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Dynamics and stability for a class of evolutionary games with time delays in strategies

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Abstract This paper investigates the modeling and stability of a class of finite evolutionary games with time delays in strategies. First, the evolutionary dynamics of a sequence of strategy profiles, named as the profile trajectory, is proposed to describe the strategy updating process of the evolutionary games with time delays. Using the semi-tensor product of matrices, the profile trajectory dynamics with two kinds of time delays are converted into their algebraic forms respectively. Then a sufficient condition is obtained to assure the stability of the delayed evolutionary potential games at a pure Nash equilibrium.

Keywords evolutionary game, time delays, potential game, semi-tensor product of matrices, stability

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1 Introduction

Evolutionary games (EGs) were firstly introduced by some biologists to study the evolution of lives in biological systems [1]. In recent years, evolutionary game theory has become a powerful tool for investigating various phenomena such as population dynamics [2], economics [3], social physics [4], engineering science [5], etc.

The first important issue in evolutionary game theory is the evolution dynamics, which describes how the frequencies of various strategies within a population change over time according to their payoffs [6]. Replicator dynamics is one of the most important continuous evolutionary dynamics [7]. The replicator dynamics lends the notion of evolutionary stability, originally introduced as a static concept, an explicit dynamic meaning [8], which indicates whether an evolution will converge to certain status [9].

It is usually assumed (as in the replicator dynamics) that the interactions between individuals take place instantaneously and their effects are immediate. But in reality, time delay phenomena always exist. Delayed evolutionary game happens in biological systems. For instance, Ref. [10] considered the delayed predator-prey model and presented a long history of the researches on such delayed model in biology. Ref. [11] discussed the combined effects of time delays for their social and biological models. In [12], the authors studied the effect of a symmetric time delay in the replicator dynamics in a population model with two strategies. Ref. [13] discussed the effect of slow time delays on the convergence of various

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evolutionary dynamics to the ESS and applied these dynamics to the medium access control. Just to name few, one sees easily that delay in evolutionary games is a long standing topic and has been studied widely, because it happens in evolutionary games pretty often.

Potential game theory plays an important role in game theoretic analysis. It has been traced back to Brown's original fictitious play [14]. Since then, potential games have been developed by many authors. For instance, Candogan et al. have studied the topological structure for finite games in strategic form and developed a canonical orthogonal decomposition of an arbitrary game into three components [15]. Morris and Ui have investigated the robust equilibria of potential games and given some sufficient conditions for the robustness of sets of equilibria in potential games [16, 17]. Sandholm has provided a new definition of potential games for the large populations of anonymous agents [18], etc.

Recently, the semi-tensor product (STP) of matrices has been proposed [19], which generalizes the conventional matrix product to two arbitrary matrices. Using STP approach, many important results have been obtained for the analysis and control design of Boolean networks [20–23], mix-valued logical networks [24], graph theory [25], dynamic analysis of networked evolutionary games [26,27], etc. Moreover, there are some recent papers considering the Boolean control networks with time delays in states and investigating their controllability [28]. Stimulated by the above mentioned researches, we consider the modeling and stability of a class of finite EGs with time delays in strategies.

The rest of this paper is organized as follows. Section 2 gives a brief review of the STP and some fundamental concepts for finite games. In Section 3, we convert the dynamics of evolutionary games with time delays into their algebraic forms by using STP approach. The stability condition of the delayed evolutionary potential game is considered in Section 4. Section 5 is a brief conclusion.

2 Preliminaries

2.1 Semi-tensor product of matrices

In this subsection, the STP method is introduced. Using the STP, the dynamics of a finite EG can be easily formulated as algebraic expressions. It has also been used for verifying whether a finite game is potential or not. We refer to [19] for all the concepts and results in this section. First, we give some notations:

- $\mathcal{M}_{m\times n}$: the set of $m\times n$ real matrices;
- $\mathcal{D}_k := \{1,2,\dots,k\}, \quad k \geqslant 2$;
- δ_n^i: the i-th column of the identity matrix I_n;
- $\Delta_k := \{\delta_k^i | i=1,\dots,k\}$;
- A matrix $L\in\mathcal{M}_{m\times n}$ is called a logical matrix if the columns of L are elements of Δ_m. Then $\mathcal{L}_{m\times n}$ denotes the set of $m\times n$ logical matrices;
- If $L\in\mathcal{L}_{m\times n}$, it can be expressed as $L=[\delta_m^{i_1},\delta_m^{i_2},\dots,\delta_m^{i_n}]$. We simply denote it as $L=\delta_m[i_1,i_2,\dots,i_n]$;
- $\mathbf{1}_k := (\underbrace{1,1,\dots,1}_{k})^{\mathrm{T}}$.

