SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

November 2016, Vol. 59 112207:1–112207:11
doi: 10.1007/s11432-015-0401-9

Shaping the PDF of the state variable based on piecewise linear control for non-linear stochastic systems

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Received November 10, 2015; accepted March 26, 2016; published online October 14, 2016

Abstract In this paper, we propose a shape-control scheme for non-linear stochastic dynamical systems to enable us to shape the probability density function (PDF) of the state variable. First, we derive the PDF analytical expression using the Fokker-Planck-Kolmogorov (FPK) equation obtained from stochastic systems. Then, we control the PDF shape by devising a piecewise linear control law whose parameters are calculated using the conjugated gradient method. Finally, we perform contrast simulation experiments to validate the effectiveness and superiority of the proposed algorithm.

Keywords non-linear stochastic systems, probability density function, shape control, piecewise linear control, Fokker-Planck-Kolmogorov (FPK) equation

Citation Wang L Z, Qian F C. Shaping the PDF of the state variable based on piecewise linear control for non-linear stochastic systems. Sci China Inf Sci, 2016, 59(11): 112207, doi: 10.1007/s11432-015-0401-9

1 Introduction

In practical industrial processes, there are several outer random disturbances, which cause systems to have random outputs. Normally, we cannot measure random disturbances accurately even when using ideal mathematical models and high-precision sensors. Therefore, the investigation of the control problem of such stochastic systems having random disturbances has gradually increased in importance in control science fields.

Over the past several decades, the main control methods employed in stochastic systems include minimum variance control, self-tuning control, linear-quadratic-Gaussian (LQG) control and so forth. Of these control methods, LQG control is the most representative one, and some studies [1–7] have extensively investigated this method and reported some meaningful and valuable achievements. LQG control has experienced several developmental stages including mean-value control, variance control [6, 8–11], cumulant control [12–14], and probability density function (PDF) shape control.

Compared with the traditional mean and variance, the PDF shape can reveal more information during the running of a system. For example, the food-machining process is a complicated process control system,

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whose key units comprise two rollers with gears on the surface. The distance between the two rollers is used to control the crushed particle size. Existing sensors for particle-size distribution can directly measure the crushed particle size distribution. The stochastic distribution control law is used to control the output PDF shape. Controlling the stochastic distribution can make the crushed food particle size follow the given distribution, which provides raw materials to meet the requirements for subsequent food machining procedures; thus improving the efficiency of the entire system and the control quality. In such problems, it is very important to control the PDF because the system PDF describes the complete statistical properties of random variables. By controlling the PDF shape, any order of the cumulant is controlled, thus including the mean and variance. Therefore, it is very important to investigate the PDF shape control.

For PDF shape control, some significant research results have been published, e.g., from Wang et al. [15–23]. They provided a series of modeling and controlling methods using the B-splines approximation method, PID control technology, and entropy theory. In 2012, Qian et al. [7] succeeded in controlling the PDF shape by adjusting each moment of the PDF of the performance index. Some of the above-mentioned control methods are limited to linear stochastic dynamical systems, and it is difficult to use them for non-linear systems. In some methods, the performance index is quadratic, and the control objective is the output PDF. In this paper, we focus on non-linear systems as the research object and the PDF shape approximation as the performance index. In addition, our control objective is the PDF of the state variable.

To control the PDF shape, we devise a suitable control scheme, making the PDF of the system state match a target PDF. Therefore, it is very important to devise a suitable control law in order to control the PDF shape for non-linear stochastic systems. In previous research work [24, 25], the polynomial control law has been preferred owing to its flexibility and continuity in mathematics. Undoubtedly, the polynomial control law will increase the computation requirements and complexity of the overall algorithm. In this paper, we devise a piecewise linear control law that is composed of two proportional coefficients and a piecewise point. The biggest advantage of a linear control is its linearity within a specific range; thus, it is mathematically simpler to deal with than a continuous non-linear polynomial control, which significantly decreases the complexity and computation requirements.

As is well known, the transition probability density of the response process of linear or nonlinear dynamic systems excited by Gaussian white noise is determined using the FPK equation [26]. Therefore, the stationary PDF can be obtained via the FPK equation. To date, the solution for the problem of the FPK equation is always difficult problem, so many approximating methods have been developed [22, 26]. In this paper, for one-dimensional (1D) non-linear systems, we derive and give the stationary analytical PDF, which is an accurate solution to the FPK equation.

