{d}t} = -\mathrm{i}[H_0, \bar{\rho}_t] - \frac{1}{2}[L, [L, \bar{\rho}_t]] - \mathrm{i}[H_1, \bar{\rho}_t]. \tag{14}$$

Consider the function

$$W(\rho_t) = \mathrm{tr}\left((\rho_t - \rho_{I_n})^2\right) = \mathrm{tr}\left({\rho_t}^2\right) - \frac{1}{n}. \tag{15}$$

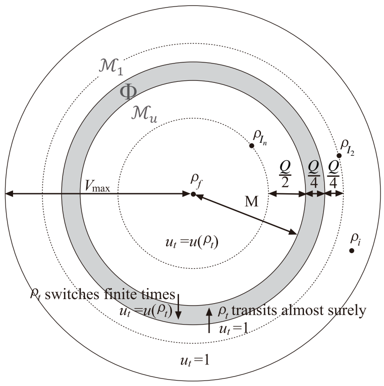

Figure 1 Diagram for the proof of Theorem 1.

According to (14), the time derivative of $W(\bar{\rho}_t)$ is

$$\begin{aligned}\frac{\mathrm{d}W(\bar{\rho}_t)}{\mathrm{d}t} &= -2\mathrm{tr}\left(\mathrm{i}\left[H_0,\bar{\rho}_t\right]\bar{\rho}_t\right) - \mathrm{tr}\left(\left[L,\left[L,\bar{\rho}_t\right]\right]\bar{\rho}_t\right) - 2\mathrm{tr}\left(\mathrm{i}\left[H_1,\bar{\rho}_t\right]\bar{\rho}_t\right)\\ &= -\mathrm{tr}\left(\left[L,\bar{\rho}_t\right]^\dagger\left[L,\bar{\rho}_t\right]\right) = -\left\|\left[L,\bar{\rho}_t\right]\right\|^2 \leqslant 0,\end{aligned}\tag{16}$$

where $\|\cdot\|$ represents the Frobenius norm.

Under the control $u_t=1$, $\bar{\rho}_t$ converges to the largest invariant set $\mathcal{M}$ contained in the set $\{\rho_t : \|[L,\rho_t]\| = 0\}$. The state trajectories in $\mathcal{M}$ are denoted as $\bar{\rho}_t^{\mathcal{M}}$. According to $\|[L,\bar{\rho}_t^{\mathcal{M}}]\|=0$, we have $[L,\bar{\rho}_t^{\mathcal{M}}]=0$. Since L is a diagonal matrix with $l_1=l_2\neq l_k$ for $k\neq 1,2$, we obtain $\bar{\rho}_t^{\mathcal{M}} = \left[\begin{smallmatrix}\rho_F & 0\\ 0 & \tilde{D}\end{smallmatrix}\right]$, in which $\rho_F = \left[\begin{smallmatrix}\rho_{F11} & \rho_{F12}\\ \rho_{F21} & \rho_{F22}\end{smallmatrix}\right]$ and $\tilde{D}=\mathrm{diag}(\tilde{d}_3,\ldots,\tilde{d}_n)$.

From Section 2, we have

$$\begin{aligned}H_{01}+uH_{11} &= \begin{bmatrix}h_{0-11} & h_{0-12}\\ h_{0-21} & Xh_{0-11}\end{bmatrix} + u\begin{bmatrix}0 & h_{12}\\ h_{21} & 0\end{bmatrix}\\ &= \begin{bmatrix}h_{0-11} & 0\\ 0 & h_{0-11}\end{bmatrix} + u\begin{bmatrix}0 & h_{12}+\frac{h_{0-12}}{u}\\ h_{21}+\frac{h_{0-21}}{u} & (X-1)\frac{h_{0-11}}{u}\end{bmatrix}.\end{aligned}$$

