SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

September 2016, Vol. 59 092207:1–092207:18
doi: 10.1007/s11432-015-5489-1

Gaussian approximate filter for stochastic dynamic systems with randomly delayed measurements and colored measurement noises

Yonggang ZHANG & Yulong HUANG*

Department of Automation, Harbin Engineering University, Harbin 150001*, China*

Received July 23, 2015; accepted September 22, 2015; published online August 23, 2016

Abstract In this paper, a new Gaussian approximate (GA) filter for stochastic dynamic systems with both one-step randomly delayed measurements and colored measurement noises is presented. For linear systems, a Kalman filter can be obtained to include one-step randomly delayed measurements and colored measurement noises. On the other hand, for nonlinear stochastic dynamic systems, different GA filters can be developed which exploit numerical methods to compute Gaussian weighted integrals involved in the proposed Bayesian solution. Existing GA filter with one-step randomly delayed measurements and existing GA filter with colored measurement noises are special cases of the proposed GA filter. The efficiency and superiority of the proposed method are illustrated in a numerical example concerning a target tracking problem.

Keywords state estimation, Gaussian approximate filter, one-step randomly delayed measurements, colored measurement noises, nonlinear stochastic dynamic systems, Bayesian estimation

Citation Zhang Y G, Huang Y L. Gaussian approximate filter for stochastic dynamic systems with randomly delayed measurements and colored measurement noises. Sci China Inf Sci, 2016, 59(9): 092207, doi: 10.1007/s11432-015-5489-1

1 Introduction

Nonlinear filtering has been widely used in signal processing, target tracking, communications and control, however the closed form solution of its posterior probability density function (PDF) is unavailable, thus optimal solution normally doesn't exist and approximate methods are necessary to obtain suboptimal nonlinear filters [1]. Gaussian approximation to such PDFs is the most conveniently and widely used by the fact that its correspondingly derived Gaussian approximate (GA) filter consistently provides a computationally cost-effective estimation for nonlinear systems with acceptable accuracy as required in many practical applications [2–9]. So far, several forms of GA filters have been developed based on different rules, as listed in Table 1.

In some applications with limited bandwidth of the communication channel and high sampling frequency, such as target tracking and control, their measurements may be randomly delayed and their measurement noises may be colored. The randomly delayed measurements are induced by limited bandwidth of the communication channel [3,27], and colored measurement noises are induced by high sampling

* Corresponding author (email: heuedu@163.com)

 info.scichina.com link.springer.com

Table 1 Existing nonlinear GA filtering algorithm

Reference	Algorithm	Rule
[10]	Extended Kalman filter (EKF)	Taylor series expansion
[11]	Central difference Kalman filter (CDKF)	Polynomial interpolation rule
[12]	Divided difference filter (DDF)	Stirling interpolation rule
[13, 14]	Unscented Kalman filter (UKF)	Unscented transform (UT)
[11]	Gauss-Hermite quadrature filter (GHQF)	Gauss-Hermite quadrature rule
[15]	Cubature Kalman filter (CKF)	Third-degree spherical-radial cubature rule
[16]	Sparse-grid quadrature filter (SGQF)	Sparse-grid theory
[17]	Transformed UKF (TUKF)	Transformed UT
[18]	High-degree CKF (HCKF)	High-degree spherical-radial cubature rule
[19]	Stochastic integration filter (SIF)	Stochastic integral rule
[20]	Cubature quadrature Kalman filter (CQKF)	Spherical radial cubature and Gauss quadrature rule
[21, 22]	Embedded CKF	Third-degree embedded cubature rule
[23]	Quasi-stochastic integration filter (QSIF)	Quasi-stochastic integration rule
[24, 25]	Spherical simplex-radial CKF (SSRCKF)	Spherical simplex-radial rule
[26]	Interpolatory CKF	Interpolatory cubature rule

frequency because the noise correlation between the successive samples of the noise can't be ignored in the case of high sampling frequency [7, 8, 28, 29]. On one hand, to solve the problem of state estimation for nonlinear stochastic dynamic systems in which the measurements are randomly delayed by one or two sampling time, an improved EKF and an improved UKF have been proposed [30, 31]. Wang et al. proposed GA filter and smoother which give general and common frameworks for addressing the state estimation problem when the measurements are randomly delayed by one sampling time [3, 4]. A GA filter for nonlinear systems with one-step randomly delayed measurements and correlated noises is proposed in [5]. Zhang et al. provided a general framework solution to state estimation of Gaussian filter for nonlinear systems with multiple step randomly delayed measurements [32]. To assess the achievable optimal performance of nonlinear state estimator with one-step randomly delayed measurements, a new conditional posterior Cramér-Rao lower bound is proposed in [33]. On the other hand, to solve the problem of state estimation for nonlinear stochastic dynamic systems in which the measurement noises are colored, Wang et al. proposed GA filter and smoother based on measurement differencing scheme [7,8,28]. However, these nonlinear GA filters mentioned above are all unsuitable for achieving the state estimation of stochastic dynamic systems with randomly delayed measurements and colored measurement noises. Moreover, the state estimation problem in this case also can not be solved by simply combining existing methods designed only for nonlinear systems with randomly delayed measurements and methods designed only for nonlinear systems with colored measurement noises, since measurement noises can't be decorrelated based on measurement differencing scheme for the case of randomly delayed measurements, as will be explained in Subsection 2.2.

To solve this problem, a new GA filter is developed under Bayesian estimation framework in this paper. For linear systems, a Kalman filter can be obtained to include one-step randomly delayed measurements and colored measurement noises. On the other hand, for nonlinear systems, different GA filters can be developed which exploit numerical methods to compute the Gaussian weighted integrals involved in the proposed Bayesian solution. The efficiency and superiority of the proposed method is illustrated in a numerical example concerning a target tracking problem.

The remainder of the paper is organized as follows. The problem formulation is given in Section 2. A new GA filter for nonlinear systems with randomly delayed measurements and colored measurement noises is derived in Section 3. Also comparisons with existing methods are provided in Section 3. Simulations are given in Section 4. Concluding remarks are drawn in Section 5.

2 Problem formulation

2.1 State-space model with one-step randomly delayed measurements and colored measurement noises

Consider the following discrete-time stochastic dynamic systems as shown by the state-space model:

$$\boldsymbol{x}_k = \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) + \boldsymbol{w}_{k-1}, \quad \text{(process equation)}, \tag{1}$$

$$\boldsymbol{z}_k = \boldsymbol{h}_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, \quad \text{(ideal measurement equation)}, \tag{2}$$

where k is the discrete time index, $\boldsymbol{x}_k \in \mathbb{R}^n$ is the state vector, $\boldsymbol{z}_k \in \mathbb{R}^m$ is the undelayed measurement vector, $\boldsymbol{w}_k \in \mathbb{R}^n$ is the process noise, $\boldsymbol{v}_k \in \mathbb{R}^{n_z}$ is the colored measurement noise with a first-order autoregressive (AR) model, which can be formulated as [8]

$$\boldsymbol{v}_k = \boldsymbol{\Psi}_{k-1}\boldsymbol{v}_{k-1} + \boldsymbol{\xi}_{k-1}, \tag{3}$$