Definition 1 ([19])**.** Let $A\in\mathcal{M}_{m\times n}$, $B\in\mathcal{M}_{p\times q}$. Denote by $t=\mathrm{lcm}\{n,p\}$ the least common multiple of n and p. Then we define the STP of A and B as

$$A \ltimes B := \left(A\otimes I_{t/n}\right)\left(B\otimes I_{t/p}\right) \in \mathcal{M}_{mt/n\times qt/p}, \tag{1}$$

where $\otimes$ is the Kronecker product.

The STP keeps almost all the major properties of the conventional matrix product available. Hence we can omit the symbol "$\ltimes$".

Proposition 1. Let $X\in\mathbb{R}^n$ be a column and M be a matrix. Then

$$X\ltimes M = (I_n\otimes M)\,X.$$

Proposition 2. Let $X \in \Delta_p$ and define a power reducing matrix $O_p^R := \delta_{p^2}[1, p+2, 2p+3, \ldots, p^2] \in \mathcal{L}_{p^2 \times p}$. Then

$$X^2 = O_p^R X.$$

Next, we define the swap matrix:

Definition 2. A matrix $W_{[m,n]} \in \mathcal{M}_{mn \times mn}$, defined by

$$W_{[m,n]} = \delta_{mn}[1, m+1, \ldots, (n-1)m+1; 2, m+2, \ldots, (n-1)m+2; \ \cdots \ ; m, 2m, \ldots, nm],$$

is called the (m,n)-th dimensional swap matrix.

Proposition 3. Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns. Then

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X.$$

Proposition 4. Let $X \in \Delta_p$ and $Y \in \Delta_q$ be two columns, and $p \geqslant 2$, $q \geqslant 2$. Define the rear and front deleting operators as

$$D_r^{[p,q]} := I_p \otimes \mathbf{1}_q^{\mathrm{T}}; \quad D_f^{[p,q]} := \mathbf{1}_p^{\mathrm{T}} \otimes I_q.$$

Then we have

$$D_r^{[p,q]} XY = X; \qquad D_f^{[p,q]} XY = Y.$$

Definition 3. (1) A function $f : \mathcal{D}_k^n \to \mathcal{D}_k$ is called a k-valued logical function. (As $k = 2$ it is called a Boolean function.)

(2) A function $c : \mathcal{D}_k^n \to \mathbb{R}$ is called a k-valued pseudo-logical function. (As $k = 2$ it is called a pseudo-Boolean function.)

If $x \in \mathcal{D}_k$, we identify $i \sim \delta_k^i$, $i = 1, 2, \ldots, k$, then we have $x \in \Delta_k$. This expression is called the vector form of x.

Theorem 1. Let $x_i \in \mathcal{D}_k$, $i = 1, \ldots, n$, $f : \mathcal{D}_k^n \to \mathbb{R}$ (or $f : \mathcal{D}_k^n \to \mathcal{D}_k$) be a k-valued pseudo-logical (or logical) function. Then there exists a unique matrix $M_f \in \mathbb{R}_{1 \times k^n}$ (or $M_f \in \mathcal{L}_{k \times k^n}$), such that in vector form we have

$$f(x_1, x_2, \ldots, x_n) = M_f \ltimes_{i=1}^n x_i, \quad x_i \in \Delta_k,$$

where M_f is called the structure matrix of f, which can be determined by

$$\mathrm{Col}_j(M_f) = f(\delta_{k^n}^j), \quad j = 1, 2, \ldots, k^n.$$

Definition 4. Let $M \in \mathcal{M}_{p \times m}$, $N \in \mathcal{M}_{q \times m}$. Then the Khatri-Rao product is defined as

$$M * N = [\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N) \cdots \mathrm{Col}_m(M) \ltimes \mathrm{Col}_m(N)].$$