Letting $H_{01}^u = \left[\begin{smallmatrix}h_{0-11} & 0\\ 0 & h_{0-11}\end{smallmatrix}\right]$ and $H_{11}^u = \left[\begin{smallmatrix}0 & h_{12}+\frac{h_{0-12}}{u}\\ h_{21}+\frac{h_{0-21}}{u} & (X-1)\frac{h_{0-11}}{u}\end{smallmatrix}\right]$, then H_0 and H_1 can be written as $H_0^u = \left[\begin{smallmatrix}H_{01}^u & 0\\ 0 & \bar{D}\end{smallmatrix}\right]$ and $H_1^u = \left[\begin{smallmatrix}H_{11}^u & H_{12}\\ H_{21} & H_{22}\end{smallmatrix}\right]$, respectively, in which $\bar{D}$ represents a diagonal matrix. It should be noted that $H_{11}^1 = \left[\begin{smallmatrix}0 & h_{12}+h_{0-12}\\ h_{21}+h_{0-21} & (X-1)\,h_{0-11}\end{smallmatrix}\right]$ with $u_t=1$. H_0^1 is also a diagonal matrix. Therefore, $[H_0^1,\bar{\rho}_t^{\mathcal{M}}]=0$. The mathematical expression of $d\bar{\rho}_t^{\mathcal{M}}$ is the same as that of $\bar{\rho}_t^{\mathcal{M}}$, so that the expression of $[H_1^1,\bar{\rho}_t^{\mathcal{M}}]$ is the same of that $\bar{\rho}_t^{\mathcal{M}}$ on the basis of (14). Because of

$$\left[H_1^1,\bar{\rho}_t^{\mathcal{M}}\right] = \begin{bmatrix}\left[H_{11}^1,\rho_F\right] & H_{12}\tilde{D}-\rho_F H_{12}\\ H_{21}\rho_F-\tilde{D}H_{21} & \left[H_{22},\tilde{D}\right]\end{bmatrix},$$

$[H_{22},\tilde{D}]$ is a diagonal matrix and $H_{12}\tilde{D}=\rho_F H_{12}$. H_{22} is connected, so $\tilde{D}$ is a diagonal matrix with the same diagonal elements and denoted as $\tilde{D}=\kappa I_{n-2}$. At this point,

$$H_{12}\tilde{D} = \kappa H_{12} = \kappa\begin{bmatrix}h_{13} & \cdots & h_{1n}\\ 0 & \cdots & 0\end{bmatrix} + \kappa\begin{bmatrix}0 & \cdots & 0\\ h_{23} & \cdots & h_{2n}\end{bmatrix},$$

$$\rho_F H_{12} = \rho_{F11}\begin{bmatrix}h_{13} & \cdots & h_{1n}\\ 0 & \cdots & 0\end{bmatrix} + \rho_{F12}\begin{bmatrix}h_{23} & \cdots & h_{2n}\\ 0 & \cdots & 0\end{bmatrix} + \rho_{F21}\begin{bmatrix}0 & \cdots & 0\\ h_{13} & \cdots & h_{1n}\end{bmatrix} + \rho_{F22}\begin{bmatrix}0 & \cdots & 0\\ h_{23} & \cdots & h_{2n}\end{bmatrix}.$$

Since $h_{k1} = h_{k2}$ and $h_{k1} = -h_{k2}$, $\forall k = 3, \ldots, n$ cannot hold simultaneously, we have

$$\begin{bmatrix} h_{13} & \cdots & h_{1n} \\ 0 & \cdots & 0 \end{bmatrix} \neq \pm \begin{bmatrix} h_{23} & \cdots & h_{2n} \\ 0 & \cdots & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \cdots & 0 \\ h_{13} & \cdots & h_{1n} \end{bmatrix} \neq \pm \begin{bmatrix} 0 & \cdots & 0 \\ h_{23} & \cdots & h_{2n} \end{bmatrix}.$$