where $\boldsymbol{\Psi}_{k-1}$ is known correlation parameter, and $\boldsymbol{\Psi}_0 = \boldsymbol{0}$ which is reasonable in practical application because undelayed measurement $\boldsymbol{z}_0$ at $k = 0$ doesn't exist, and $\boldsymbol{w}_k \in \mathbb{R}^n$ and $\boldsymbol{\xi}_k \in \mathbb{R}^m$ are zero-mean Gaussian white noise vectors satisfying $\mathrm{E}[\boldsymbol{w}_k\boldsymbol{w}_l^{\mathrm{T}}] = \boldsymbol{Q}_k\delta_{kl}$ and $\mathrm{E}[\boldsymbol{\xi}_k\boldsymbol{\xi}_l^{\mathrm{T}}] = \boldsymbol{R}_k\delta_{kl}$ respectively, where δ_{kl} is the Kronecker delta function, and the initial state $\boldsymbol{x}_0$ is a Gaussian random vector with mean $\hat{\boldsymbol{x}}_{0|0}$ and covariance matrix $\boldsymbol{P}_{0|0}$, which is independent of $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$.

Remark 1. First-order AR model formulated in (3) has been widely accepted and used to represent the colored noise in many practical applications, such as target tracking [29], multipath parameters estimation of weak GPS signal [34], inertial navigation [35], channel and spectral estimation [36, 37], and speech processing [38].

Assuming that the first measurement, $\boldsymbol{z}_1$ arrives on time, but for $k \geqslant 2$, the ideal measurement $\boldsymbol{z}_k$ may be randomly delayed by one sample time and, in that case, a previous measurement $\boldsymbol{z}_{k-1}$ is used. Thus the available (real) measurements $\boldsymbol{y}_k$ at time k can be mathematically described as [3, 30]

$$\boldsymbol{y}_k = (1 - \gamma_k)\boldsymbol{z}_k + \gamma_k\boldsymbol{z}_{k-1}, \quad (k \geqslant 2), \tag{4}$$

where $\gamma_1 = 0$ and γ_k $(k \geqslant 2)$ is a Bernoulli random variable taking the value of zero or one with latency probability $p(\gamma_k = 1) = p_k$ $(k \geqslant 2)$ and $p_k \in [0, 1]$. Moreover, we assume that $\{\gamma_k, k \geqslant 2\}$, $\boldsymbol{x}_0$, $\{\boldsymbol{w}_k, k \geqslant 0\}$ and $\{\boldsymbol{\xi}_k, k \geqslant 0\}$ are mutually independent.

Remark 2. In many network-based engineering applications, such as vehicle management system of future generation aircraft [39], signal receiving process of a mobile phone based on network [40], interconnected network security assessment of power system [41], and GPS/INS integrated system for relative navigation in formation flight [3], their measurements may be subject to ineluctable random sensor delays. These random delays may be induced by multiplexed data communication networks in distributed control systems [39–41] or multiple sensor systems [3]. The one-step randomly delayed measurement model has been widely accepted and used to model these induced delays in many application based on the reasonable supposition that these induced delays are usually restricted so as not to exceed the sampling period [3, 39–41].

2.2 Difficulties of state estimation for stochastic dynamic systems formulated in (1)–(4)

Firstly, for the case of randomly delayed measurements and colored measurement noises, the real measurement noise is more intricate as compared with the case of randomly delayed measurements or colored measurement noises. Using (2) and (3) in (4), randomly delayed measurement equation can be rewritten as

$$\boldsymbol{y}_k = [(1 - \gamma_k)\boldsymbol{h}_k(\boldsymbol{x}_k) + \gamma_k\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1})] + [(1 - \gamma_k)\boldsymbol{v}_k + \gamma_k\boldsymbol{v}_{k-1}] = \boldsymbol{h}'_k(\boldsymbol{x}_k, \boldsymbol{x}_{k-1}) + \boldsymbol{v}'_k, \tag{5}$$

where $\boldsymbol{v}'_k$ is the real measurement noise. It can be seen from (5) that $\boldsymbol{v}'_k$ consists of Bernoulli random variable γ_k and colored measurement noises $\boldsymbol{v}_k$ and $\boldsymbol{v}_{k-1}$. Consequently, the real measurement noise $\boldsymbol{v}'_k$ is non-Gaussian and colored, which increases the difficulty of state estimation under Bayesian estimation framework.

Secondly, the non-Gaussianity of systems formulated in (1)–(4) is increased due to the non-Gaussianity of real measurement noise $\boldsymbol{v}'_k$. Thirdly, the cross-correlation between colored measurement noise $\boldsymbol{v}_k$ and available measurements $\boldsymbol{y}_k$ becomes more intricate, for example, $\boldsymbol{P}^{vz}_{k,k-1|k-1}$ is incurred by randomly delayed measurements and colored measurement noises, however, it is inexistent in the case of randomly delayed measurements or colored measurement noises. Fourthly, existing standard GA filter [15], GA filter with randomly delayed measurements [3], GA filter with colored measurement noises [7] are all not suitable for achieving the sate estimation of stochastic dynamic systems with randomly delayed measurements and colored measurement noises because they use incorrect measurement model.

Finally, available measurements $\boldsymbol{y}_k$ can not be decorrelated based on existing measurement differencing scheme [8]. A new measurement is constructed based on measurement difference scheme as follows:

$$\boldsymbol{y}^*_k = \boldsymbol{y}_k - \boldsymbol{\Psi}_{k-1}\boldsymbol{y}_{k-1}. \tag{6}$$

Substituting (4) into (6), we can obtain

$$\boldsymbol{y}^*_k = [(1-\gamma_k)\boldsymbol{z}_k - (1-\gamma_{k-1})\boldsymbol{\Psi}_{k-1}\boldsymbol{z}_{k-1}] + [\gamma_k\boldsymbol{z}_{k-1} - \gamma_{k-1}\boldsymbol{\Psi}_{k-1}\boldsymbol{z}_{k-2}]. \tag{7}$$

Using (2) and (3) in (7), Eq. (7) can be formulated as

$$\begin{aligned}\boldsymbol{y}^*_k &= [(1-\gamma_k)\boldsymbol{h}_k(\boldsymbol{x}_k) - (1-\gamma_{k-1})\boldsymbol{\Psi}_{k-1}\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1}) + \gamma_k\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1}) - \gamma_{k-1}\boldsymbol{\Psi}_{k-1}\boldsymbol{h}_{k-2}(\boldsymbol{x}_{k-2})] \\ &\quad + [(\gamma_{k-1}-\gamma_k)\boldsymbol{\Psi}_{k-1}\boldsymbol{v}_{k-1} + (1-\gamma_k)\boldsymbol{\xi}_{k-1} + \gamma_k\boldsymbol{v}_{k-1} - \gamma_{k-1}\boldsymbol{\Psi}_{k-1}\boldsymbol{v}_{k-2}] \\ &= \boldsymbol{h}^*_k(\boldsymbol{x}_k, \boldsymbol{x}_{k-1}, \boldsymbol{x}_{k-2}) + \boldsymbol{v}^*_k.\end{aligned} \tag{8}$$

It is seen from (8) that $\boldsymbol{y}^*_k$ is still colored measurement. However, if all measurements can arrive on time, i.e. $\gamma_k = 0 (k \geqslant 1)$, then (8) can be rewritten as

$$\boldsymbol{y}^*_k = \boldsymbol{h}_k(\boldsymbol{x}_k) - \boldsymbol{\Psi}_{k-1}\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1}) + \boldsymbol{\xi}_{k-1}. \tag{9}$$

It can be seen from (9) that $\boldsymbol{y}^*_k$ is uncolored measurement when all measurements arrive on time. Thus, measurement difference method is not suitable for systems formulated in (1)–(4) due to the random measurement delay, and the state estimation problem can not be solved by simply combining existing methods designed only for systems with randomly delayed measurements and methods designed only for systems with colored measurement noises. These difficulties represent the main motivation and significance of this paper.