Although piecewise linear control has been studied in literature [22] there are differences from three main perspectives: 1) The control objective in the literature is the output PDF without measurement noise, while ours is the PDF of the state variable. 2) The PDF is approximated in [22], while we solve for an accurate PDF. 3) Although we adopt the same control structure, different methods are used to determine control gains. In this paper, we adopt the conjugated gradient method to obtain control gains. Because the characteristics of the problem being studied is linear or of polynomial form, when solving such optimization problems, the use of the conjugate-gradient method may be optimal in finite steps, and the efficiency is very high. This is one of the contributions of our paper.

The remainder of the paper is organized as follows. In Section 2, we state the problem and derive the stationary PDF expression. Then, in Section 3, the piecewise linear control design scheme is introduced. In Section 4, we performed some contrast simulation experiments to verify the algorithm, while in Section 5, we conclude the paper.

2 Problem statement

Consider a 1D non-linear stochastic system, and its process is described by the following Itô differential equation:

$$\mathrm{d}x(t) = [f(x) + u(x)]\mathrm{d}t + \mathrm{d}\xi(t), \tag{1}$$

where $x(t) \in R$ is the state variable of the system, $f(x)$ is the deterministic non-linear function of $x(t)$, and $u(x) \in R$ is the control. $\xi(t) \in R$ is a standard Wiener process with zero mean and the self-correlation coefficients $E[\mathrm{d}\xi(t)\mathrm{d}\xi(\tau)] = 2D(t-\tau)$ $(t > \tau)$, where $D = \pi S_0$, and S_0 is the spectral density of $\mathrm{d}\xi(t)$.

In traditional stochastic control such as representative LQG control, only the expected value of the performance index is considered, which fails to limit the fluctuations caused by random factors in every realization for the control algorithm. To resolve this problem, variance control is introduced. If the PDF of the performance index is Gaussian or symmetric, large fluctuations can be realized by restraining the variance. However, the comprehensive performance index of LQG is neither Gaussian nor symmetric, so effectively controlling the PDF shape is key. Our goal is to shape the PDF of the state variable in System (1), or to make the actual PDF match the target PDF by devising the control law $u(x)$.

The FPK equation corresponding to System (1) is [27]

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\{[f(x)+u(x)]p(x,t)\} + \frac{1}{2}\frac{\partial^2[2\pi S_0 p(x,t)]}{\partial x^2}, \tag{2}$$

where $p(x,t)$ is the PDF of the random variable $x(t)$. Generally, when $x(t)$ is in steady state, it is no longer a stochastic process but a stochastic variable. Therefore, $p(x,t)$ is only related to x, and not to t, which means $\frac{\partial p(x)}{\partial t} = 0$. For simplicity, take $2\pi S_0 = 1$. If it does not meet the condition $2\pi S_0 = 1$, then there exists a constant in the equation that does not influence the subsequent derivation of the formula. When $2\pi S_0 = 1$ holds, the stationary PDF $p(x)$ should satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x}\{[f(x)+u(x)]p(x)\} - \frac{1}{2}\frac{\mathrm{d}^2 p(x)}{\mathrm{d}x^2} = 0. \tag{3}$$

Theorem 1. The stationary PDF of the state variable in a 1D stochastic system (1) is

$$p(x) = C\exp\left\{\int 2[f(x)+u(x)]\mathrm{d}x\right\}.$$

Proof. Let $M(x) = f(x) + u(x)$, Eq. (3) is rewritten as

$$p''(x) - 2M(x)p'(x) - 2M'(x)p(x) = 0, \tag{4}$$

where $p'(x)$ is the first-order partial derivative of $p(x)$ to x, and $p''(x)$ is the second-order partial derivative of $p(x)$ to x. $M'(x)$ is the first-order partial derivative of $M(x)$ to x.

