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Optimal control on special Euclidean group via natural gradient algorithm

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Abstract Considering the optimal control problem about the control system of the special Euclidean group whose output only depends on its input is meaningful in practical applications. The optimal control considered here is described as the output matrix is as close as possible to the target matrix by adjusting the system input. The geodesic distance is adopted as the measure of the difference between the output matrix and the target matrix, and the trajectory of the control input obtained in the process is achieved. Furthermore, some numerical simulations are shown to illustrate our outcomes based on the natural gradient descent algorithm for optimizing the control system of the special Euclidean group.

Keywords system control, natural gradient, special Euclidean group, geodesic distance, numerical simulation

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1 Introduction

The general linear group $GL(n,\mathbb{R})$ with a differentiable structure is a manifold. It has attracted more and more attention not only in the study of theory but also in the study of application recently. The optimal control for the system consisting of the matrix groups is widely studied for various applications such as in biomedicine, signal processing, robotics control [1–4].

In this paper, the optimal control problem on the special Euclidean group $SE(3)$ [5] will be discussed using the natural gradient method [6, 7]. In fact, the traditional gradient does not represent the steepest descent direction in the Riemannian space while the natural gradient does since the latter overcomes the problem of poor convergence of the former. Moreover, S. Amari has proved that the natural gradient is asymptotically Fisher-efficient for the maximum likelihood estimation, implying that it has the same performance as the optimal batch estimation of the parameters. Based on the advantages of the natural gradient algorithm, researchers can solve various problems more effectively in different fields such as neural network, optimal control [8–10].

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The paper is organized as follows. Firstly, some basic definitions and results about the general linear group $GL(n,\mathbb{R})$ and the special Euclidean group $SE(n)$ are introduced without proofs since the detailed information can be referred to in many references [5,11]. With the help of the left invariant metric on the general linear group, the geodesics and their distances between any two points not far away on the special Euclidean group has been introduced [11]. Secondly, for simplicity, the control system is described as follows: suppose that the output is only determined by the control input; the geodesic distance between the output matrix and the target matrix is taken as the cost function and the natural gradient descent algorithm is proposed to evaluate control trajectory of the input matrix when the output matrix changes from the initial value to the final one. Finally, two cases whether the target matrix lies on the output matrix manifold or not are considered as examples in the simulations to illustrate our discussions.

2 Geometric structure of the special Euclidean group $SE(n)$

In this section, basic materials including some definitions and results about the general linear group $GL(n,\mathbb{R})$ and the special Euclidean group $SE(n)$ are reviewed. These will be used throughout the paper.

The general linear group $GL(n,\mathbb{R})$ consisting of all invertible $n\times n$ matrices with real entries is homeomorphic to $\mathbb{R}^{n\times n}$. Thus it has a differentiable manifold structure. The group multiplication of $GL(n,\mathbb{R})$ is the usual matrix multiplication, the inverse map of the group multiplication takes a matrix A to its inverse A^{-1}, and the identity of the group is the identity matrix I. The Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ of $GL(n,\mathbb{R})$ turns out to be the set of $n\times n$ real matrices.

All the other real matrix Lie groups are the subgroups of $GL(n,\mathbb{R})$, whose group operators are subgroup restrictions of the ones on $GL(n,\mathbb{R})$. The exponential mapping is the mapping

$$\begin{aligned} &\exp: \mathfrak{gl}(n,\mathbb{R}) \to GL(n,\mathbb{R}),\\ &X \mapsto \sum_{n=0}^{\infty}\frac{X^n}{n!}, \quad X\in\mathfrak{gl}(n,\mathbb{R}). \end{aligned} \tag{1}$$

It is well known that $\exp(X)\exp(Y)=\exp(X+Y)$ if X and Y commute.

The logarithms of A in $GL(n,\mathbb{R})$ are solutions of the matrix equation $\exp X = A$ [12]. When A does not have eigenvalues in the (closed) negative real line, there exists a unique real logarithm, called the principal logarithm and denoted by $\log A$ when the special condition is satisfied.

Furthermore, if $\parallel A-I\parallel<1$, where $\parallel . \parallel$ is the given matrix norm and I denotes the identity matrix, then the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(A-I)^n}{n}$ converges to $\log A$, and therefore

$$\log(A)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(A-I)^n}{n} \tag{2}$$

is obtained. In general, $\log(AB)\neq\log(A)+\log(B)$. Another important fact is that [12]

$$\log(A^{-1}BA)=A^{-1}(\log B)A.$$

This fact is also true when log in the above is replaced with an analytic matrix function.

For simplicity, assume that all matrices for the matrix-valued function $\log(\cdot)$ exist throughout this paper. The exponential mapping and the logarithmic mapping provide a method to transform information between the Lie group and the Lie algebra. Moreover, the Lie algebra is a linear space which is easier to deal with than the Lie group so that it is a useful tool of studying the Lie group.