2.3 Preliminary of the proposed GA filter

In order to design a GA filter for the system formulated in (1)–(4), we require to obtain Gaussian approximation to the posterior PDF $p(\boldsymbol{x}_k|\boldsymbol{Y}_k)$, where $\boldsymbol{Y}_l = \{\boldsymbol{y}_i\}_{i=1}^l$ denotes the delayed measurements as formulated in (4). Then, the state estimation and corresponding estimation error covariance matrix are obtained by computing the first two moments of $p(\boldsymbol{x}_k|\boldsymbol{Y}_k)$, i.e.

$$p(\boldsymbol{x}_k|\boldsymbol{Y}_k) = N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k}, \boldsymbol{P}_{k|k}), \tag{10}$$

where $\hat{\boldsymbol{x}}_{k|k}$ and $\boldsymbol{P}_{k|k}$ are computed as follows:

$$\hat{\boldsymbol{x}}_{k|k} = \mathrm{E}[\boldsymbol{x}_k|\boldsymbol{Y}_k], \quad \boldsymbol{P}_{k|k} = \mathrm{E}[\tilde{\boldsymbol{x}}_{k|k}\tilde{\boldsymbol{x}}^{\mathrm{T}}_{k|k}|\boldsymbol{Y}_k], \tag{11}$$

where the expectation $\mathrm{E}[\cdot|\boldsymbol{Y}_k]$ is with respect to PDF $p(\boldsymbol{x}_k|\boldsymbol{Y}_k)$, $\tilde{\boldsymbol{x}}$ denotes the state estimation error vector and $(\cdot)^{\mathrm{T}}$ denotes vector transpose.

Similar to that in [3], we present two assumptions to deal with colored measurement noises and one-step randomly delayed measurements.

Assumption 1. The one-step predictive PDF of the state vector $\boldsymbol{x}_k$ conditioned on $\boldsymbol{Y}_{k-1}$ is Gaussian, i.e.

$$p(\boldsymbol{x}_k|\boldsymbol{Y}_{k-1}) = N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k-1}, \boldsymbol{P}_{k|k-1}), \tag{12}$$

where in the minimum mean square error (MMSE) sense, the state prediction vector $\hat{\boldsymbol{x}}_{k|k-1}$ and corresponding prediction error covariance matrix $\boldsymbol{P}_{k|k-1}$ denote the first two moments of $p(\boldsymbol{x}_k|\boldsymbol{Y}_{k-1})$ respectively i.e.

$$\hat{\boldsymbol{x}}_{k|k-1} = \mathrm{E}[\boldsymbol{x}_k|\boldsymbol{Y}_{k-1}], \tag{13}$$

$$\boldsymbol{P}_{k|k-1} = \mathrm{E}[\tilde{\boldsymbol{x}}_{k|k-1}\tilde{\boldsymbol{x}}^{\mathrm{T}}_{k|k-1}|\boldsymbol{Y}_{k-1}], \tag{14}$$

the expectation $\mathrm{E}[\cdot|\boldsymbol{Y}_{k-1}]$ is with respect to PDF $p(\boldsymbol{x}_{k-1}|\boldsymbol{Y}_{k-1})$ and $N(\cdot)$ denotes the normal distribution.

Assumption 2. The one-step predictive PDF of delayed measurement vector $\boldsymbol{y}_k$ conditioned on $\boldsymbol{Y}_{k-1}$ is Gaussian, i.e.

$$p(\boldsymbol{y}_k|\boldsymbol{Y}_{k-1}) = N(\boldsymbol{y}_k; \hat{\boldsymbol{y}}_{k|k-1}, \boldsymbol{P}^{yy}_{k|k-1}), \tag{15}$$

where in the MMSE sense, the delayed measurement prediction vector $\hat{\boldsymbol{y}}_{k|k-1}$ and corresponding prediction error covariance matrix $\boldsymbol{P}^{yy}_{k|k-1}$ denote the first two moments of $p(\boldsymbol{y}_k|\boldsymbol{Y}_{k-1})$ respectively i.e.

$$\hat{\boldsymbol{y}}_{k|k-1} = \mathrm{E}[\boldsymbol{y}_k|\boldsymbol{Y}_{k-1}], \tag{16}$$

$$\boldsymbol{P}^{yy}_{k|k-1} = \mathrm{E}[\tilde{\boldsymbol{y}}_{k|k-1}\tilde{\boldsymbol{y}}^{\mathrm{T}}_{k|k-1}|\boldsymbol{Y}_{k-1}], \tag{17}$$

Remark 3. Assumptions 1 and 2 are reasonable in many applications with mild nonlinearity because Gaussian random vector has the largest entropy among all random vectors of equal mean vector and covariance matrix, i.e. Gaussian distribution is the most random or the least structured of all distributions [42]. Moreover, these two assumptions have been widely accepted and used to design GA filters.