Assume that one particular solution to (4) is of the form $p_1(x) = \exp[N(x)]$, where $N(x)$ is to be settled. From $p_1(x)$, we derive

$$p_1'(x) = N'(x)\exp[N(x)], \quad p_1''(x) = \{N''(x) + [N'(x)]^2\}\exp[N(x)],$$

substituting $p_1(x)$, $p_1'(x)$ and $p_1''(x)$ into (4), it yields

$$N''(x) + [N'(x)]^2 - 2M(x)N'(x) - 2M'(x) = 0. \tag{5}$$

Consequently, seeking the particular solution $p_1(x)$ is transformed into solving (5). Now, we will work out another particular solution $p_2(x)$, which is linearly independent of $p_1(x)$. Generally, the variation of constants technique is commonly used.

Assume that $p_2(x) = c(x)\exp[N(x)]$ and $c(x) \neq 0$. Then,

$$p_2'(x) = [c'(x) + c(x)N'(x)]\exp[N(x)],$$

$$p_2''(x) = \{c''(x) + 2c'(x)N'(x) + c(x)N''(x) + c(x)[N'(x)]^2\}\exp[N(x)].$$

After substituting $p_2(x)$, $p_2'(x)$ and $p_2''(x)$ into (4), we get

$$c(x)[N''(x) + (N'(x))^2 - 2M(x)N'(x) - 2M'(x)] + c''(x) + 2c'(x)N'(x) - 2M(x)c'(x) = 0. \tag{6}$$

According to (5), $N''(x) + (N'(x))^2 - 2M(x)N'(x) - 2M'(x) = 0$, so

$$c''(x) + 2c'(x)N'(x) - 2M(x)c'(x) = 0. \tag{7}$$

Eq. (7) is a second-order linear homogeneous differential equation, and it can be lowered to a first-order one $c'(x) = \exp\{\int -[2N'(x) - 2M(x)]\mathrm{d}x\}$. Then, we have

$$c(x) = \int \exp\left\{\int -[2N'(x) - 2M(x)]\mathrm{d}x\right\}\mathrm{d}x, \tag{8}$$

hence, another particular solution to (4) is

$$p_2(x) = \exp[N(x)]\int \exp\left\{\int -[2N'(x) - 2M(x)]\mathrm{d}x\right\}\mathrm{d}x.$$

The ratio of $p_2(x)$ to $p_1(x)$ is

$$\frac{p_2(x)}{p_1(x)} = c(x) = \int \exp\left\{\int -[2N'(x) - 2M(x)]\mathrm{d}x\right\}\mathrm{d}x.$$

For

$$c(x) = \int \exp\left\{\int -[2N'(x) - 2M(x)]\mathrm{d}x\right\}\mathrm{d}x \neq 0, \quad \frac{p_2(x)}{p_1(x)} \neq 0,$$

which means that the two particular solutions of (4) are linearly independent of each other; thus, the general solution of (4) is

$$p(x) = Cp_1(x) + C'p_2(x).$$

C and C' are arbitrary constants.

To solve for the specific expression of the solution $p(x)$, we need to find $N(x)$ that satisfies (5). It is easy to know that $N(x) = \int 2M(x)\mathrm{d}x$ is a solution of (5). Therefore, the general solution of (4) is

$$p(x) = C\exp\left[\int 2M(x)\mathrm{d}x\right] + C'\exp[2M(x)]\int \exp\left\{\int -[2N'(x) - 2M(x)]\mathrm{d}x\right\}\mathrm{d}x. \tag{9}$$

As is commonly known, because of the complexity of the problem, as in other studies, we are also concerned primarily with a particular solution of (4). Taking $C' = 0$ or $C = 0$,

$$p(x) = C\exp\left[\int 2M(x)\mathrm{d}x\right], \tag{10}$$

or

$$p(x) = C'\exp[2M(x)]\int \exp\left\{\int -[2N'(x) - 2M(x)]\mathrm{d}x\right\}\mathrm{d}x \tag{11}$$

is the solution to (4). We can use any of the two solutions (10) and (11), but in this paper, we only use the solution in (10) .

As $p(x)$ is a PDF, $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$, if and only if

$$C = \frac{1}{\int_{-\infty}^{\infty} \exp[\int 2M(x)\mathrm{d}x]\mathrm{d}x},$$

where C is a normalized constant.