Definition 1. For the general linear group $GL(n,\mathbb{R})$, the left translation L_A at $A\in GL(n,\mathbb{R})$ is defined by $L_A(B)=AB$ for every $B\in GL(n,\mathbb{R})$ and the tangent mapping is obtained as $(L_A)_*X=AX$ for every $X\in T_AGL(n,\mathbb{R})$. The right translation and the corresponding tangent mapping are similarly defined as the left translation. The left invariant metric is given by

$$\langle X,Y\rangle_A=\left\langle A^{-1}X,A^{-1}Y\right\rangle_I, \tag{3}$$

with $X,Y\in T_AGL(n,\mathbb{R})$, where $\langle A^{-1}X,A^{-1}Y\rangle_I := \mathrm{tr}((A^{-1}X)^{\mathrm{T}}A^{-1}Y)$ is the Frobenius inner product.

Similarly, the right invariant metric can be defined. It has been proved that there exists the left invariant metrics on all matrix Lie groups so that Definition 1 is well defined. Furthermore, compact Lie groups possess bi-invariant metrics. That is to say the left invariant metric on it is the right invariant metric simultaneously. It is worthy to note that the Euclidean group is noncompact [13].

Let $\gamma:[0,1]\to GL(n,\mathbb{R})$ be a sufficiently smooth curve on $GL(n,\mathbb{R})$. It is the geodesic when it satisfies the second-order differentiable equation $\ddot{\gamma}(t)+\Gamma_{\gamma(t)}(\dot{\gamma}(t),\dot{\gamma}(t))=0$ with a suitable parametrization, where the one over-dot and two over-dot denote the first and second derivatives with respect to the parameter t respectively and $\Gamma_{\cdot}(\cdot,\cdot)$ stands for the Christoffel operator.

Definition 2. Let $\gamma:[0,1]\to GL(n,\mathbb{R})$ be a geodesic. The length of $\gamma(t)$ is defined by

$$\mathcal{L}:=\int_0^1\sqrt{\langle\dot{\gamma}(t),\dot{\gamma}(t)\rangle_{\gamma(t)}}\mathrm{d}t=\int_0^1\sqrt{\mathrm{tr}\{(\gamma(t)^{-1}\dot{\gamma}(t))^{\mathrm{T}}\gamma(t)^{-1}\dot{\gamma}(t)\}}\mathrm{d}t. \tag{4}$$

The geodesic distance between two matrices A and B on $GL(n,\mathbb{R})$ is expressed as

$$d(A,B):=\inf\{\mathcal{L}(\gamma)|\gamma:[0,1]\to GL(n,\mathbb{R})\ \text{ with }\ \gamma(0)=A,\ \gamma(1)=B\}. \tag{5}$$

According to the Hopf-Rinow theorem [14], there exists a geodesic connecting any two points on the completely connected Lie group and the geodesic can extend to infinite. Although the special Euclidean group is not complete, there still exists a geodesic connecting two points which are not so far.

The special Euclidean group $SE(n)$ in $GL(n,\mathbb{R})$ is the semidirect product of the special orthogonal group $SO(n)$ with $\mathbb{R}^n$ itself [11, 13], that is, $SE(n)=SO(n)\ltimes\mathbb{R}^n$. The representation of elements of $SE(n)$ is

$$SE(n)=\left\{\begin{pmatrix}A & b\\ 0 & 1\end{pmatrix}\middle| A\in SO(n),\ b\in\mathbb{R}^n\right\}. \tag{6}$$

On a Lie group, the tangent space at the group identity defines a Lie algebra. The Lie algebra $\mathfrak{se}(n)$ of $SE(n)$ can be denoted by

$$\mathfrak{se}(n)=\left\{\begin{pmatrix}X & v\\ 0 & 0\end{pmatrix}\middle| X\in \mathfrak{so}(n),\ v\in\mathbb{R}^n\right\}. \tag{7}$$

Specially, $SE(n)$ becomes $SE(3)$ when $n=3$ [5]. $SE(3)$ is the special Euclidean group of rigid body transformations in three dimensional space denoted by

$$SE(3)=\left\{\begin{pmatrix}R & d\\ 0 & 1\end{pmatrix}\middle| R\in\mathbb{R}^{3\times3},\ d\in\mathbb{R}^3,\ R^{\mathrm{T}}R=I,\ \det(R)=1\right\}. \tag{8}$$

It is easy to show that $SE(3)$ is a group for the standard matrix multiplication and that it is a manifold. It is therefore a Lie group.

The Lie algebra of $SE(3)$, denoted by $\mathfrak{se}(3)$, is given by

$$\mathfrak{se}(3)=\left\{\begin{pmatrix}\Omega & v\\ 0 & 0\end{pmatrix}\middle| \Omega\in\mathbb{R}^{3\times3},\ v\in\mathbb{R}^3,\ \Omega^{\mathrm{T}}=-\Omega\right\}. \tag{9}$$

A 3×3 skew-symmetric matrix Ω can be uniquely identified with a vector $\omega\in\mathbb{R}^3$ so that for an arbitrary vector $x\in\mathbb{R}^3$, $\Omega x=\omega\times x$, where $\times$ is the vector cross product operation in $\mathbb{R}^3$.