In view of $h_{1k} \neq 0$ and $h_{2k} \neq 0$ for $k = 3, \ldots, n$, $H_{12}\tilde{D} = \rho_F H_{12}$ holds only when $\rho_{F11} = \rho_{F22} = \kappa$ and $\rho_{F12} = \rho_{F21} = 0$, i.e., $\rho_F = \left[\begin{smallmatrix} \kappa & 0 \\ 0 & \kappa \end{smallmatrix}\right]$. Due to $\tilde{D} = \kappa I_{n-2}$ and the fact that $\mathrm{tr}(\bar{\rho}_t^{\mathcal{M}}) = 1$, we have $\kappa = \frac{1}{n}$ and $\bar{\rho}_t^{\mathcal{M}} = \frac{1}{n} I_n$, namely $\mathcal{M} = \{\rho_{I_n}\}$. Thus, $\lim_{t\to\infty} \mathrm{E}(\rho_t) = \lim_{t\to\infty} \bar{\rho}_t = \rho_{I_n}$. The proof of Lemma 1 is completed.

Lemma 1 indicates that the system state will converge to the state ρ_{I_n} under the constant control $u_t = 1$. Since ρ_{I_n} is included in $\mathcal{Q}_{\leqslant M} \subset \mathcal{M}_u$, the proof of the first part of Step 1 is finished. Next, we prove the second part of Step 1 by Lemma 2.

Lemma 2. Let $\tau_{\rho_0}(\mathcal{Q}_{>M})$ be the first exit time from the set $\mathcal{Q}_{>M}$ of the system trajectories starting from ρ_0, and then $\tau_{\rho_0}(\mathcal{Q}_{>M}) < \infty$ holds almost surely.

Proof. Because $V(\rho_t)$ is continuous and $\lim_{t\to\infty} \mathrm{E}(\rho_t) = \rho_{I_n}$, we have

$$\lim_{t\to\infty} \mathrm{E}(V(\rho_t)) = V(\lim_{t\to\infty}(\rho_t)) = 1 - \frac{1}{n} + c\,\mathrm{tr}^2(P\rho_{I_n}), \tag{17}$$

$M - V(\rho_{I_n}) = Q/2 > 0$, thus there exists $T > 0$ so that $|\mathrm{E}(V(\rho_t)) - V(\rho_{I_n})| < Q/2$ for all $t \geqslant T$. Consequently, for all $t \geqslant T$,

$$\mathrm{E}(V(\rho_t)) < V(\rho_{I_n}) + \frac{Q}{2} = M. \tag{18}$$

Define $T(\rho_0) = \inf\{T : \mathrm{E}(V(\rho_t)) < M, t \geqslant T\}$ and $\bar{T} = \sup_{\rho_0 \in \mathcal{S}} T(\rho_0)$. Due to the continuity of $\mathrm{E}(V(\rho_t))$, $T(\rho_0)$ is continuous with respect to ρ_0, so $\bar{T} < \infty$ must hold, i.e., there exists $\bar{T} < \infty$ so that $\mathrm{E}(V(\rho_t)) < M$, $\forall t \geqslant \bar{T}$, $\forall \rho_0 \in \mathcal{S}$ under the control $u_t = 1$. Then we can show $\tau_{\rho_0}(\mathcal{Q}_{>M}) < \infty$ almost surely by the similar analysis in [26]. Lemma 2 holds.

Lemma 2 indicates that ρ_t will leave the set $\mathcal{Q}_{>M}$ under the control $u_t = 1$ in a finite time. According to Lemmas 1 and 2, Step 1 holds and the proof of Step 1 is completed.