Therefore, one solution to (3) is

$$p(x) = C\exp\left\{\int 2[f(x) + u(x)]\mathrm{d}x\right\}. \tag{12}$$

The stationary PDF of the state variable in System (1) is $p(x) = C\exp\{\int 2[f(x) + u(x)]\mathrm{d}x\}$.

3 Control scheme

Our goal is to attempt to make the PDF of the state variable follow the target PDF in the control time by optimizing the parameters of the control law. Over the past two years, we have made several attempts to study those non-linear systems using polynomial control law, which results in significant computational processing. Therefore, in this paper, we propose a piecewise linear control instead of the polynomial non-linear control.

3.1 Piecewise linear control

In this paper, we investigate only some special non-linear systems that are of polynomial expression.

$$f(x) = a_0 + a_1 x + \cdots + a_p x^p, \tag{13}$$

where $a_0,\ a_1, \dots, a_p$ are coefficients of the nonlinear function $f(x)$, and p is the highest order of $f(x)$.

A piecewise control is given by

$$u(x) = \begin{cases} k_1(x + k_3), & x \leqslant k_3, \\ k_2(x + k_3), & x > k_3, \end{cases} \tag{14}$$

where k_1 and k_2 are proportional coefficients, and k_3 is the piecewise point of the linear control.

Substituting $f(x)$ in (13) and $u(x)$ in (14) into (12), we get

$$p(x) = \begin{cases} C\exp\{\int 2[a_0 + k_1 k_3 + (a_1 + k_1)x + a_2 x^2 + \cdots + a_p x^p]\mathrm{d}x\}, & x \leqslant k_3, \\ C\exp\{\int 2[a_0 + k_2 k_3 + (a_1 + k_2)x + a_2 x^2 + \cdots + a_p x^p]\mathrm{d}x\}, & x > k_3. \end{cases} \tag{15}$$

In order to compare the performances of piecewise linear control and non-linear polynomial control, we assume that the polynomial control is

$$u_n(x) = c_0 + c_1 x + \cdots + c_n x^n, \tag{16}$$

where $c_0,\ c_1, \dots, c_n$ are gains of the polynomial control. For $n = 2$, $u_2(x)$ is the second-order polynomial control, while for $n = 3$, $u_3(x)$ is the cubic-order polynomial control. The corresponding PDF of the state variable is

$$p(x) = C\exp\left\{\int 2[a_0 + c_0 + (a_1 + c_1)x + (a_2 + c_2)x^2 + \cdots + a_p x^p]\mathrm{d}x\right\}, \quad n = 2, \tag{17}$$

or

$$p(x) = C\exp\left\{\int 2[a_0 + c_0 + (a_1 + c_1)x + (a_2 + c_2)x^2 + (a_3 + c_3)x^3 + \cdots + a_p x^p]\mathrm{d}x\right\}, \quad n = 3. \tag{18}$$

Define the performance index function as $J = \int_{-\infty}^{\infty}[p(x) - p_{tg}(x)]^2\mathrm{d}x$, and $p_{tg}(x)$ is the target PDF. Obviously, the index function is a nonlinear function of x, when $p(x)$ matches the target PDF, J is minimized by means of the conjugate gradient technique. Then, we simultaneously calculated the gradient and the optimal value of the parameters in the control law.

From two kinds of control (14) and (16), although the linear control is discontinuous, as it shows linearity at some range in form, so it is more easily addressed than the polynomial control. Moreover, the linear control requires that we determine no more than three parameters, while the polynomial control requires that we determine no less than three parameters as a non-linear one, which reduces the complexity when minimizing the index function.

For the piecewise linear control, the determination of the piecewise point k_3 is very important for the overall control algorithm. Like the control gains, the piecewise point is regarded as a decision variable in the optimization problem J. By solving this optimization problem, we obtain the optimal solution of the piecewise point.

3.2 Conjugate gradient method

The conjugate-gradient method is between the steepest descent method and the Newton method. It not only overcomes the slow convergence defect of the steepest descent method, but also voids the shortcoming of storing and calculating the Hessen matrix and solving the inverse matrix using only first-order derivative information. The conjugate-gradient method is one of the most useful ways of solving large linear equations, and it is also one of the most effective ways of solving large nonlinear optimization algorithms. Further, this method has many advantages such as the fact that its algorithm is simple and easily programmed, it does not require the calculation of the second derivative, and it requires only a small amount of memory. However, it should be noted that using only the gradient-based method guarantees local optimality, and we should not expect a global result.