Denote $U,\ V\in SE(n)$ by

$$U=\begin{pmatrix}A_1 & b_1\\ 0 & 1\end{pmatrix},\quad V=\begin{pmatrix}A_2 & b_2\\ 0 & 1\end{pmatrix}. \tag{10}$$

Taking the corresponding exponential mappings on manifold $SO(n)$ and vector space $\mathbb{R}^n$ into consideration, the geodesic $\gamma_{U,V}$ connecting U and V on the Lie group $SE(n)$ is given by [11]

$$\gamma_{U,V}(t)=\begin{pmatrix}A_1(A_1^{\mathrm{T}}A_2)^{\mathrm{T}} & b_1+(b_2-b_1)t\\ 0 & 1\end{pmatrix},\quad t\in[0,1]. \tag{11}$$

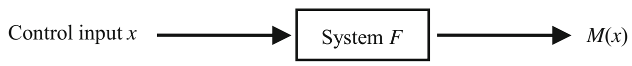

Figure 1 The considered control system.

Based on (1) and (11), the following lemma is obtained [11].

Lemma 1. The geodesic distance between two points U and V on $SE(n)$ induced by the left invariant metric (3) is given by

$$d(U,V) = (\| \log(A_1^{\mathrm{T}} A_2)\|_F^2 + \|b_2 - b_1\|_F^2)^{\frac{1}{2}}. \tag{12}$$

3 The control on the special Euclidean group $SE(n)$

As mentioned in the first section, the purpose of control is to control the system input so that the output approximates the target to the greatest extent. There are lots of ways to describe the concept 'approximate' in practical problems in order to measure the distance between the output and the target [9, 15]. In this paper, the geodesic distance on manifold $SE(n)$ is chosen as a cost function to solve the control problem on $SE(n)$.

Suppose that F is the considered control system with the input variable x and the output matrix $M(x)$ [9,16–18], where x plays the role of vector-valued parameter and the output manifold $N = \{M(x)\}$ is a submanifold of $SE(n)$ (See Figure 1).

Our goal is to design the input x so that the output matrix $M(x)$ is as close as possible to the target matrix T which is also a element of $SE(n)$ and give the learning trajectory of the control input x while the output matrix develops from the initial matrix $M(x_1)$ to the optimal matrix $M(x_*)$.

In order to achieve the above purpose, two important problems should be considered as follows:

I. Define a distance function to measure the difference between the output matrix and the target matrix of the system.

II. Describe the learning trajectory of input x so as to make the final output matrix of the system approximate to the target matrix as close as possible.

As discussed in Section 2, the geodesic distance (12) is adopted to measure the difference between the matrix $M(x)$ and the target matrix T. Therefore, the cost function $J(x)$ is defined as

$$J(x) = d^2_{M(x)\in N}(M(x), T), \tag{13}$$

consequently, the optimal point x_* on the system is obtained by

$$x_* = \arg \min_{M(x)\in N} J(x), \tag{14}$$

where

$$N = \{M(x) | x = (x^1, x^2, \ldots, x^m) \in \Theta \subset \mathbb{R}^k\} \tag{15}$$

is the output matrix manifold.

Next, consider the control problem proposed above by using the method of the natural gradient descent algorithm. Note that there are two cases of the control problem according to the target matrix T lies on manifold N or not and the natural gradient algorithm is suitable for both cases. For the purpose of getting the gradient on manifold $SE(n)$, we introduce a lemma as follows.

Lemma 2 ([13]). Let $X(t)$ be a matrix-valued function of the real variable t and A, B be constant valued matrices. Suppose $X(t)$ is an invertible matrix which does not have eigenvalues on the closed negative real line for all t in its domain, then the following equations hold:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr}\left(X(t)^{\mathrm{T}} X(t)\right) = 2 \operatorname{tr}\left(X(t)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}t} X(t)\right), \\ &\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr}(\log X(t)) = \operatorname{tr}\left(X(t)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} X(t)\right), \\ &\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr}(A X(t) B) = \operatorname{tr}\left(A \frac{\mathrm{d}}{\mathrm{d}t} X(t) B\right). \end{aligned} \tag{16}$$

Whether the target matrix T belongs to the output manifold N or not, we always take the advantage of the geodesic equation as trajectory and the negative gradient of the cost function $J(x)$ as the direction to give the iterative formula. Moreover, readers can refer to [8–11, 19, 20] for details.

The iterative formula is displayed as the following theorem using the natural gradient descent algorithm on $SE(n)$.