Next, we prove Step 2. Step 1 indicates that the system state transfers to $\mathcal{M}_u$ from $\mathcal{M}_1$ in a finite time $T_{1\text{st}}$. For $t > T_{1\text{st}}$, there are two cases: the state stays in $\mathcal{M}_u$ permanently (the occurrence probability of this case is denoted as $\mathcal{P}_{\mathcal{M}_u}$); and the state returns to $\mathcal{M}_1$ with probability $1 - \mathcal{P}_{\mathcal{M}_u}$. Furthermore, the state ρ_t may transit back and forth between $\mathcal{M}_1$ and $\mathcal{M}_u$ repeatedly. Thus the proof of Step 2 also contains two parts: (1) The switching number of the state ρ_t between $\mathcal{M}_1$ and $\mathcal{M}_u$ is finite under the control u_t. This conclusion will be given accurately by Proposition 1. (2) After finite switches, the system state ρ_t will stay in $\mathcal{M}_u$ finally. In the premise that the switching number is finite, this conclusion can be obtained by combining Step 1. Now, we give Proposition 1 and its proof.

Proposition 1. The system state switches between $\mathcal{M}_1$ and $\mathcal{M}_u$ have a finite number of times.

Proof. According to the Lyapunov stability theorem for SME, we have

$$\mathbb{P}\left(\sup_{0\leqslant t<\infty} V(\rho_t) \geqslant \lambda\right) \leqslant \frac{V(\rho_0)}{\lambda}.$$

Taking $\lambda = M + \frac{Q}{4}$, then $\mathbb{P}(\sup_{0\leqslant t<\infty} V(\rho_t) \geqslant M + \frac{Q}{4}) \leqslant \frac{M}{M+Q/4} = 1 - p < 1$. $V(\rho_0) \leqslant M$ leads to $\mathbb{P}(\sup_{0\leqslant t<\infty} V(\rho_t) < M + \frac{Q}{4}) \geqslant p$. Therefore, when the system state is located in $\mathcal{Q}_{\leqslant M}$, it stays in $\mathrm{c}\mathcal{Q}_{<M+Q/4}$ with a probability more than or equal to $p = 1 - \frac{M}{M+Q/4}$.

Denoting the event $\mathcal{B}_m = \{\text{switch } m \text{ times between } \mathcal{M}_u \text{ and } \mathcal{M}_1\}$, $m = 1, 2, \ldots$, then the probability of $\mathcal{B}_m$ satisfies $\mathbb{P}(\mathcal{B}_m) \leqslant (1-p)^m$. Thus, we have

$$\sum_{m=1}^{\infty} \mathbb{P}(\mathcal{B}_m) \leqslant \sum_{m=1}^{\infty} (1-p)^m = \frac{1-p}{p} \leqslant \infty, \tag{19}$$

which indicates that the switching number between $\mathcal{M}_u$ and $\mathcal{M}_1$ is finite from the Borel-Cantelli lemma [30]. The proof of Proposition 1 is completed.

Finally, we finish the proof of Step 3. Steps 1 and 2 show that the system state will stay in $\mathcal{M}_u$ permanently after a finite time. According to the LaSalle's invariance theorem for SME, the state ρ_t will converge to the invariant set $\mathcal{R}$ under the control $u_t = u(\rho_t)$ (see Subsection 3.2). Here, we show that M in Theorem 1 exists as long as that c in (3) satisfies certain conditions. For the states in $\mathcal{R}$, we have

$$\begin{aligned} V(\rho_{I_n}) &= 1 - \frac{1}{n} + c\left(\frac{1}{n}\right)^2 \mathrm{tr}^2(P), \\ V(\rho_i) &= 1, \quad i = 3, \ldots, n, \\ V(\rho_{I_2}) &= 1 - \frac{1}{2} + c\left(\frac{1}{2}\right)^2 \mathrm{tr}^2(P_1), \\ V(\rho_h) &= 1 - \frac{1}{2} - 2\Re(h_{12}b^*) + c\left(\frac{1}{2}\mathrm{tr}(P_1) + 2\Re(P_{12}h_{21})\right)^2. \end{aligned} \tag{20}$$

According to $V(\rho_{I_n}) < V(\rho_t)$ with $\rho_t \in \mathcal{R}$, the following conditions hold.