The basic idea of the conjugate-gradient method is to combine the steepest descent method with the Newton method. We construct a set of conjugate directions using the gradient of the known point, search along these directions, and solve the minimum of the objective function.

Theorem 2. In linear control law (14), the gradient of k_1 is $\nabla e(k_1)$, the gradient of k_2 is $\nabla e(k_2)$, and the gradient of k_3 is $\nabla e(k_3)$. Assume that all of the following infinite integrals are bounded.

When $x \leqslant k_3$, $\nabla e(k_1) = G_1$, $\nabla e(k_2) = 0$, $\nabla e(k_3) = S_1$; when $x > k_3$, $\nabla e(k_1) = 0$, $\nabla e(k_2) = G_2$, $\nabla e(k_3) = S_2$, where

$$\begin{aligned}
G_1 &= \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1+k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]\mathrm{d}xH_1,\\
H_1 &= 2C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1+k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\}\int (k_3+x)\mathrm{d}x,\\
G_2 &= \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_2k_3 + (a_1+k_2)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]\mathrm{d}xH_2,\\
H_2 &= 2C\exp\left\{\int 2[a_0 + k_2k_3 + (a_1+k_2)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\}\int (k_3+x)\mathrm{d}x,\\
S_1 &= \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1+k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]\mathrm{d}xQ_1(x),\\
Q_1(x) &= 2C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1+k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\}\int k_1\mathrm{d}x,\\
S_2 &= \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_2k_3 + (a_1+k_2)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]\mathrm{d}xQ_2(x),\\
Q_2(x) &= 2C\exp\left\{\int 2[a_0 + k_2k_3 + (a_1+k_2)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\}\int k_2\mathrm{d}x.
\end{aligned}$$

Proof. When $x \leqslant k_3$,

$$p(x) = C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1+k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\}, \tag{19}$$

$$e = \int_{-\infty}^{\infty}[p(x) - p_{tg}(x)]^2\mathrm{d}x, \tag{20}$$

then

$$\begin{aligned}
\frac{\partial e}{\partial k_1} &= \int_{-\infty}^{\infty} 2[p(x) - p_{tg}(x)]\mathrm{d}x\frac{\partial p(x)}{\partial k_1}\\
&= \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1+k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]\mathrm{d}x\frac{\partial p(x)}{\partial k_1},
\end{aligned} \tag{21}$$

and

$$\frac{\partial p(x)}{\partial k_1} = \frac{\partial C\exp\{\int 2[a_0 + k_1k_3 + (a_1+k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\}}{\partial k_1}$$

$$= 2C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\}\int(k_3 + x)\mathrm{d}x. \tag{22}$$

Let $H_1 = 2C\exp\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\}\int(k_3 + x)\mathrm{d}x$. Then, we have

$$\frac{\partial e}{\partial k_1} = \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]\mathrm{d}xH_1. \tag{23}$$

Let $G_1 = \int_{-\infty}^{\infty}[C\exp\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\} - p_{tg}(x)]\mathrm{d}xH_1$. Then, the gradient of k_1 is

$$\nabla e(k_1) = \frac{\partial e}{\partial k_1} = G_1. \tag{24}$$

Eq. (19) does not conclude k_2, so

$$\nabla e(k_2) = \frac{\partial e}{\partial k_2} = 0. \tag{25}$$

$$\begin{aligned}\nabla e(k_3) &= \frac{\partial e}{\partial k_3} = \int_{-\infty}^{\infty} 2[p(x) - p_{tg}(x)]\mathrm{d}x\frac{\partial p(x)}{\partial k_3}\\ &= \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]\mathrm{d}x\frac{\partial p(x)}{\partial k_3},\end{aligned} \tag{26}$$

and

$$\begin{aligned}\frac{\partial p(x)}{\partial k_3} &= \frac{\partial C\exp\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\}}{\partial k_3}\\ &= 2C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\}\int k_1\mathrm{d}x.\end{aligned} \tag{27}$$