Theorem 1. Let the output matrix $M(x)$, which is corresponding to the control input variable x, and the target matrix T be points on a submanifold of $SE(n)$. Here $M(x)$ and T are in the form of

$$M(x) = \begin{pmatrix} A_x & b_x \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} A_T & b_T \\ 0 & 1 \end{pmatrix}, \tag{17}$$

where

$$\begin{aligned} A_x \in SO(n), \quad b_x \in \mathbb{R}^n, \\ A_T \in SO(n), \quad b_T \in \mathbb{R}^n. \end{aligned} \tag{18}$$

Suppose the submanifold is an m-dimensional manifold, the coordinates of A_x and A_T are $(x^1, x^2, \ldots, x^k)$ and $(x_0^1, x_0^2, \ldots, x_0^k)$, the coordinates of b_x and b_T are $(x^{k+1}, \ldots, x^m)$ and $(x_0^{k+1}, \ldots, x_0^m)$ $(k < m)$, respectively. Thus, the control input x and the target T can be denoted by the coordinates $(x^1, x^2, \ldots, x^m)$ and $(x_0^1, x_0^2, \ldots, x_0^m)$.

Then, the cost function of the iterative process based on the geodesic distance is given by

$$J(x_t) = \| \log(A_T^{\mathrm{T}} A_x)\|_F^2 + \|b_T - b_x\|_F^2, \tag{19}$$

where

$$x_{t+1} = x_t - \eta G(x_t)^{-1} \nabla J(x_t). \tag{20}$$

$G(x_t)^{-1}$ is the inverse of the Riemannian metric $G(x_t)$ at the point x_t and η is the step length. The components of the gradient $\nabla J(x_t)$ satisfy

$$\frac{\partial}{\partial x_t^i} J(x_t) = \begin{cases} 2\,\mathrm{tr}\left(\log(A_T^{\mathrm{T}} A_{x_t}) A_{x_t}^{\mathrm{T}} \frac{\partial}{\partial x_t^i} A_{x_t}\right), & i = 1, 2, \ldots, k, \\ 2(x_t^i - x_0^i), & i = k+1, \ldots, m. \end{cases} \tag{21}$$

Proof. As in [7], let P_t and P_{t+1} be two points on a submanifold of $SE(n)$ named N for convenient in this proof. Corresponding to the functions $J(x_t)$ and $J(x_{t+1})$, whose coordinates are given by x_t and x_{t+1} respectively.

Assume that the vector $\overrightarrow{P_t P_{t+1}} \in T_{P_t} N$ has a fixed length expressed as $\|\overrightarrow{P_t P_{t+1}}\|^2 = \varepsilon^2$ and the positive number ε is small enough. Denote $\overrightarrow{P_t P_{t+1}} = \varepsilon \boldsymbol{v}$, and $\boldsymbol{v} = \boldsymbol{a}^{\mathrm{T}} \nabla J(x) \in T_{P_t} N$ can be treated as a tangent vector of $T_{P_t} N$ at P_t, where $\boldsymbol{a}$ is a constant vector. Suppose that the tangent vector $\boldsymbol{v}$ satisfies the following constraints

$$|\boldsymbol{v}|^2 = \langle \boldsymbol{v}, \boldsymbol{v} \rangle = \boldsymbol{a}^{\mathrm{T}} G(x_t) \boldsymbol{a} = 1, \tag{22}$$

then the relation between $J(x_t)$ and $J(x_{t+1})$ can be denoted by

$$J(x_{t+1}) = J(x_t) + \varepsilon \boldsymbol{a}^{\mathrm{T}} \nabla J(x_t). \tag{23}$$

Now we search for the vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)^{\mathrm{T}}$ which minimizes (23) under the constraint (22). Setting the Lagrange function

$$F(\boldsymbol{a}) = J(x_t) + \varepsilon \boldsymbol{a}^{\mathrm{T}} \nabla J(x_t) + \lambda(\boldsymbol{a}^{\mathrm{T}} G(x_t) \boldsymbol{a} - 1), \tag{24}$$

and from this expression, we get

$$\boldsymbol{a} = -\frac{\varepsilon}{2\lambda} G(x_t)^{-1} \nabla J(x_t). \tag{25}$$

Let $\eta = \frac{\varepsilon}{2\lambda}$, then the iterative formula (20) is obtained.

Furthermore, combining Theorem 1 and Lemma 2, the components of $\nabla J(x_t)$ are shown as

$$\frac{\partial}{\partial x_t^i}J(x_t) = 2\,\mathrm{tr}\left(\log(A_T^{\mathrm{T}}A_{x_t})\frac{\partial}{\partial x_t^i}\left(\log(A_T^{\mathrm{T}}A_{x_t})\right)\right) = 2\,\mathrm{tr}\left(\log(A_T^{\mathrm{T}}A_{x_t})A_{x_t}^{\mathrm{T}}\frac{\partial}{\partial x_t^i}A_{x_t}\right),\ i = 1,2,\ldots,k, \quad (26)$$

and

$$\frac{\partial}{\partial x_t^i}J(x_t) = \frac{\partial}{\partial x_t^i}(x_t^i - x_0^i)^2 = 2(x_t^i - x_0^i), \qquad i = k+1,\ldots,m.$$

Based on the discussion above, the algorithm for the optimal control problem on $SE(n)$ is obtained as follows.