(1) For Cases 1–3, $\mathcal{R} = \{\rho_f, \rho_{I_2}$ and $\rho_i, i = 3, \ldots, n\}$,

(a) In the light of $V(\rho_{I_n}) < V(\rho_i)$, one can get that $c < N$, where $N = n/\mathrm{tr}^2(P)$;

(b) In the light of $V(\rho_{I_n}) < V(\rho_{I_2})$, one can get that $c > 2N/(n+2)$.

Thus, the constraint for c is $2N/(n+2) < c < N$ in this case. Owing to $2/(n+2) < 1$, c exists, which guarantees the existence of M.

(2) For Case 4, $\mathcal{R} = \{\rho_f, \rho_h, \rho_{I_2}$ and $\rho_i, i = 3, \ldots, n\}$,

(a) In the light of $V(\rho_{I_n}) < \min\{V(\rho_i), V(\rho_{I_2})\}$, one can get that $2N/(n+2) < c < N$;

(b) In the light of $V(\rho_{I_n}) < V(\rho_h)$, one can get that

$$\max\left\{\left(\frac{n}{n+2} - \frac{8n\Re(h_{12}b^*)}{4+n^2}\right)N, 0\right\} < c < N.$$

Therefore, the constraint for c is $\max\{(\frac{n}{n+2} - \frac{8n\Re(h_{12}b^*)}{4+n^2})N, \frac{2}{n+2}N\} < c < N$. When $\frac{n}{n+2} - \frac{8n\Re(h_{12}b^*)}{4+n^2} < 1$, i.e., $\Re(h_{12}b^*) > -\frac{4+n^2}{8(n+2)}$, c exists, which also guarantees the existence of M.

Based on the constraints for c, we have

$$\min\{V(\rho_t), \rho_t \in \mathcal{R} \text{ and } \rho_t \neq \rho_f\} = V(\rho_{I_n}) + Q = M + \tfrac{Q}{2} > M + \tfrac{Q}{4},$$

and $\rho_t \notin \mathcal{Q}_{\leqslant M+Q/4}$ for any state $\rho_t \neq \rho_f$ in $\mathcal{R}$. Thus,

$$\{\rho_t : \rho_t \in \mathcal{R} \cap \mathcal{Q}_{\leqslant M+Q/4}\} = \{\rho_f\}, \tag{21}$$

which means that there is only the target state in the domain $\mathcal{Q}_{\mathrm{cd}}$ for the states in $\mathcal{R}$, and ρ_t converges to ρ_f under the control $u(\rho_t)$, i.e., $\mathbb{P}(\lim_{t\to\infty}\rho_t = \rho_f) = 1$.

Combining the three steps, one knows that the system state ρ_t is steered to $\mathcal{M}_u$ from $\mathcal{M}_1$ under the constant control u_c in a finite time. Although ρ_t may move between $\mathcal{M}_1$ and $\mathcal{M}_u$, the switching number is finite, the state stays in $\mathcal{M}_u$ finally under switching control u_t, and finally ρ_t converges to target state ρ_f under the control $u(\rho_t)$, i.e., the target state ρ_f is globally asymptotically stable in probability under the control u_t for any initial state. The proof of Theorem 1 is completed.