Proposition 5. Let $u : \mathcal{D}_k^n \to \mathcal{D}_p$ and $v : \mathcal{D}_k^n \to \mathcal{D}_q$ be expressed in their algebraic forms respectively as

$$u = M_u \ltimes_{i=1}^n x_i; \quad v = M_v \ltimes_{i=1}^n x_i, \quad x_i \sim \Delta_k,$$

where $M_u \in \mathcal{L}_{p \times k^n}$ and $M_v \in \mathcal{L}_{q \times k^n}$. Then

$$uv = (M_u * M_v) \ltimes_{i=1}^n x_i.$$

2.2 Finite game

Definition 5 ([29]). A finite game, denoted by $G=(N,S,C)$, consists of three fundamental ingredients:

(i) $N=\{1,2,\dots,n\}$ is the set of players;

(ii) $S=\prod_{i=1}^n S_i$ is called the strategy profile, where $S_i=\{1,2,\dots,k\}$ is the set of strategies for every player $i\in N$. The strategies of all players but the i-th one are denoted by $x_{-i}\in S_{-i}:=\prod_{j\neq i}S_j$;

(iii) $C=(c_1,\dots,c_n)\in\mathbb{R}^n$ with $c_i:S\to\mathbb{R}$ defined as

$$c_i:=c_i(x_1,\dots,x_n)=V_i^c\ltimes_{j=1}^n x_j,\quad x_j\in S_j,\quad j=1,\dots,n,\quad i=1,\dots,n, \tag{2}$$

is called the payoff function of player i.

In game theory, the basic solution of a noncooperative game is Nash equilibrium. A pure Nash equilibrium is a strategy profile from which no player can unilaterally deviate and increase its payoff. Formally, a strategy profile $x^*=(x_1^*,\dots,x_n^*)$ is a Nash equilibrium if

$$c_i(x_i^*,x_{-i}^*)\geqslant c_i(x_i,x_{-i}^*),\quad \forall x_i\in S_i,\ i=1,\dots,n.$$

As a special class of games, the potential game has many nice properties, hence has been used in many control problems. The following definition and properties are from [30].

Definition 6. G is said to be an (exact) potential game, if there exists a function $P:S\to\mathbb{R}$, such that for every $i\in N$ and every $x_{-i}\in S_{-i}$,

$$c_i(x_i,x_{-i})-c_i(\bar{x}_i,x_{-i})=P(x_i,x_{-i})-P(\bar{x}_i,x_{-i}),\quad \forall x_i,\bar{x}_i\in S_i.$$

P is called the potential function of G.

In the following we present a useful method to verify whether a game is potential and give an easy formula to calculate the potential function and refer to [31] for details.

Define

$$\psi_i=I_{k^{i-1}}\otimes\mathbf{1}_k\otimes I_{k^{n-i}},\quad i=1,\dots,n,$$

and set

$$\xi_i\in\mathbb{R}^{k^{n-1}},\quad i=1,\dots,n;\quad b_i:=(V_i^c-V_1^c)^{\mathrm{T}}\in\mathbb{R}^{k^n},\quad i=2,\dots,n,$$

where V_i^c is the structure vector of the payoff c_i. Then we construct a linear system, called the potential equation as

$$\Psi\xi=b, \tag{3}$$

where

$$\Psi=\begin{bmatrix}-\psi_1 & \psi_2 & 0 & \cdots & 0\\ -\psi_1 & 0 & \psi_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ -\psi_1 & 0 & 0 & \cdots & \psi_n\end{bmatrix};\quad \xi=\begin{bmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{bmatrix};\quad b=\begin{bmatrix}b_2\\ b_3\\ \vdots\\ b_n\end{bmatrix}.$$

Theorem 2 ([31]). A finite game G is potential, if and only if the potential equation (3) has solution. Moreover, the potential P can be calculated by

$$V_P=V_1^c-\xi_1^{\mathrm{T}}(\mathbf{1}_k^{\mathrm{T}}\otimes I_k), \tag{4}$$

where $k=|S_i|$ denotes the number of strategies for each player.

3 Dynamics of delayed EGs

In this section, we consider a class of finite evolutionary games where the strategies of all players have two kinds of time delays. First, we give the problem formulation.