For the coordinate $(x^1, x^2, \ldots, x^m)$ of the considered special Euclidean group system, the natural gradient descent algorithm is given by:

Algorithm 1 Optimal control on special Euclidean group

- 1. Set $(x_1^1, x_1^2, \ldots, x_1^m)$ as the coordinate of the initial output matrix $M(x_1)$ and choose a tolerance $\varepsilon > 0$ and the step-length $\eta > 0$;
- 2. Compute the gradient of the cost function $\nabla J(x_t)$;
- 3. If $\|\nabla J(x_t)\|_F < \varepsilon$ then stop;
- 4. Update $x_{t+1} = x_t - \eta G(x_t)^{-1}\nabla J(x_t)$ and go to step 2.

Remark 1. If the target matrix lies on the output matrix manifold N, a geodesic connecting the initial output matrix $M(x_1)$ and the optimal output matrix $M(x_*)$ can always be found on N, so does the optimal output matrix $M(x_*)$ and the target matrix T. Therefore, Algorithm 1 realizes the optimal trajectory of the control input variable x.

Remark 2. If the target matrix T doesn't lie on the output matrix manifold N but on manifold $SE(n)$, the geodesics between T and the output matrix $M(x)$ on N are measured by the geodesics on $SE(n)$. In the iterative process, from the initial output matrix $M(x_1)$ to the optimal output matrix $M(x_*)$, the minimal distance between T and manifold N is obtained. In fact, the optimal output matrix $M(x_*)$ is the geodesic projection of T on N. The optimal trajectory of the control input variable x using Algorithm 1 can be realized.

4 Numerical simulations

In this section, some examples are given to illustrate our results about the control trajectory on the special Euclidean matrix system $SE(3)$ based on the natural gradient descent algorithm for cases that whether the target matrix belongs to the output matrix manifold N or not respectively.

In order to express the designed control system, we consider the rigid body motion of the object on $SE(3)$. Suppose that the coordinate of the center of gravity of the rigid body is $c_{RO} \in \mathbb{R}^3$, and the optimal trajectory from the configuration U to V is the curve $C(t)$ such that

$$\begin{pmatrix} C(t) \\ 1 \end{pmatrix} = \gamma_{U,V}(t)\begin{pmatrix} c_{RO} \\ 1 \end{pmatrix}, \quad (27)$$

where $t \in [0,1]$ and $\gamma_{U,V}(t)$ denotes the geodesic connecting U and V on $SE(3)$.

4.1 The case that the target matrix is on the output matrix manifold

If the target matrix T is on the output matrix manifold N, then the rigid body can move in the whole manifold, which means it can approach the target along the geodesic in the submanifold N (See Figure 2).

Example 1. Suppose that the target matrix T belongs to the output manifold N_1, which is in the following form

$$N_1 = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \middle| A \in SO(3), b \in \mathbb{R}^3 \right\}, \quad (28)$$

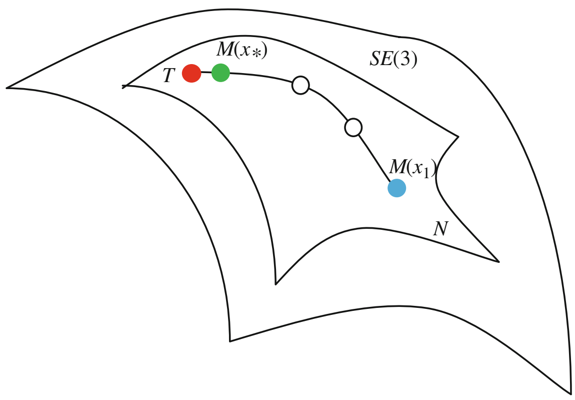

Figure 2 (Color online) Control system for T on N.

Table 1 The relation between algorithm efficiency and step length

Step length	Iteration	Step length	Iteration
0.1	140	0.6	63
0.2	73	0.7	95
0.3	56	0.8	231
0.4	47	0.9	1000
0.5	52		

where

$$A = \begin{pmatrix} \cos\omega_x \cos\omega_y & -\sin\omega_x \cos\omega_z - \cos\omega_x \sin\omega_y \sin\omega_z & \sin\omega_x \sin\omega_z - \cos\omega_x \sin\omega_y \cos\omega_z \\ \sin\omega_x \cos\omega_y & \cos\omega_x \cos\omega_z - \sin\omega_x \sin\omega_y \sin\omega_z & -\cos\omega_x \sin\omega_z - \sin\omega_x \sin\omega_y \cos\omega_z \\ \sin\omega_y & \cos\omega_y \sin\omega_z & \cos\omega_y \cos\omega_z \end{pmatrix}, \tag{29}$$

$$b^{\mathrm{T}} = (v_x, v_y, v_z).$$

$\omega_x, \omega_y, \omega_z$ represent the Euler angles along x-axis, y-axis and z-axis, respectively.