Let $Q_1(x) = 2C\exp\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\}\int k_1\mathrm{d}x$. Then, we have

$$\frac{\partial p(x)}{\partial k_3} = \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]\mathrm{d}xQ_1(x). \tag{28}$$

Let $S_1 = \int_{-\infty}^{\infty} 2[C\exp\{\int 2[a_0 + k_1k_3 + (a_1 + k_1)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\} - p_{tg}(x)]\mathrm{d}xQ_1(x)$. Then, the gradient of k_3 is

$$\nabla e(k_3) = \frac{\partial p(x)}{\partial k_3} = S_1. \tag{29}$$

Therefore, when $x \leqslant k_3$, $\nabla e(k_1) = G_1$, $\nabla e(k_2) = 0$, $\nabla e(k_3) = S_1$.

In the same way, we can obtain the gradients of k_1, k_2 and k_3 in another case: when $x > k_3$, $\nabla e(k_1) = 0$, $\nabla e(k_2) = G_2$, $\nabla e(k_3) = S_2$.

$$\begin{aligned}&G_2 = \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_2k_3 + (a_1 + k_2)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]H_2\mathrm{d}x,\\ &H_2 = 2C\exp\left\{\int 2[a_0 + k_2k_3 + (a_1 + k_2)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\}\int(k_3 + x)\mathrm{d}x,\\ &S_2 = \int_{-\infty}^{\infty} 2\left[C\exp\left\{\int 2[a_0 + k_2k_3 + (a_1 + k_2)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\} - p_{tg}(x)\right]\mathrm{d}xQ_2(x),\\ &Q_2(x) = 2C\exp\left\{\int 2[a_0 + k_2k_3 + (a_1 + k_2)x + a_2x^2 + \cdots + a_px^p]\mathrm{d}x\right\}\int k_2\mathrm{d}x.\end{aligned}$$

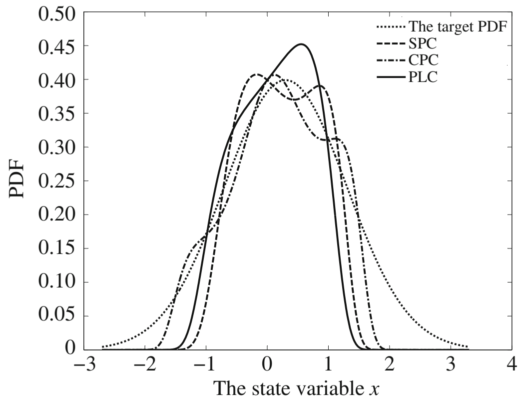

Figure 1 Tracking uni-modal PDF.

4 Example simulations

Take the error function as $e = J$. When the actual PDF shape is as close as possible to the target PDF shape, we use the conjugate-gradient method to find k_1, k_2 and k_3, which makes the error function $e(k)(k = [k_1, k_2, k_3])$ be a minimum. According to Theorem 2, to solve the minimum $e(k)$, the algorithm steps are as follows:

Step 1. Give an initial point k^0 and precision $\varepsilon > 0$.

Step 2. If the gradient of $e(k)$ at the initial point is $\|\nabla e(k^0)\| \leqslant \varepsilon$, the algorithm ends and the minimum point is k^0. Otherwise, go to Step 3.

Step 3. Take the negative gradient direction as the initial searching direction $P^0 = -\nabla e(k^0)$, and set $l = 0$.

Step 4. Get the step size t_l using the 1D searching method, making $e(k^l + t_l P^l) = \min_{t\geqslant 0} e(k^l + t_l P^l)$. Let $k^{l+1} = k^l + t_l P^l$, and go to Step 5.

Step 5. If the gradient $\|\nabla e(k^{l+1})\| \leqslant \varepsilon$, the algorithm ends and the minimum point is k^{l+1}. Otherwise, go to Step 6.

Step 6. If $l + 1 = 3$, let $k^0 = k^3$, and go to Step 3. Otherwise, go to Step 7.

Step 7. Renew the searching direction $P^{l+1} = -\nabla e(k^{l+1}) + \lambda_l P^l$, where the step size $\lambda_l = \frac{\|\nabla e(k^{l+1})\|^2}{\|\nabla e(k^l)\|^2}$. Let $l = l + 1$, and go to Step 4.