Before proposing a new GA filter, we define

$$\begin{cases} \hat{\boldsymbol{v}}_{k|k-1} = \mathrm{E}[\boldsymbol{v}_k|\boldsymbol{Y}_{k-1}], \quad \boldsymbol{P}^{vv}_{k|k-1} = \mathrm{E}[\tilde{\boldsymbol{v}}_{k|k-1}\tilde{\boldsymbol{v}}^{\mathrm{T}}_{k|k-1}|\boldsymbol{Y}_{k-1}], \quad \boldsymbol{P}^{xv}_{k|k-1} = \mathrm{E}[\tilde{\boldsymbol{x}}_{k|k-1}\tilde{\boldsymbol{v}}^{\mathrm{T}}_{k|k-1}|\boldsymbol{Y}_{k-1}], \\ \boldsymbol{P}^{xy}_{k|k-1} = \mathrm{E}[\tilde{\boldsymbol{x}}_{k|k-1}\tilde{\boldsymbol{y}}^{\mathrm{T}}_{k|k-1}|\boldsymbol{Y}_{k-1}], \quad \boldsymbol{P}^{vy}_{k|k-1} = \mathrm{E}[\tilde{\boldsymbol{v}}_{k|k-1}\tilde{\boldsymbol{y}}^{\mathrm{T}}_{k|k-1}|\boldsymbol{Y}_{k-1}], \\ \boldsymbol{P}^{xz}_{k|k-1} = \mathrm{E}[\tilde{\boldsymbol{x}}_{k|k-1}\tilde{\boldsymbol{z}}^{\mathrm{T}}_{k|k-1}|\boldsymbol{Y}_{k-1}], \quad \boldsymbol{P}^{vz}_{k|k-1} = \mathrm{E}[\tilde{\boldsymbol{v}}_{k|k-1}\tilde{\boldsymbol{z}}^{\mathrm{T}}_{k|k-1}|\boldsymbol{Y}_{k-1}], \\ \boldsymbol{P}^{zz}_{k-d|k-1} = \mathrm{E}[\tilde{\boldsymbol{z}}_{k-d|k-1}\tilde{\boldsymbol{z}}^{\mathrm{T}}_{k-d|k-1}|\boldsymbol{Y}_{k-1}], \quad \boldsymbol{P}^{xz}_{k,k-1|k-1} = \mathrm{E}[\tilde{\boldsymbol{x}}_{k|k-1}\tilde{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1}|\boldsymbol{Y}_{k-1}], \\ \boldsymbol{P}^{vz}_{k,k-1|k-1} = \mathrm{E}[\tilde{\boldsymbol{v}}_{k|k-1}\tilde{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1}|\boldsymbol{Y}_{k-1}], \quad \boldsymbol{P}^{xv}_{k|k} = \mathrm{E}[\tilde{\boldsymbol{x}}_{k|k}\tilde{\boldsymbol{v}}^{\mathrm{T}}_{k|k}|\boldsymbol{Y}_{k-1}], \quad \hat{\boldsymbol{z}}_{k-d|k-1} = \mathrm{E}[\boldsymbol{z}_{k-d}|\boldsymbol{Y}_{k-1}], \end{cases} \tag{18}$$

where $d = 1, 2$.

With Assumptions 1 and 2 and definitions in (18), a new GA filter for nonlinear systems with one-step randomly delayed measurements and colored measurement noises as formulated in (1)–(4) can be derived.

3 GA filter for nonlinear systems with randomly delayed measurements and colored measurement noises

In this section, the derivation of the proposed GA filter for the system formulated in (1)–(4) will be separated into three steps including one-step predictions of state and measurement noise given in Theorem 1, one-step prediction of delayed measurement given in Theorem 2, and the filtering updates of state and measurement noise given in Theorem 3. Before proposing Theorems 1–3, a standard result about Gaussian functions is first reviewed.

Lemma 1. If $\boldsymbol{F}$, $\boldsymbol{d}$, $\boldsymbol{\Gamma}$, $\boldsymbol{m}$ and $\boldsymbol{P}$ have appropriate dimensions and $\boldsymbol{\Gamma}$ and $\boldsymbol{P}$ are positive definition, it can be obtained that [4]

$$\int N(\boldsymbol{x}; \boldsymbol{F}\boldsymbol{\lambda} + \boldsymbol{d}, \boldsymbol{\Gamma})N(\boldsymbol{\lambda}; \boldsymbol{m}, \boldsymbol{P})\mathrm{d}\boldsymbol{\lambda} = N(\boldsymbol{x}; \boldsymbol{F}\boldsymbol{m} + \boldsymbol{d}, \boldsymbol{F}\boldsymbol{P}\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{\Gamma}). \tag{19}$$

3.1 One-step predictions of state and measurement noise

Theorem 1. With Assumption 1 and Gaussian approximation of $p(\boldsymbol{x}_{k-1}, \boldsymbol{v}_{k-1}|\boldsymbol{Y}_{k-1})$, the Gaussian approximation of $p(\boldsymbol{x}_k, \boldsymbol{v}_k|\boldsymbol{Y}_{k-1})$ has one-step prediction estimations $\hat{\boldsymbol{x}}_{k|k-1}$ and $\hat{\boldsymbol{v}}_{k|k-1}$, and prediction error covariance matrixes $\boldsymbol{P}_{k|k-1}$ and $\boldsymbol{P}^{vv}_{k|k-1}$, and cross-covariance matrix $\boldsymbol{P}^{xv}_{k|k-1}$ as the unified form:

$$\hat{\boldsymbol{x}}_{k|k-1} = \int_{\mathbb{R}^n} \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) N(\boldsymbol{x}_{k-1}; \hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{P}_{k-1|k-1}) \mathrm{d}\boldsymbol{x}_{k-1}, \tag{20}$$

$$\hat{\boldsymbol{v}}_{k|k-1} = \boldsymbol{\Psi}_{k-1} \hat{\boldsymbol{v}}_{k-1|k-1}, \tag{21}$$

$$\boldsymbol{P}_{k|k-1} = \int_{\mathbb{R}^n} \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) \boldsymbol{f}^{\mathrm{T}}_{k-1}(\boldsymbol{x}_{k-1}) N(\boldsymbol{x}_{k-1}; \hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{P}_{k-1|k-1}) \mathrm{d}\boldsymbol{x}_{k-1} - \hat{\boldsymbol{x}}_{k|k-1} \hat{\boldsymbol{x}}^{\mathrm{T}}_{k|k-1} + \boldsymbol{Q}_{k-1}, \tag{22}$$

$$\boldsymbol{P}^{vv}_{k|k-1} = \boldsymbol{\Psi}_{k-1} \boldsymbol{P}^{vv}_{k-1|k-1} \boldsymbol{\Psi}^{\mathrm{T}}_{k-1} + \boldsymbol{R}_{k-1}, \tag{23}$$

$$\boldsymbol{P}^{xv}_{k|k-1} = \left[\int_{\mathbb{R}^n} \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) \hat{\boldsymbol{v}}^{\mathrm{T}}_{x,k-1|k-1} N(\boldsymbol{x}_{k-1}; \hat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{P}_{k-1|k-1}) \mathrm{d}\boldsymbol{x}_{k-1}\right] \boldsymbol{\Psi}^{\mathrm{T}}_{k-1} - \hat{\boldsymbol{x}}_{k|k-1} \hat{\boldsymbol{v}}^{\mathrm{T}}_{k|k-1}, \tag{24}$$

$$\hat{\boldsymbol{v}}_{x,k-1|k-1} = \hat{\boldsymbol{v}}_{k-1|k-1} + (\boldsymbol{P}^{xv}_{k-1|k-1})^{\mathrm{T}} \boldsymbol{P}^{-1}_{k-1|k-1} (\boldsymbol{x}_{k-1} - \hat{\boldsymbol{x}}_{k-1|k-1}), \tag{25}$$

Proof. Considering that $\boldsymbol{w}_{k-1} \sim N(\boldsymbol{w}_{k-1}; \boldsymbol{0}, \boldsymbol{Q}_{k-1})$ is independent of $\boldsymbol{Y}_{k-1}$ and according to the definition of $\hat{\boldsymbol{x}}_{k|k-1}$ and $\boldsymbol{P}_{k|k-1}$ in (13)–(14), we have

$$\hat{\boldsymbol{x}}_{k|k-1} = \mathrm{E}[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Y}_{k-1}], \tag{26}$$

$$\boldsymbol{P}_{k|k-1} = \mathrm{E}[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) \boldsymbol{f}^{\mathrm{T}}_{k-1}(\boldsymbol{x}_{k-1})|\boldsymbol{Y}_{k-1}] - \hat{\boldsymbol{x}}_{k|k-1} \hat{\boldsymbol{x}}^{\mathrm{T}}_{k|k-1} + \boldsymbol{Q}_{k-1}. \tag{27}$$

Substituting (10) into (26) and (27) obtains (20) and (22).