Remark 3. To study conveniently, the following conventions are made throughout these numerical examples not just this example:

(1) The rigid body is represented by the segment $\boldsymbol{p} + t\boldsymbol{q}$, where $\boldsymbol{p} = (0, 0, 0)^{\mathrm{T}}$ and $\boldsymbol{q} = (0.1, 0.1, 0.1)^{\mathrm{T}}$.

(2) The initial values $\boldsymbol{\omega^0} = (\omega_x^0, \omega_y^0, \omega_z^0)$ and $\boldsymbol{v^0} = (v_x^0, v_y^0, v_z^0)$ are chosen as $(0, \frac{\pi}{3}, \frac{\pi}{6})$ and $(0.1, 0.2, 0.1)$ respectively.

(3) The target values $\boldsymbol{\omega^1} = (\omega_x^1, \omega_y^1, \omega_z^1)$ and $\boldsymbol{v^1} = (v_x^1, v_y^1, v_z^1)$ are selected as $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ and $(0.2, 0.3, 0.4)$.

Therefore, the initial location of the rigid body is $S = \boldsymbol{p^0} + t\boldsymbol{q^0}$, where $\boldsymbol{p^0} = (0.1, 0.2, 0.1)^{\mathrm{T}}$ and $\boldsymbol{q^0} = (0.0317, 0.2366, 0.2549)^{\mathrm{T}}$. Besides, the target location of the rigid body is $T = \boldsymbol{p^1} + t\boldsymbol{q^1}$, where $\boldsymbol{p^1} = (0.2, 0.3, 0.4)^{\mathrm{T}}$ and $\boldsymbol{q^1} = (0.1793, 0.2793, 0.5707)^{\mathrm{T}}$. In the whole section, $t \in [0, 1]$ exists for all corresponding expressions.

In fact, the rigid object can move in the whole output manifold N_1 freely. According to Algorithm 1, the trajectory of the input $\boldsymbol{\omega}^0, \boldsymbol{v}^0$ from the initial location S of the rigid body to the final location is achieved (See Figure 3) and the parameters of the optimal control output are $\omega^* = (0.7854, 0.7854, 0.7854)$, $v^* = (0.2, 0.3, 0.4)$ with the error tolerance 10^{-4} and step length 0.1. In this example, the metric matrix in (20) is $\mathrm{diag}(2\boldsymbol{I}, \boldsymbol{I})$, where $\boldsymbol{I}$ denotes the 3-ordered identity matrix.

Furthermore, the efficiency and convergence of Algorithm 1 are obtained under the step length $\eta = 0.1, 0.2, 0.4$ respectively (See Figure 4). More results about numerical experiments are shown in Table 1 with the different step length and the same error tolerance.

4.2 The case that the target matrix is not on the output matrix manifold

Here, we consider the case that the target matrix T is not on the output matrix manifold N but on manifold $SE(3)$ (See Figure 5). In this case, the geodesics between T and the points on N are measured

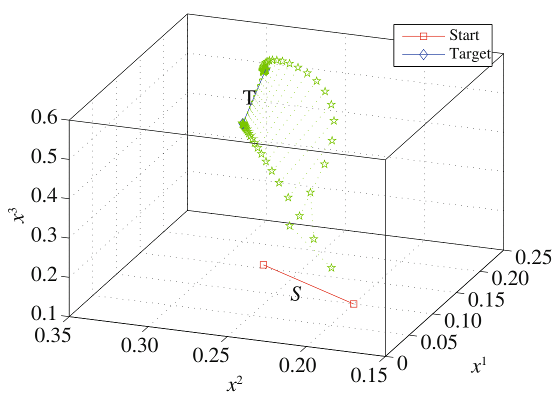

Figure 3 (Color online) Trajectories of the rigid object motion on N_1.

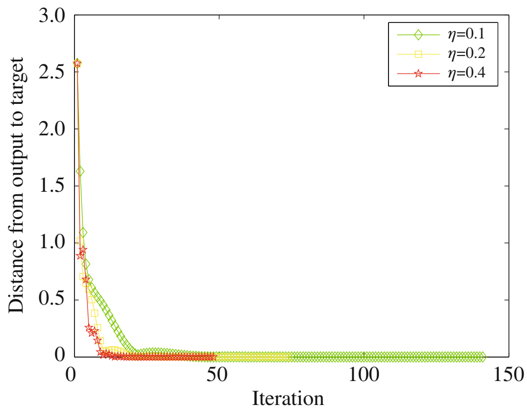

Figure 4 (Color online) Convergence of natural gradient.