Next, we used the above algorithm to shape the PDF of the state variable. Using the conjugate-gradient method, the optimal solutions of the parameters in the control law are found, making the error function e a minimum. In other words, the actual PDF is closest to the target PDF.

We adopted the same system in literature [28], and the non-linear function is $f(x) = 0.5(x - x^3 - \varepsilon x^5)$. ε represents the nonlinear degree of $f(x)$. Here, we take $\varepsilon = 4$, which means that the system is a strong nonlinear one. The coefficients in the non-linear function (13) are respectively: $a_0 = 0$, $a_1 = 0.5$, $a_2 = 0$, $a_3 = -0.5$, $a_4 = 0$, and $a_5 = -2$. Take the following three various probability distributions to be the target PDF.

(1) The target PDF is a uni-modal distribution: $p_{tg}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{(x-\mu)^2}{2\sigma^2}]$, $\mu = 0.3$, $\sigma = 1$. Using the conjugate gradient method, we acquire different control laws. The non-linear second-order control is $u_2(x) = -0.1036 - 0.8157x + 1.5616x^2$, the non-linear cubic-order control is $u_3(x) = 0.1585 - 1.8238x - 0.0214x^2 + 2.7102x^3$, and the piecewise linear control is $u(x) = \begin{cases} 1.0000(x-0.2854), & x\leqslant -0.2854 \\ 0.5139(x-0.2854), & x> -0.2854 \end{cases}$. The result of the $p(x)$ tracking target PDF is as shown in Figure 1. Meanwhile, the run time of the programmes and error e are recorded in Table 1.

(2) The target PDF is a symmetrical bimodal distribution: $p_{tg}(x) = 0.24\exp(0.5x^2 - 0.25x^4)$. The non-linear second-order control is $u_2(x) = 0.1304x$, which is a linear control. The non-linear cubic-order control is $u_3(x) = 0.2142x - 0.1745x^3$, and the piecewise linear control is $u(x) = \begin{cases} 1.0000x, & x\leqslant 0 \\ 0.1304x, & x>0 \end{cases}$. The tracking result is shown in Figure 2, and running time and error are listed in Table 2.

Table 1 Comparison between linear control and polynomial control when the target PDF is uni-modal

Control law	Run time (s)	Error
Non-linear second-order control	5.433080	0.2695
Non-linear cubic-order control	9.312476	0.0946
Piecewise linear control	1.205621	0.3222

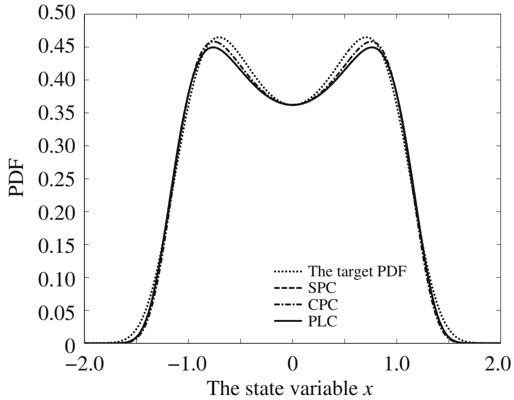

Figure 2 Tracking the symmetrical bimodal PDF.

Table 2 Comparison of the linear control and polynomial control when the target PDF is symmetrical bimodal

Control law	Run time (s)	Error
Non-linear second-order control	8.286222	0.0607
Non-linear cubic-order control	18.211304	0.0488
Piecewise linear control	2.163758	0.0607

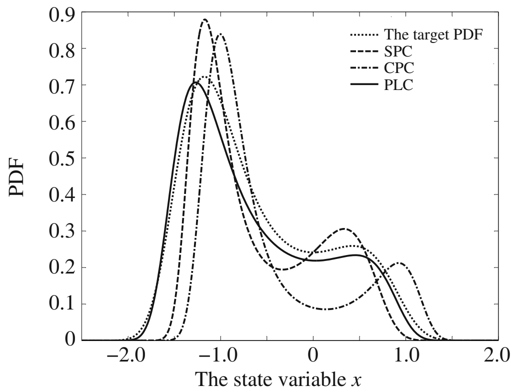

Figure 3 Tracking the asymmetrical bimodal PDF.