According to the Bayesian theorem and Markov properties of the colored measurement noise, we have

$$\begin{aligned} p(\boldsymbol{v}_k|\boldsymbol{Y}_{k-1}) &= \int_{\mathbb{R}^{n_z}} p(\boldsymbol{v}_k, \boldsymbol{v}_{k-1}|\boldsymbol{Y}_{k-1}) \mathrm{d}\boldsymbol{v}_{k-1} = \int_{\mathbb{R}^{n_z}} p(\boldsymbol{v}_k|\boldsymbol{v}_{k-1}) p(\boldsymbol{v}_{k-1}|\boldsymbol{Y}_{k-1}) \mathrm{d}\boldsymbol{v}_{k-1} \\ &= \int_{\mathbb{R}^{n_z}} N(\boldsymbol{v}_k; \boldsymbol{\Psi}_{k-1} \boldsymbol{v}_{k-1}, \boldsymbol{R}_{k-1}) p(\boldsymbol{v}_{k-1}|\boldsymbol{Y}_{k-1}) \mathrm{d}\boldsymbol{v}_{k-1}. \end{aligned} \tag{28}$$

Using Gaussian approximation of $p(\boldsymbol{v}_{k-1}|\boldsymbol{Y}_{k-1})$ and according to Lemma 1, $p(\boldsymbol{v}_k|\boldsymbol{Y}_{k-1})$ can be computed as Gaussian, i.e.

$$\begin{aligned} p(\boldsymbol{v}_k|\boldsymbol{Y}_{k-1}) &= \int_{\mathbb{R}^m} N(\boldsymbol{v}_k; \boldsymbol{\Psi}_{k-1} \boldsymbol{v}_{k-1}, \boldsymbol{R}_{k-1}) N(\boldsymbol{v}_{k-1}; \hat{\boldsymbol{v}}_{k-1|k-1}, \boldsymbol{P}^{vv}_{k-1|k-1}) \mathrm{d}\boldsymbol{v}_{k-1} \\ &= N(\boldsymbol{v}_k; \hat{\boldsymbol{v}}_{k|k-1}, \boldsymbol{P}^{vv}_{k|k-1}), \end{aligned} \tag{29}$$

where $\hat{\boldsymbol{v}}_{k|k-1}$ and $\boldsymbol{P}^{vv}_{k|k-1}$ are given in (21) and (23).

According to the definition of $\boldsymbol{P}^{xv}_{k|k-1}$ in (18) and using (1) and (3), we can obtain

$$\begin{aligned} \boldsymbol{P}^{xv}_{k|k-1} &= \mathrm{E}[\boldsymbol{x}_k \boldsymbol{v}^{\mathrm{T}}_k|\boldsymbol{Y}_{k-1}] - \hat{\boldsymbol{x}}_{k|k-1} \hat{\boldsymbol{v}}^{\mathrm{T}}_{k|k-1} \\ &= \mathrm{E}[(\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) + \boldsymbol{w}_{k-1})(\boldsymbol{\Psi}_{k-1} \boldsymbol{v}_{k-1} + \boldsymbol{\xi}_{k-1})^{\mathrm{T}}|\boldsymbol{Y}_{k-1}] - \hat{\boldsymbol{x}}_{k|k-1} \hat{\boldsymbol{v}}^{\mathrm{T}}_{k|k-1}. \end{aligned} \tag{30}$$

By the fact that $\boldsymbol{w}_{k-1}$ is uncorrelated with $\boldsymbol{v}_{k-1}$ and $\boldsymbol{\xi}_{k-1}$, and $\boldsymbol{\xi}_{k-1} \sim N(\boldsymbol{\xi}_{k-1}; \boldsymbol{0}, \boldsymbol{R}_{k-1})$ is uncorrelated with $\boldsymbol{Y}_{k-1}$, Eq. (30) can be rewritten as

$$\boldsymbol{P}^{xv}_{k|k-1} = \mathrm{E}[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) \boldsymbol{v}^{\mathrm{T}}_{k-1}|\boldsymbol{Y}_{k-1}] \boldsymbol{\Psi}^{\mathrm{T}}_{k-1} - \hat{\boldsymbol{x}}_{k|k-1} \hat{\boldsymbol{v}}^{\mathrm{T}}_{k|k-1}. \tag{31}$$

Using Gaussian approximations of $p(\boldsymbol{x}_{k-1}, \boldsymbol{v}_{k-1}|\boldsymbol{Y}_{k-1})$ and according to marginal integral formula in Appendix A, $\boldsymbol{P}^{xv}_{k|k-1}$ can be computed as (24), where $\hat{\boldsymbol{v}}_{x,k-1|k-1}$ is given in (25).

With Gaussian approximations of $p(\boldsymbol{x}_k|\boldsymbol{Y}_{k-1})$ and $p(\boldsymbol{v}_k|\boldsymbol{Y}_{k-1})$ in (12) and (29), the joint PDF of $\boldsymbol{x}_k$ and $\boldsymbol{v}_k$ conditioned on $\boldsymbol{Y}_{k-1}$ is also Gaussian, i.e

$$p(\boldsymbol{x}_k, \boldsymbol{v}_k|\boldsymbol{Y}_{k-1}) = N\left(\begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{v}_k \end{bmatrix}; \begin{bmatrix} \hat{\boldsymbol{x}}_{k|k-1} \\ \hat{\boldsymbol{v}}_{k|k-1} \end{bmatrix}, \begin{bmatrix} \boldsymbol{P}_{k|k-1} & \boldsymbol{P}^{xv}_{k|k-1} \\ (\boldsymbol{P}^{xv}_{k|k-1})^{\mathrm{T}} & \boldsymbol{P}^{vv}_{k|k-1} \end{bmatrix}\right). \tag{32}$$

3.2 One-step prediction of delayed measurement

Theorem 2. With Assumptions 1, 2 and Gaussian approximations of $p(\boldsymbol{x}_{k-1},\boldsymbol{v}_{k-1}|\boldsymbol{Y}_{k-1})$ and $p(\boldsymbol{x}_k,\boldsymbol{v}_k|\boldsymbol{Y}_{k-1})$, the Gaussian approximation of $p(\boldsymbol{y}_k|\boldsymbol{Y}_{k-1})$ has the one-step prediction estimation $\hat{\boldsymbol{y}}_{k|k-1}$, prediction error covariance matrix $\boldsymbol{P}^{yy}_{k,k|k-1}$, and cross-covariance matrixes $\boldsymbol{P}^{xy}_{k,k|k-1}$ and $\boldsymbol{P}^{vy}_{k,k|k-1}$ as the unified form:

$$\hat{\boldsymbol{y}}_{k|k-1}=(1-p_k)\hat{\boldsymbol{z}}_{k|k-1}+p_k\hat{\boldsymbol{z}}_{k-1|k-1}, \tag{33}$$