3.1 Problem formulation

Assuming that a finite game G is played repetitively, denote it by G_∞. The evolutionary process of G_∞ is determined by certain specialized strategy updating rule, which is used to describe how a player updates its strategy at each time moment. Assume the strategy of player i at time $t+1$, denoted as $x_i(t+1)$, depends on all previous strategies of all players, that is,

$$x_i(t+1) = f_i(x_1(t), \ldots, x_n(t), x_1(t-1), \ldots, x_n(t-1), \ldots, x_1(0), \ldots, x_n(0)), \quad i = 1, 2, \ldots, n.$$

In this paper, we only consider one-step memory dynamics, which is described as

$$x_i(t+1) = f_i(x_1(t), \ldots, x_n(t)), \quad i = 1, 2, \ldots, n, \tag{5}$$

where $x_i(t) \in \mathcal{D}_k$ is the strategy of player i at time t, and $f_i : \mathcal{D}_k^n \to \mathcal{D}_k$ is the k-valued logical function, which is determined by the strategy updating rule. Then G_∞ is called a finite evolutionary game (EG), and Eq. (5) is called its strategy dynamics. In fact, the strategy dynamics (5) is a standard k-valued logical dynamic system.

First, we consider the strategy dynamics of an EG (5) with time invariant delay τ, which is expressed as follows:

$$x_i(t+1) = f_i(x_1(t-\tau), \ldots, x_n(t-\tau)), \quad i = 1, 2, \ldots, n, \tag{6}$$

where τ is a nonnegative integer delay.

Note that in many cases, time delays are time varying. Similar to (6), a more general EG with time varying delays is described as

$$x_i(t+1) = f_i(x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))), \quad i = 1, 2, \ldots, n, \tag{7}$$

where $\tau(t)$ are the nonnegative integers satisfying the following:

A1. There exists a smallest integer τ such that $\max_{t \geqslant 0}\{\tau(t)\} \leqslant \tau < +\infty$.

The Eqs. (6) and (7) are both called the delayed evolutionary game dynamics. Next, we provide the algebraic forms of the delayed evolutionary game dynamics.

3.2 Algebraic forms of delayed EGs

If we use the vector form to express the logical variables, that is, set the equivalence $\mathcal{D}_k \sim \Delta_k$, (precisely, $i \sim \delta_k^i$, $i = 1, \ldots, k$), then for each logical function f_i in (5), we can find its structure matrix $M_i \in \mathcal{L}_{k \times k^n}$, such that the dynamics (5) can be converted into its algebraic form as

$$x(t+1) = Mx(t), \tag{8}$$

where $x(t) = \ltimes_{i=1}^n x_i(t)$, $M = M_1 * M_2 * \cdots * M_n \in \mathcal{L}_{k^n \times k^n}$ is called the transition matrix of (8).

Similarly, the logical equations in (6) can be represented equivalently as

$$x(t+1) = Lx(t-\tau), \tag{9}$$

where $x(t-\tau) = \ltimes_{i=1}^n x_i(t-\tau)$, $L \in \mathcal{L}_{k^n \times k^n}$ is called the transition matrix of (9).

We can also convert (7) into its algebraic form as

$$x(t+1) = \widetilde{L}x(t-\tau(t)). \tag{10}$$

For the sake of convenience in analysis, we intend to transform (9) and (10) to the standard discrete time dynamic systems without time delays by certain transformations.

Introducing a new index as $\alpha = T(\tau+1)$, $T = 0, 1, \ldots$, we define

$$y(T) := \ltimes_{i=T(\tau+1)}^{T(\tau+1)+\tau} x(i) = \ltimes_{i=\alpha}^{\alpha+\tau} x(i) \in \Delta_{k^{(\tau+1)n}}, \quad T = 0, 1, \ldots.$$

Using the strategy profile dynamics (9), we can obtain that

$$\begin{aligned} y(T+1) &= \ltimes_{i=(T+1)(\tau+1)}^{(T+1)(\tau+1)+\tau} x(i) = \ltimes_{i=\alpha+\tau+1}^{\alpha+2\tau+1} x(i) = x(\alpha+\tau+1)x(\alpha+\tau+2)\cdots x(\alpha+2\tau+1) \\ &= Lx(\alpha)Lx(\alpha+1)\cdots Lx(\alpha+\tau) \\ &= L(I_{k^n}\otimes L)(I_{k^{2n}}\otimes L)\cdots(I_{k^{\tau n}}\otimes L)x(\alpha)\cdots x(\alpha+\tau) := L_\tau y(T). \end{aligned}$$

Then we have the following result.