Remark. The Lyapunov function (3) and the switching control u_t in Theorem 1 can also be applied to the case of eigenstates ρ_i for $i = 3, \ldots, n$. When the target state ρ_f is an eigenstate ρ_j for $j \in \{3, \ldots, n\}$, $V(\rho_f) = 0$. We analyze the states $\rho_{\mathcal{R}}$ which make $g(\rho_{\mathcal{R}}) = 0$. Considering that $[\rho_{\mathcal{R}}, \rho_f] = 0$, we have $\mathrm{tr}(H_1[\rho_{\mathcal{R}}, \rho_f]) = 0$, which means that there is no state $\rho_{\mathcal{R}}$ in $\mathcal{R}_2$. Combined with $\mathcal{R}_3$, the invariant set $\mathcal{R} \subseteq R_{\mathrm{Max}} = \{\mathcal{R}_1, \rho_f, \rho_i, i = 3, \ldots, n \text{ and } i \neq j\}$. In order to make $[\rho_{\mathcal{R}}, P] = 0$, one can take $P_1 = k_P I_2$. Then, $\mathcal{R} = \mathcal{R}_{\mathrm{Max}}$ when $P_1 = k_P I_2$, otherwise $\mathcal{R} \subset \mathcal{R}_{\mathrm{Max}}$. Since $V(\rho_{\mathcal{R}}) = 1 + c\mathrm{tr}^2(P_1\rho_{\mathcal{R}1})$, the constraint for c can be obtained as $0 \leqslant c < N$ according to $V(\rho_{I_n}) < V(\rho_{I_2})$ and $V(\rho_{I_n}) < V(\rho_i)$ for $i = 3, \ldots, n$ with $i \neq j$. Thereby, M in Theorem 1 exists for the case of the eigenstate ρ_j. Under the switching control in Theorem 1, the target eigenstate is globally stable, which can be proved by using a proof similar to that of Theorem 1. Thus, either $\rho_f = \left[\begin{smallmatrix} \rho_{f1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ or $\rho_f = \rho_j$ is globally stable under the switching control u_t in Theorem 1.

5 Numerical simulation experiments

The numerical experiments are performed on a three-dimensional stochastic quantum system ($n = 3$) in this section, in which the switching control proposed in Section 3 is used to make the target state globally stable.

The system parameters in the experiment are set as

$$H_0 = \begin{bmatrix} 1 & -2/3 & 0 \\ -2/3 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0 & 1 & \mathrm{i} \\ 1 & 0 & 0.5\mathrm{i} \\ -\mathrm{i} & -0.5\mathrm{i} & 0 \end{bmatrix},$$

$L = \mathrm{diag}\{1, 1, 2\}$, and $\eta = 0.5$. It is obvious that the system parameters satisfy the settings in Section 2. So, Theorem 1 is applicable to this system.

The initial and target states are given as $\rho_0 = \mathrm{diag}\,\{0, 1, 0\}$ and

$$\rho_f = \begin{bmatrix} 0.8 & 0.4 & 0 \\ 0.4 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

respectively. In this setting, ρ_f is included in the case of $\Im\,(b) = 0$ and $a \neq 0.5$. P is designed as

$$P = 0.2 \cdot \begin{bmatrix} 1 & -2+\mathrm{i} & 0 \\ -2-\mathrm{i} & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which satisfies the condition (1) for P_1 in Theorem 1, i.e., $\frac{\Im(h_{21}p_{12})}{\Im(h_{12})} \neq \frac{p_{22}-p_{11}}{2a-1}b$. c is designed as $c = 1.5 \in (0.4N, N)$ with $N = \frac{n}{\mathrm{tr}^2(P)} = 3$, in which $n = 3$, and $k_v = 5$, so that $M = 0.85$ and $Q = 0.42$. For the selected system, the control (13) in Theorem 1 becomes

(1) If $V\,(\rho_t) \geqslant 0.91$ or when $\rho_t \in \Phi = \mathcal{Q}_{<0.91} \cap \mathcal{Q}_{>0.85}$ and ρ_t enters Φ through the boundary $\mathcal{Q}_{0.91}$, the control law is $u_t = u_c = 1$;

(2) If $V\,(\rho_t) \leqslant 0.85$ or when $\rho_t \in \Phi$ and ρ_t enters Φ through the boundary $\mathcal{Q}_{0.85}$ the control law is

$$u_t = u\,(\rho_t) = -\frac{2c\mathrm{tr}\,(P\rho_t)\,\mathrm{tr}\,(P\alpha) + c\mathrm{tr}^2\,(P\delta)}{\mathrm{tr}\,(-\mathrm{i}H_1\,[\rho_t, 2c\mathrm{tr}\,(P\rho_t)\,P - \rho_f])} - k_v\mathrm{tr}\,(-\mathrm{i}H_1\,[\rho_t, 2c\mathrm{tr}\,(P\rho_t)\,P - \rho_f])\,.$$