$$\boldsymbol{P}^{yy}_{k|k-1}=(1-p_k)\boldsymbol{P}^{zz}_{k|k-1}+p_k\boldsymbol{P}^{zz}_{k-1|k-1}+(1-p_k)p_k(\hat{\boldsymbol{z}}_{k|k-1}-\hat{\boldsymbol{z}}_{k-1|k-1})(\hat{\boldsymbol{z}}_{k|k-1}-\hat{\boldsymbol{z}}_{k-1|k-1})^{\mathrm{T}}, \tag{34}$$

$$\boldsymbol{P}^{xy}_{k|k-1}=(1-p_k)\boldsymbol{P}^{xz}_{k|k-1}+p_k\boldsymbol{P}^{xz}_{k,k-1|k-1}, \tag{35}$$

$$\boldsymbol{P}^{vy}_{k|k-1}=(1-p_k)\boldsymbol{P}^{vz}_{k|k-1}+p_k\boldsymbol{P}^{vz}_{k,k-1|k-1}, \tag{36}$$

$$\hat{\boldsymbol{z}}_{k|k-1}=\int_{\mathbb{R}^n}\boldsymbol{h}_k(\boldsymbol{x}_k)N(\boldsymbol{x}_k;\hat{\boldsymbol{x}}_{k|k-1},\boldsymbol{P}_{k|k-1})\mathrm{d}\boldsymbol{x}_k+\hat{\boldsymbol{v}}_{k|k-1}, \tag{37}$$

$$\boldsymbol{P}^{xz}_{k|k-1}=\int_{\mathbb{R}^n}\boldsymbol{x}_k\boldsymbol{h}^{\mathrm{T}}_k(\boldsymbol{x}_k)N(\boldsymbol{x}_k;\hat{\boldsymbol{x}}_{k|k-1},\boldsymbol{P}_{k|k-1})\mathrm{d}\boldsymbol{x}_k+\boldsymbol{P}^{xv}_{k|k-1}-\hat{\boldsymbol{x}}_{k|k-1}(\hat{\boldsymbol{z}}_{k|k-1}-\hat{\boldsymbol{v}}_{k|k-1})^{\mathrm{T}}, \tag{38}$$

$$\begin{aligned}\boldsymbol{P}^{zz}_{k|k-1}&=\int_{\mathbb{R}^n}[\boldsymbol{h}_k(\boldsymbol{x}_k)+\hat{\boldsymbol{v}}_{x,k|k-1}][\boldsymbol{h}_k(\boldsymbol{x}_k)+\hat{\boldsymbol{v}}_{x,k|k-1}]^{\mathrm{T}}N(\boldsymbol{x}_k;\hat{\boldsymbol{x}}_{k|k-1},\boldsymbol{P}_{k|k-1})\mathrm{d}\boldsymbol{x}_k\\&\quad+\boldsymbol{\Omega}_{k|k-1}-\hat{\boldsymbol{z}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k|k-1},\end{aligned} \tag{39}$$

$$\boldsymbol{P}^{vz}_{k|k-1}=\int_{\mathbb{R}^n}\hat{\boldsymbol{v}}_{x,k|k-1}\boldsymbol{h}^{\mathrm{T}}_k(\boldsymbol{x}_k)N(\boldsymbol{x}_k;\hat{\boldsymbol{x}}_{k|k-1},\boldsymbol{P}_{k|k-1})\mathrm{d}\boldsymbol{x}_k+\boldsymbol{P}^{vv}_{k|k-1}-\hat{\boldsymbol{v}}_{k|k-1}(\hat{\boldsymbol{z}}_{k|k-1}-\hat{\boldsymbol{v}}_{k|k-1})^{\mathrm{T}}, \tag{40}$$

$$\hat{\boldsymbol{v}}_{x,k|k-1}=\hat{\boldsymbol{v}}_{k|k-1}+(\boldsymbol{P}^{xv}_{k|k-1})^{\mathrm{T}}\boldsymbol{P}^{-1}_{k|k-1}(\boldsymbol{x}_k-\hat{\boldsymbol{x}}_{k|k-1}), \tag{41}$$

$$\boldsymbol{\Omega}_{k|k-1}=\boldsymbol{P}^{vv}_{k|k-1}-(\boldsymbol{P}^{xv}_{k|k-1})^{\mathrm{T}}\boldsymbol{P}^{-1}_{k|k-1}\boldsymbol{P}^{xv}_{k|k-1}, \tag{42}$$

$$\hat{\boldsymbol{z}}_{k-1|k-1}=\int_{\mathbb{R}^n}\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1})N(\boldsymbol{x}_{k-1};\hat{\boldsymbol{x}}_{k-1|k-1},\boldsymbol{P}_{k-1|k-1})\mathrm{d}\boldsymbol{x}_{k-1}+\hat{\boldsymbol{v}}_{k-1|k-1}, \tag{43}$$

$$\begin{aligned}\boldsymbol{P}^{zz}_{k-1|k-1}&=\int_{\mathbb{R}^n}[\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1})+\hat{\boldsymbol{v}}_{x,k-1|k-1}][\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1})+\hat{\boldsymbol{v}}_{x,k-1|k-1}]^{\mathrm{T}}\\&\quad\times N(\boldsymbol{x}_{k-1};\hat{\boldsymbol{x}}_{k-1|k-1},\boldsymbol{P}_{k-1|k-1})\mathrm{d}\boldsymbol{x}_{k-1}+\boldsymbol{\Omega}_{k-1|k-1}-\hat{\boldsymbol{z}}_{k-1|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1},\end{aligned} \tag{44}$$

$$\boldsymbol{\Omega}_{k-1|k-1}=\boldsymbol{P}^{vv}_{k-1|k-1}-(\boldsymbol{P}^{xv}_{k-1|k-1})^{\mathrm{T}}\boldsymbol{P}^{-1}_{k-1|k-1}\boldsymbol{P}^{xv}_{k-1|k-1}, \tag{45}$$

$$\begin{aligned}\boldsymbol{P}^{xz}_{k,k-1|k-1}&=\int_{\mathbb{R}^n}\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})[\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1})+\hat{\boldsymbol{v}}_{x,k-1|k-1}]^{\mathrm{T}}N(\boldsymbol{x}_{k-1};\hat{\boldsymbol{x}}_{k-1|k-1},\boldsymbol{P}_{k-1|k-1})\mathrm{d}\boldsymbol{x}_{k-1}\\&\quad-\hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1},\end{aligned} \tag{46}$$

$$\begin{aligned}\boldsymbol{P}^{vz}_{k,k-1|k-1}&=\boldsymbol{\Psi}_{k-1}\int_{\mathbb{R}^n}\hat{\boldsymbol{v}}_{x,k-1|k-1}\boldsymbol{h}^{\mathrm{T}}_{k-1}(\boldsymbol{x}_{k-1})N(\boldsymbol{x}_{k-1};\hat{\boldsymbol{x}}_{k-1|k-1},\boldsymbol{P}_{k-1|k-1})\mathrm{d}\boldsymbol{x}_{k-1}\\&\quad+\boldsymbol{\Psi}_{k-1}\boldsymbol{P}^{vv}_{k-1|k-1}-\hat{\boldsymbol{v}}_{k|k-1}(\hat{\boldsymbol{z}}_{k-1|k-1}-\hat{\boldsymbol{v}}_{k-1|k-1})^{\mathrm{T}}.\end{aligned} \tag{47}$$

Proof. Considering the independence assumptions in Section 2 and substituting the delayed measurement function in (4) into the definition of $\hat{\boldsymbol{y}}_{k|k-1}$ in (16), we obtain (33).