Table 3 Comparison of the linear control and polynomial control when the target PDF is asymmetrical bimodal

Control law	Run time (s)	Error
Non-linear second-order control	15.391581	4.1221
Non-linear cubic-order control	34.879117	11.7150
Piecewise linear control	3.207369	0.6640

(3) The target PDF is asymmetrical bimodal distribution:$p_{tg}(x) = 0.2413\exp(x^2 - x^3 - x^4)$. The non-linear second-order control is $u_2(x) = 0.5111 - 0.8503x - 4.6184x^2$, the non-linear cubic-order control is $u_3(x) = -0.3555 + 1.6929x - 1.3505x^2 - 1.0683x^3$, and the piecewise linear control is $u(x) = \begin{cases} 1.0000(x-0.5961), & x\leqslant -0.5961 \\ 0.5260(x-0.5961), & x>-0.5961 \end{cases}$. The tracking result is shown in Figure 3, and the run time and error are listed in Table 3.

In the above three contrast simulation results, SPC is the second-order polynomial control, CPC is the cubic-order polynomial control, and PLC is the piecewise linear control.

From Figure 1, although the error in the linear control is larger than that of the others in three controls, the PDF shape is most similar with the target PDF. Note that the two non-linear controls make the actual PDF appear as two modals, and not one modal.

In Figure 2, all three controls can adjust the PDF shape of the state variable. Of these, the result of the cubic-order polynomial control is a little better than that of the others. The PDF of the second-order polynomial control and the PDF of the piecewise linear control coincide completely. In addition, in Table 3, the error of the linear control is the same as the second-order polynomial control. Meanwhile, we observe that when realizing the second-order polynomial control, the control is in fact a linear one, and this is not related only to the chosen system, but also to the target PDF. In other words, for this system, when the target PDF is symmetrical bimodal, we can shape the PDF of the state variable using only a linear control.

From Figure 3 and Table 3, the control result of the piecewise linear control is observed to be better than that of the others, and the error is also the smallest of the three cases.

In addition, from Tables 1–3, regardless of the target PDF, the run time of the linear control is shorter than that of the others, demonstrating the superiority of the linear control.

In fact in this paper, we aim to describe some statistical characteristics such as the mean, variance, and cumulants through the PDF. However, in some cases, the control error of the PDF does not affect the mean, variance, and even the PDF shape, so this is why the PDF shape is used as our control goal. In the first example, the error of the non-linear cubic-order control is the smallest of the three controls, but the shape of the PDF is completely different from the target PDF. Of course, if the shape is similar, we expect a smaller control error. In the acceptable error range, we expect a shorter computation time.

On the whole, the piecewise linear control is found to be helpful for controlling the PDF shape for nonlinear stochastic systems, especially when the target PDF is bimodal.

5 Conclusion

In this paper, we proposed a scheme for shaping the probability density function of the state in stochastic nonlinear systems. We devised a piecewise linear control law with two coefficients and a piecewise point. The simulation results illustrate that the piecewise control is effective in shaping the PDF of the state variable, and it has an even better control performance for an asymmetrical bimodal target PDF compared with the two non-linear polynomial controls. Moreover, because the linear control maintains its linear characteristic over some range, it makes the algorithm simpler, decreasing the run time in all three cases. The non-linearity of the system in the examples demonstrates that the proposed algorithm is applicable to a strong non-linear system.

From our research, we propose the following open problems for further investigations:

1) Firstly, the PDF shape control for an n-dimension stochastic system should be studied. As is commonly known, an n-dimension system can be divided into several one-dimension subsystems. If there exists coupling among subsystems, it can be dealt with. Otherwise, it is very difficult.

2) Secondly, in future, it is worth investigating a piecewise linear algorithm with more than one switching points.

3) Finally, in future studies, we need to consider how to deal with the case where the steady state of a nonlinear system switches to another steady state over some distance.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61273127, U1534208), Key Laboratory for Fault Diagnosis and Maintenance of Spacecraft in Orbit (Grant No. SDML_OF2015004), Key Program of National Natural Science Foundation of China (Grant No. 61533014), and Scientific Research Plan Projects of Shannxi Education Department (Grant Nos. 16JK1690).

Conflict of interest The authors declare that they have no conflict of interest.

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