Proposition 6. The dynamics (6) (or (9)) is equivalent to

$$y(T+1) := L_\tau y(T), \quad T=0,1,\ldots, \tag{11}$$

where $L_\tau = \prod_{i=1}^{\tau+1}(I_{k^{(i-1)n}}\otimes L) \in \mathcal{L}_{k^{(\tau+1)n}\times k^{(\tau+1)n}}$ is called the time invariant transition matrix of (11).

Since $y(T)=\ltimes_{i=\alpha}^{\alpha+\tau}x(i)$ is a one-to-one correspondence, according to Proposition 4, we have

$$x(i) = D_r^{[k^n,k^{\tau n}]} W_{[k^{(i-\alpha)n},k^n]} y(T), \quad i=\alpha,\ldots,\alpha+\tau. \tag{12}$$

For the dynamics (10) with time varying delays $\tau(t)$ satisfying A1, which have the lowest upper bound τ, we define

$$J_T = [T(\tau+1),(T+1)(\tau+1)), \quad T=0,1,\ldots,$$

as the T-th period of $\tau+1$. Then it is clear that $y(T+1)$ is uniquely determined by $y(T)$. We therefore have the following result.

Proposition 7. The dynamics (7) (or (10)) is equivalent to

$$y(T+1) := L(T)y(T), \quad T=0,1,\ldots, \tag{13}$$

where $L(T)\in\mathcal{L}_{k^{(\tau+1)n}\times k^{(\tau+1)n}}$ is called the time varying transition matrix of (13).

Similar to (12), we have

$$x(T(\tau+1)+j) = D_r^{[k^n,k^{\tau n}]} W_{[k^{jn},k^n]} y(T), \quad j=0,1,\ldots,\tau; \quad T=0,1,\ldots. \tag{14}$$

Because of the expression of $y(T)=\ltimes_{i=\alpha}^{\alpha+\tau}x(i)$, we call $x(i)=\ltimes_{k=1}^{n}x_k(i)$ the strategy profile and $X(t)=(x(t),x(t+1),\ldots,x(t+\tau))$ the profile trajectory of length $\tau+1$. The profile trajectory means a sequence of strategy profiles of length $\tau+1$. By building a bijective mapping $\ltimes_{i=t}^{t+\tau}: \Delta_{k^n}\longrightarrow\Delta_{k^{(\tau+1)n}}$, we can replace $X(t)$ with $y(T)$.

It is obvious that the properties of an EG are uniquely determined by its strategy profile dynamics. The profile dynamics is determined by the strategy updating rule (SUR). In this paper, we only consider the myopic best response adjustment rule (MBRAR) [32], which is described as follows.

Construct a set of optimal response set of strategies at t as

$$BR_i(t) = \mathrm{argmax}_{x_i\in S_i}\, c_i(x_i,x_{-i}(t)).$$

Then

- Case 1. If $x_i(t)\in BR_i(t)$, then $x_i(t+1)=x_i(t)$;
- Case 2. If $x_i(t)\notin BR_i(t)$, then (i) Deterministic model: choose the smallest j, such that $x_j\in BR_i(t)$, and set $x_i(t+1)=x_j$; (ii) Stochastic model: choose any $x_j\in BR_i(t)$, with equal probability $p=1/|BR_i(t)|$.

For the delayed evolutionary game dynamics, if we use MBRAR as its strategy updating rule, considering the different updating moments, we can obtain the different evolutionary dynamics.

(1) Parallel MBRAR. All the players update their strategies simultaneously. This SUR will lead to (11) (or (13)) directly.

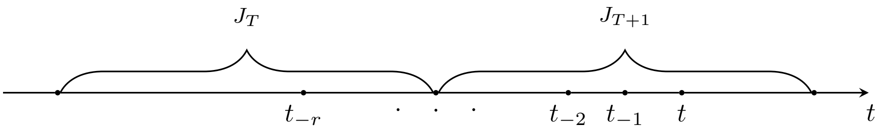

Figure 1 Time steps.

(2) Sequential MBRAR. At each time only one player (randomly chosen) updates its strategy. Precisely, for (6), the dynamics can be described as

$$\begin{cases} x_i(t+1) = f_i(x_1(t-\tau), \ldots, x_n(t-\tau)), \quad i \in N, \\ x_j(t+1) = x_j(t-\tau), \quad j \neq i, \end{cases} \tag{15}$$

where f_i obeys the MBRAR as described in above.