The experiments results are shown in Figure 2, in which Figure 2(a)–(e) are the functions of u_t, ρ_{11}, ρ_{12}, ρ_{22} and V over time in 5 experiments, respectively, while Figure 2(f) is the average value of V in 10 experiments. From Figure 2(a) one can see that under the control u_t, the functions of ρ_{11}, ρ_{12} and ρ_{22} with time tend to be 0.8, 0.4, and 0.2, respectively, while the curves of u_t, V and the average value of V all tend to be zero, which demonstrate that the system state is steered to the desired target state, and the control is effective. According to the results in 5 experiments, the state paths in every experiment are different for the same initial state and target state, which shows the randomness of the controlled system. Notice that the curves of V do not decrease monotonously, which is different from the deterministic system.

6 Conclusion

The global stabilization control of arbitrary eigenstates for finite dimensional stochastic quantum systems with non-diagonal free Hamiltonian and non-regular measurement operator has been studied in this paper. We have adopted a switching feedback control combined with the control designed based on the Lyapunov stability theorem and a constant control to steer the system state to the target eigenstate from any initial

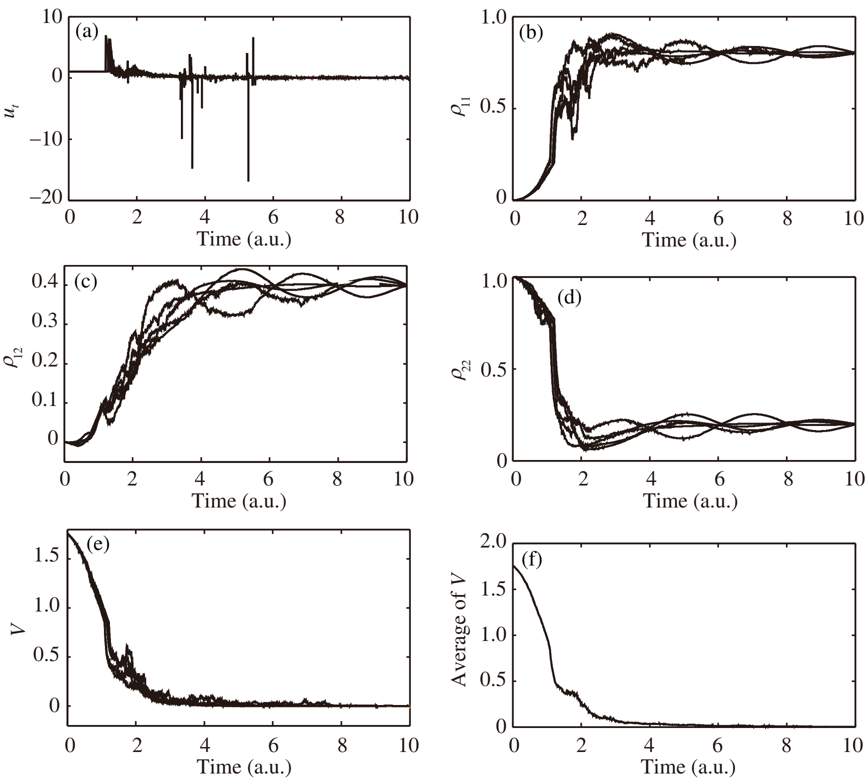

Figure 2 Experiments results. Functions of (a) u_t, (b) ρ_{11}, (c) ρ_{12}, (d) ρ_{22}, and (e) V over time in 5 experiments; (f) average value of V in 10 experiments.

state in the state space. The convergence of the control has been proved based on the Lyapunov stability theorem and LaSalle invariance principle for the quantum SME. Moreover, we have used the switching control to steer system state to the target state for a three-dimensional stochastic quantum system in the simulation experiments, which have verified the effectiveness of the proposed control laws.

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Conflict of interest The authors declare that they have no conflict of interest.

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