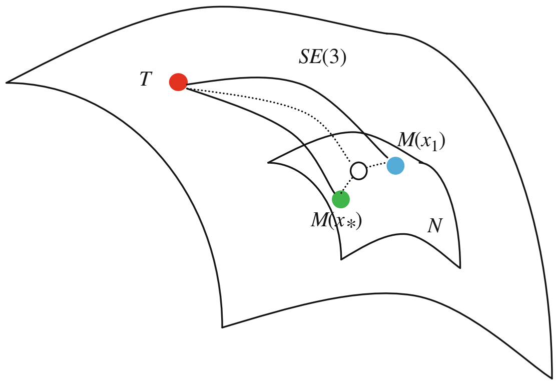

Figure 5 (Color online) Control system for T outside of N.

by the geodesics on $SE(3)$. Once given an initial control input variable x_1, we have the initial output matrix $M(x_1)$ on N, the geodesic between T and $M(x_1)$ exists and the next point on N is generated according to (20). And this iterative process (from the initial output matrix $M(x_1)$ to the optimal output matrix $M(x_*)$) continues until the distance between T and manifold N is minimal within the given error. In fact, the optimal output point $M(x_*)$ on N is the geodesic projection of T on N. In the process, the Riemannian metric matrix G used in the natural gradient Algorithm 1 must be the metric on N rather than $SE(3)$ in order to ensure that the new points produced by iteration are still on N. Four examples are used to explain the situation.

Remark 4. In the following four examples, the trajectories of the input $(\boldsymbol{\omega}^0, \boldsymbol{v}^0)$ from the initial output location S (according to $M(x_1)$) of the rigid body to the final output location (according to $M(x_*)$) are obtained denoted by the green trajectories and the geodesic projection trajectories from the target matrixes T to manifolds are exhibited as the black ones in the figures.

Example 2. When the system keeps the parameter $\boldsymbol{b} = (0.1, 0.2, 0.1)^{\mathrm{T}}$ in (29) invariant, the output matrix manifold is denoted by N_2. The optimal output parameters $\omega^* = (0.7854, 0.7854, 0.7853)$ and $\boldsymbol{v}^* = (0.1, 0.2, 0.1)$ are achieved according to our proposed method. Here, the metric matrix in (20) is $\mathrm{diag}(2\boldsymbol{I}, \boldsymbol{0})$, where $\boldsymbol{I}$ and $\boldsymbol{0}$ denote the 3-ordered identity matrix and the 3-ordered null matrix respectively. The associated information, namely, the rigid body rotates on the manifold N_2, can be illustrated in Figure 6.

Example 3. Similarly, when we consider the parameter $\boldsymbol{\omega} = (0, \frac{\pi}{3}, \frac{\pi}{6})^{\mathrm{T}}$ in (29) fixed, the output matrix manifold is called N_3, whose metric matrix is $\mathrm{diag}(\boldsymbol{0}, \boldsymbol{I})$, where $\boldsymbol{I}$ and $\boldsymbol{0}$ denote the 3-ordered identity matrix and the 3-ordered null matrix respectively. The optimal output of the control system can be denoted by parameters $\boldsymbol{\omega}^* = (0, 1.0472, 0.5236)$ and $\boldsymbol{v}^* = (0.2, 0.3, 0.4)$ (See Figure 7). From this figure, we find that the rigid body only translates when restricted to manifold N_3.

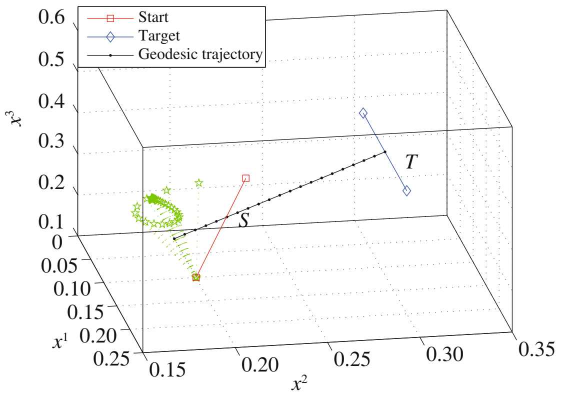

Figure 6 (Color online) Trajectories of the rigid object motion on N_2.

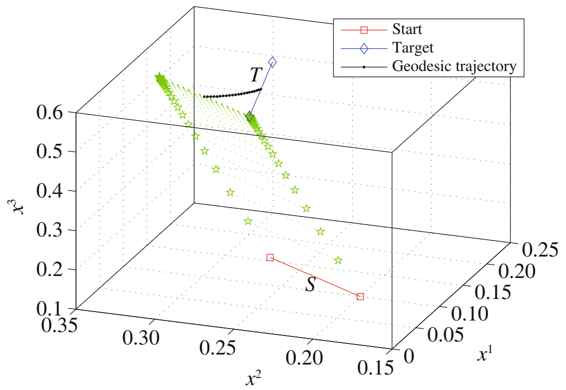

Figure 7 (Color online) Trajectories of the rigid object motion on N_3.