As for (7), we have the dynamics as

$$\begin{cases} x_i(t+1) = f_i(x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))), \quad i \in N, \\ x_j(t+1) = x_j(t-\tau(t)), \quad j \neq i. \end{cases} \tag{16}$$

4 Stability of delayed EPG

A finite potential game has finite improvement property [30]. That is, assume at each time moment, only one player, say i, is allowed to update its strategy to maximize its payoff. Correspondingly, the potential function is increasing at each updating step.

$$\begin{aligned} c_i(x_i(t+1), x_{-i}(t)) \geqslant c_i(x_i(t), x_{-i}(t)) \Longrightarrow P(x_i(t+1), x_{-i}(t)) \geqslant P(x_i(t), x_{-i}(t)), \\ \forall x_i(t+1), x_i(t) \in S_i, \quad x_{-i}(t) \in S_{-i}. \end{aligned} \tag{17}$$

The following is an essential result of evolutionary potential games (EPGs) without time delay [30].

Lemma 1. Consider a finite EPG without time delay. If its SUR is the sequential MBRAR, the EPG without time delay converges to a pure Nash equilibrium after finite step evolutions.

Now we give the definition of a delayed EPG.

Definition 7. Let G be a finite potential game. Consider an evolutionary game G_∞, where the strategy profile dynamics has time delays such as (6) (or (7)), then this EPG is called a delayed EPG.

Next, we will investigate the stability of a delayed EPG.

Theorem 3. Consider an EPG with time varying delays satisfying A1. If its SUR is the sequential MBRAR as described in (16), the profile dynamics (7) will converge to a pure Nash equilibrium x^* after finite updating steps.

Proof. From A1, we assume $\tau(t) \leqslant \tau$ and set $d_T = \min\{P(x(t)) \mid t \in J_T\}$, where P is the potential function of G.

Now consider an arbitrary $t \in J_{T+1}$, and denote by $t_{-1} := t - \tau(t) - 1$, $t_{-2} := t_{-1} - \tau(t_{-1}) - 1, \ldots$, which are a sequence of predecessors of t. Then there exists a smallest $r > 0$ such that $t_{-r} \in J_T$ (See Figure 1). As long as $x(\xi)$ are not constant, where $\xi \in J_T \cup J_{T+1}$, we have

$$P(x(t)) > P(x(t_{-1})) > \cdots > P(x(t_{-r})) \geqslant d_T. \tag{18}$$

It follows that

$$d_{T+1} > d_T. \tag{19}$$

So as long as $x(\xi)$ are not constant, $\xi \in J_T \cup J_{T+1}$, d_T is strictly increasing.

Since the set of strategy profiles is finite, i.e., $|S| < \infty$, before converging to a fixed point, we have

$$d_{T+1} - d_T \geqslant \epsilon > 0. \tag{20}$$

Table 1 Payoff matrix of example 1

Payoffs\profiles	111	112	121	122	211	212	221	222
c_1	0	1	1	−2	−2	1	1	0
c_2	0	1	−2	1	1	−2	1	0
c_3	0	−2	1	1	1	1	−2	0
f_1	1	1	1	2	1	2	2	2
f_2	1	1	1	2	1	2	2	2
f_3	1	1	1	2	1	2	2	2

But

$$\max_{T\geqslant 0} d_T \leqslant \max_{x\in S} P(x) < \infty.$$

It follows that Eq. (20) holds for only finite T. That is, after finite steps (each step is of length $\tau+1$), the profile dynamics will reach a fixed point x^*.

Finally, we have only to prove that x^* is a (pure) Nash equilibrium. Assume x^* is not a Nash equilibrium. Note that by the convergent property, there exists a t_0 such that

$$x(t) = x^*, \quad t \geqslant t_0.$$

Now we can find a T^0, such that

$$T^0_{-1} = T^0 - \tau(T^0) - 1 \geqslant t_0.$$

Then at T^0 the players have the information about the strategy profile as

$$x(T^0_{-1}) = x^*.$$

By assumption, x^* is not a Nash equilibrium. Hence, there is at least one player, who can change his strategy to improve his payoff. This fact leads to that

$$x(T^0+1) \neq x(T^0),$$

which is a contradiction. We conclude that x^* is a Nash equilibrium.