Using (4) and (33), the prediction error of delayed measruement $\tilde{\boldsymbol{y}}_{k|k-1}$ can be written as follows:

$$\tilde{\boldsymbol{y}}_{k|k-1}=(1-\gamma_k)\tilde{\boldsymbol{z}}_{k|k-1}+\gamma_k\tilde{\boldsymbol{z}}_{k-1|k-1}+(p_k-\gamma_k)(\hat{\boldsymbol{z}}_{k|k-1}-\hat{\boldsymbol{z}}_{k-1|k-1})^{\mathrm{T}}. \tag{48}$$

Considering that γ_k is independent of the measurements and substituting (48) into the definitions of $\boldsymbol{P}^{yy}_{k|k-1}$ in (17) and $\boldsymbol{P}^{xy}_{k|k-1}$ and $\boldsymbol{P}^{vy}_{k|k-1}$ in (18), we can obtain (34)–(36).

According to the definitions of $\hat{\boldsymbol{z}}_{k|k-1}$ and $\boldsymbol{P}^{xz}_{k|k-1}$ in (18), we can obtain (37) and (38). With Gaussian approximation of $p(\boldsymbol{x}_k, \boldsymbol{v}_k|\boldsymbol{Y}_{k-1})$ and marginal integral formula in Appendix A, and according to the definitions of $\boldsymbol{P}^{zz}_{k|k-1}$ and $\boldsymbol{P}^{vz}_{k|k-1}$ in (18), we have (39) and (40), where $\hat{\boldsymbol{v}}_{x,k|k-1}$ and $\boldsymbol{\Omega}_{k|k-1}$ are given by (41) and (42).

Similarly, according to the definition of $\hat{\boldsymbol{z}}_{k-1|k-1}$ in (18), we have (43). With Gaussian approximation of $p(\boldsymbol{x}_{k-1}, \boldsymbol{v}_{k-1}|\boldsymbol{Y}_{k-1})$ and marginal integral formula in Appendix A, and according to the definition of $\boldsymbol{P}^{zz}_{k-1|k-1}$ in (18), we have (44), where $\hat{\boldsymbol{v}}_{x,k-1|k-1}$ is given by (25) and $\boldsymbol{\Omega}_{k-1|k-1}$ is given by (45).

According to the definitions of $\boldsymbol{P}^{xz}_{k,k-1|k-1}$ and $\boldsymbol{P}^{vz}_{k,k-1|k-1}$ in (18) and considering that both $\boldsymbol{w}_{k-1}$ and $\boldsymbol{\xi}_{k-1}$ are independent of $\boldsymbol{Y}_{k-1}$, we have

$$\begin{aligned}\boldsymbol{P}^{xz}_{k,k-1|k-1} &= \mathrm{E}[\boldsymbol{x}_k\boldsymbol{z}^{\mathrm{T}}_{k-1}|\boldsymbol{Y}_{k-1}] - \hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1}\\ &= \mathrm{E}[(\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) + \boldsymbol{w}_{k-1})\boldsymbol{z}^{\mathrm{T}}_{k-1}|\boldsymbol{Y}_{k-1}] - \hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1}\\ &= \mathrm{E}[\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})\boldsymbol{z}^{\mathrm{T}}_{k-1}|\boldsymbol{Y}_{k-1}] - \hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1},\end{aligned} \tag{49}$$

$$\begin{aligned}\boldsymbol{P}^{vz}_{k,k-1|k-1} &= \mathrm{E}[\boldsymbol{v}_k\boldsymbol{z}^{\mathrm{T}}_{k-1}|\boldsymbol{Y}_{k-1}] - \hat{\boldsymbol{v}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1}\\ &= \mathrm{E}[(\boldsymbol{\Psi}_{k-1}\boldsymbol{v}_{k-1} + \boldsymbol{\xi}_{k-1})\boldsymbol{z}^{\mathrm{T}}_{k-1}|\boldsymbol{Y}_{k-1}] - \hat{\boldsymbol{v}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1}\\ &= \boldsymbol{\Psi}_{k-1}\mathrm{E}[\boldsymbol{v}_{k-1}\boldsymbol{z}^{\mathrm{T}}_{k-1}|\boldsymbol{Y}_{k-1}] - \hat{\boldsymbol{v}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1}.\end{aligned} \tag{50}$$

With Gaussian approximation of $p(\boldsymbol{x}_{k-1}, \boldsymbol{v}_{k-1}|\boldsymbol{Y}_{k-1})$ and using marginal integral formula in Appendix A, we can obtain (46) and (47), where $\hat{\boldsymbol{v}}_{x,k-1|k-1}$ is given by (25).

3.3 Filtering updates of state and measurement noise

Theorem 3. With Assumptions 1, 2 and Gaussian approximations of $p(\boldsymbol{x}_k, \boldsymbol{v}_k|\boldsymbol{Y}_{k-1})$, the Gaussian approximation of $p(\boldsymbol{x}_k, \boldsymbol{v}_k|\boldsymbol{Y}_k)$ has the filtering estimations $\hat{\boldsymbol{x}}_{k|k}$ and $\hat{\boldsymbol{v}}_{k|k}$, and estimation error covariance matrixes $\boldsymbol{P}_{k|k}$ and $\boldsymbol{P}^{vv}_{k|k}$, and cross-covariance matrix $\boldsymbol{P}^{xv}_{k|k}$ as the unified form:

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}^x_k(\boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k|k-1}), \tag{51}$$

$$\boldsymbol{K}^x_k = \boldsymbol{P}^{xy}_{k|k-1}(\boldsymbol{P}^{yy}_{k|k-1})^{-1}, \tag{52}$$

$$\boldsymbol{P}_{k|k} = \boldsymbol{P}_{k|k-1} - \boldsymbol{K}^x_k\boldsymbol{P}^{yy}_{k|k-1}(\boldsymbol{K}^x_k)^{\mathrm{T}}, \tag{53}$$

$$\hat{\boldsymbol{v}}_{k|k} = \hat{\boldsymbol{v}}_{k|k-1} + \boldsymbol{K}^v_k(\boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k|k-1}), \tag{54}$$

$$\boldsymbol{K}^v_k = \boldsymbol{P}^{vy}_{k|k-1}(\boldsymbol{P}^{yy}_{k|k-1})^{-1}, \tag{55}$$

$$\boldsymbol{P}^{vv}_{k|k} = \boldsymbol{P}^{vv}_{k|k-1} - \boldsymbol{K}^v_k\boldsymbol{P}^{yy}_{k|k-1}(\boldsymbol{K}^v_k)^{\mathrm{T}}, \tag{56}$$

$$\boldsymbol{P}^{xv}_{k|k} = \boldsymbol{P}^{xv}_{k|k-1} - \boldsymbol{K}^x_k\boldsymbol{P}^{yy}_{k|k-1}(\boldsymbol{K}^v_k)^{\mathrm{T}}. \tag{57}$$