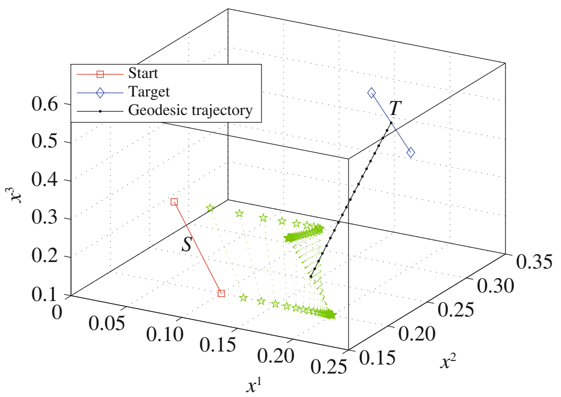

Figure 8 (Color online) Trajectories of the rigid object motion on N_4.

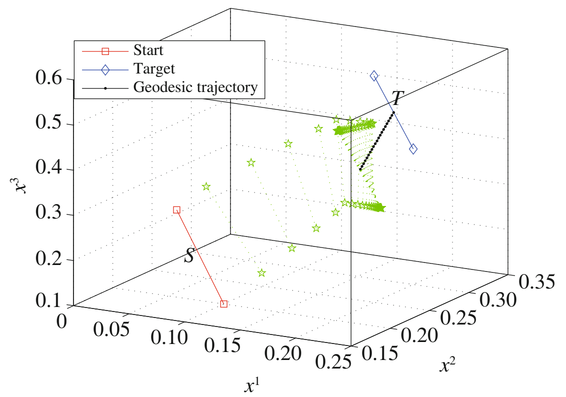

Figure 9 (Color online) Trajectories of the rigid object motion on N_5.

Example 4. Let the output matrix manifold N_4 be a manifold with $\omega_z = \frac{\pi}{6}$, $v_y = 0.2$ and $v_z = 0.1$ in (29) denoted by

$$N_4 = \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \middle| A \in SO(3), b \in \mathbb{R}^3 \right\},$$

where

$$A = \begin{pmatrix} \cos\omega_x \cos\omega_y & -\frac{\sqrt{3}}{2}\sin\omega_x - \frac{1}{2}\cos\omega_x \sin\omega_y & \frac{1}{2}\sin\omega_x - \frac{\sqrt{3}}{2}\cos\omega_x \sin\omega_y \\ \sin\omega_x \cos\omega_y & \frac{\sqrt{3}}{2}\cos\omega_x - \frac{1}{2}\sin\omega_x \sin\omega_y & -\frac{1}{2}\cos\omega_x - \frac{\sqrt{3}}{2}\sin\omega_x \sin\omega_y \\ \sin\omega_y & \frac{1}{2}\cos\omega_y & \frac{\sqrt{3}}{2}\cos\omega_y \end{pmatrix},$$

$$b^{\mathrm{T}} = (v_x, 0.2, 0.2),$$

here the metric matrix on N_4 is diag$(2, 2, 0, 1, 0, 0)$. The optimal output is represented by the parameters $\boldsymbol{\omega}^* = (0.7854, 0.7853, 0.5236)$ and $\boldsymbol{v}^* = (0.2, 0.2, 0.1)$. In Figure 8, we identify that the rigid body restricted on N_4 keeps rotating and translating when moving towards the given target.

Example 5. For the fixed parameters in (29)

$$\begin{aligned} &\omega_x \in \left[-\frac{\pi}{6}, \frac{\pi}{3}\right], \quad \omega_y \in \left[0, \frac{\pi}{2}\right], \quad \omega_z \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right], \\ &v_x \in [0, 0.3], \quad v_y \in [0.15, 0.25], \quad v_z \in [0, 0.3], \end{aligned} \tag{30}$$

the output matrix manifold is named N_5. The optimal output parameters are $\boldsymbol{\omega}^* = (0.7854, 0.7853, 0.5236)$ and $\boldsymbol{v}^* = (0.2, 0.2489, 0.2999)$. In this example, the metric matrix is diag$(2k_1, 2k_2, 2k_3, k_4, k_5, k_6)$, where $k_i = 1$ when the corresponding coordinate belongs to the given interval and $k_i = 0$ for others $(i = 1, \dots, 6)$. In fact, the motion of the rigid body is too complex to be described on manifold N_5 (See Figure 9).

5 Conclusion

An application of the natural gradient descent algorithm is applied to solve the control problem on the special Euclidean group. To discuss the descent direction problem on manifold $SE(n)$, we choose the geodesic distance as the cost function, the geodesic equation as the trajectory and the natural gradient as the gradient direction to give the iterative formula. Among them, the geodesic distance is induced by the left invariant matric. Two kinds of cases are considered on the optimal control problem: one is for the case that the target matrix belongs to the output matrix manifold; the other is the situation that the target matrix doesn't lie on the output matrix manifold but on $SE(n)$. At the end of the paper, simulations are given to illustrate our results efficiently.

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Conflict of interest The authors declare that they have no conflict of interest.

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