Finally, because of the delayed information, it is possible that the trajectory goes from one Nash equilibrium to another Nash equilibrium. But according to (20), the profile dynamics can not converge to a limit cycle consisting of Nash equilibria.

We give a simple example to illustrate our theoretical result.

Example 1. Consider the Palm Up-Down Game G with $N=\{1,2,3\}$ and $S_1=S_2=S_3=\{1,2\}$, which has the payoff matrix as in Table 1. Using MBRAR, the best responding strategies f_i can be figured out in Table 1. According to Theorem 2, it is easily verified that G is potential and its potential function is $P(x) = V_p x + c_0 = (0,-2,-2,-2,-2,-2,-2,0)x + c_0$, where c_0 is an arbitrary number.

Assume that the time delays are time varying and the SUR is the sequential MBRAR as (16). Player i is chosen to update its strategy at time t and the dynamics can be described as

$$\begin{cases} x_i(t+1) = L_i x(t-\tau(t)), \\ x_j(t+1) = x_j(t-\tau(t)), \quad i\in N; \quad j\neq i, \end{cases}$$

where $L_1 = L_2 = L_3 = \delta_8[1,1,1,2,1,2,2,2]$. Then we have

$$x(t+1) = M_i x(t-\tau(t)),$$

where

$$\begin{aligned} M_1 &= L_1(I_2\otimes O_4^R) = \delta_8[1,2,3,8,1,6,7,8]; \\ M_2 &= (I_2\otimes L_2)O_2^R(I_4\otimes O_2^R) = \delta_8[1,2,1,4,5,8,7,8]; \\ M_3 &= (I_4\otimes L_3)O_4^R = \delta_8[1,1,3,4,5,6,8,8]. \end{aligned}$$

Now assume that the time delays are described as

$$\tau(t) = \begin{cases} 0, & t \text{ is even;} \\ 1, & t \text{ is odd.} \end{cases} \tag{21}$$

Using the sequential MBRAR in periodical type, that is, three players update their strategies periodically by turn, we have

$$\begin{cases} x(1) = M_1 x(0), \\ x(2) = M_2 x(0), \\ x(3) = M_3 x(2), \\ x(4) = M_1 x(2), \\ x(5) = M_2 x(4), \\ x(6) = M_3 x(4), \\ \qquad \vdots \end{cases} \tag{22}$$

According to (13), we can obtain the profile trajectory dynamics as

$$\begin{cases} y(T+1) = T_1 y(T), & T = 3p, \\ y(T+1) = T_2 y(T), & T = 3p+1, \\ y(T+1) = T_3 y(T), & T = 3p+2, \ p = 0, 1, \ldots, \end{cases} \tag{23}$$

where

$$T_1 = M_2(I_8 \otimes M_3 M_2) O_8^R D_r^{[2^3,2^3]};$$
$$T_2 = M_1(I_8 \otimes M_2 M_1) O_8^R D_r^{[2^3,2^3]};$$
$$T_3 = M_3(I_8 \otimes M_1 M_3) O_8^R D_r^{[2^3,2^3]}.$$

It is easy to check that

$$y(k) = y(3) = T_3 T_2 T_1 y(0) = T y(0), \quad k \geqslant 3, \tag{24}$$

where

$$\begin{aligned} T = \delta_{64} \ [&1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ &1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64 \\ &1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64 \\ &64\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64\ 64]. \end{aligned}$$

According to (14), we have

$$x(t) = x(6) = D_r^{[2^3,2^3]} y(3) = D_r^{[2^3,2^3]} T_3 T_2 T_1 y(0) = \widehat{T} y(0), \quad t \geqslant 6,$$

where

$$\begin{aligned} \widehat{T} = \delta_{64} \ [&1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8 \\ &1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8]. \end{aligned}$$

Hence, the EPG with time varying delays as (21) will converge to x^* after three updating steps (each step is of length 2), where $x^* = \delta_8^1 \sim (1,1,1)$ and $x^* = \delta_8^8 \sim (2,2,2)$ are two pure Nash equilibria of G.

5 Conclusion

In this paper, we considered the dynamics and stability of evolutionary games with time delays in strategies. Using the STP of matrices, the profile trajectory dynamics with time delays can be converted into the standard discrete time dynamic systems, and then the mathematical models of the delayed EGs were obtained. Finally, a sufficient condition for the stability of the delayed EPG was provided.

Conflict of interest The authors declare that they have no conflict of interest.

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