Proof. With the Gaussian approximations of $p(\boldsymbol{x}_k|\boldsymbol{Y}_{k-1})$ in (12), $p(\boldsymbol{y}_k|\boldsymbol{Y}_{k-1})$ in (15) and $p(\boldsymbol{v}_k|\boldsymbol{Y}_{k-1})$ in (29), both joint PDFs $p(\boldsymbol{x}_k, \boldsymbol{y}_k|\boldsymbol{Y}_{k-1})$ and $p(\boldsymbol{v}_k, \boldsymbol{y}_k|\boldsymbol{Y}_{k-1})$ are Gaussian, i.e.

$$p(\boldsymbol{x}_k, \boldsymbol{y}_k|\boldsymbol{Y}_{k-1}) = N\left(\begin{bmatrix}\boldsymbol{x}_k\\ \boldsymbol{y}_k\end{bmatrix}; \begin{bmatrix}\hat{\boldsymbol{x}}_{k|k-1}\\ \hat{\boldsymbol{y}}_{k|k-1}\end{bmatrix}, \begin{bmatrix}\boldsymbol{P}_{k|k-1} & \boldsymbol{P}^{xy}_{k|k-1}\\ (\boldsymbol{P}^{xy}_{k|k-1})^{\mathrm{T}} & \boldsymbol{P}^{yy}_{k|k-1}\end{bmatrix}\right), \tag{58}$$

$$p(\boldsymbol{v}_k, \boldsymbol{y}_k|\boldsymbol{Y}_{k-1}) = N\left(\begin{bmatrix}\boldsymbol{v}_k\\ \boldsymbol{y}_k\end{bmatrix}; \begin{bmatrix}\hat{\boldsymbol{v}}_{k|k-1}\\ \hat{\boldsymbol{y}}_{k|k-1}\end{bmatrix}, \begin{bmatrix}\boldsymbol{P}^{vv}_{k|k-1} & \boldsymbol{P}^{vy}_{k|k-1}\\ (\boldsymbol{P}^{vy}_{k|k-1})^{\mathrm{T}} & \boldsymbol{P}^{yy}_{k|k-1}\end{bmatrix}\right). \tag{59}$$

According to Gaussian update rule in Appendix B, both $p(\boldsymbol{x}_k|\boldsymbol{Y}_k)$ and $p(\boldsymbol{v}_k|\boldsymbol{Y}_k)$ are computed as Gaussian in (60) and (61):

$$p(\boldsymbol{x}_k|\boldsymbol{Y}_k) = N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k}, \boldsymbol{P}_{k|k}), \tag{60}$$

$$p(\boldsymbol{v}_k|\boldsymbol{Y}_k) = N(\boldsymbol{v}_k; \hat{\boldsymbol{v}}_{k|k}, \boldsymbol{P}^{vv}_{k|k}), \tag{61}$$

where $\hat{\boldsymbol{x}}_{k|k}$, $\boldsymbol{P}_{k|k}$, $\hat{\boldsymbol{v}}_{k|k}$ and $\boldsymbol{P}^{vv}_{k|k}$ are given by (51)–(56).

Using (51) and (54) yields

$$\tilde{\boldsymbol{x}}_{k|k} = \tilde{\boldsymbol{x}}_{k|k-1} - \boldsymbol{K}^x_k \tilde{\boldsymbol{y}}_{k|k-1}, \tag{62}$$

$$\tilde{\boldsymbol{v}}_{k|k} = \tilde{\boldsymbol{v}}_{k|k-1} - \boldsymbol{K}^v_k \tilde{\boldsymbol{y}}_{k|k-1}. \tag{63}$$

With (52) and (55), we have

$$\boldsymbol{K}^x_k (\boldsymbol{P}^{vy}_{k|k-1})^{\mathrm{T}} = \boldsymbol{P}^{xy}_{k|k-1} (\boldsymbol{K}^v_k)^{\mathrm{T}} = \boldsymbol{K}^x_k \boldsymbol{P}^{yy}_{k|k-1} (\boldsymbol{K}^v_k)^{\mathrm{T}}. \tag{64}$$

Substituting (62) and (63) into the definition of $\boldsymbol{P}^{xv}_{k|k}$ in (18) and using (64), we can obtain (57).

With Gaussian approximations of $p(\boldsymbol{x}_k|\boldsymbol{Y}_k)$ and $p(\boldsymbol{v}_k|\boldsymbol{Y}_k)$ in (60) and (61), the joint PDF of $\boldsymbol{x}_k$ and $\boldsymbol{v}_k$ conditioned on $\boldsymbol{Y}_k$ is also Gaussian, i.e

$$p(\boldsymbol{x}_k, \boldsymbol{v}_k|\boldsymbol{Y}_k) = N\left(\begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{v}_k \end{bmatrix}; \begin{bmatrix} \hat{\boldsymbol{x}}_{k|k} \\ \hat{\boldsymbol{v}}_{k|k} \end{bmatrix}, \begin{bmatrix} \boldsymbol{P}_{k|k} & \boldsymbol{P}^{xv}_{k|k} \\ (\boldsymbol{P}^{xv}_{k|k})^{\mathrm{T}} & \boldsymbol{P}^{vv}_{k|k} \end{bmatrix} \right). \tag{65}$$

The proposed GA filter for nonlinear systems with randomly delayed measurements and colored measurement noises operates by combining the analytical computations in (21), (23), (25), (33)–(36), (41), (42), (45) and (51)–(57) with the Gaussian weighted integrals in (20), (22), (24), (37)–(40), (43), (44), (46) and (47). On one hand, in the derivation of the proposed GA filter, if both the functions $\boldsymbol{f}_{k-1}(\cdot)$ and $\boldsymbol{h}_k(\cdot)$ in (1) and (2) are linear, the proposed filter can automatically reduce to the linear Kalman filter with one-step randomly delayed measurements and colored measurement noises. On the other hand, for nonlinear systems, different GA filters can be obtained by utilizing different numerical methods to compute these Gaussian weighted integrals, such as EKF with randomly delayed measurements and colored measurement noises based on the first-order Taylor series expansion and UKF with randomly delayed measurements and colored measurement noises based on UT. Appendices C and D show how these two filters are developed from the proposed GA filter framework formulated in Theorems 1–3.

The proposed GA filter is designed for stochastic dynamic systems with randomly delayed measurements and colored measurement noises. If Assumptions 1 and 2 hold, the posterior PDF $p(\boldsymbol{x}_k|\boldsymbol{Y}_k)$ is updated as Gaussian by Theorem 3, as shown in (60). In the MMSE sense, the state estimation of the proposed GA filter is optimal when Assumptions 1 and 2 hold. Thus, for stochastic dynamic systems with randomly delayed measurements and colored measurement noises, under Assumptions 1 and 2, the proposed GA filter outperforms existing methods. Existing GA filter with one-step randomly delayed measurements is designed for stochastic dynamic systems with randomly delayed measurements, and its state estimation is optimal in the MMSE sense under Assumptions that one-step predictive PDFs of state and white delayed measurement are Gaussian [3]. Existing GA filter with colored measurement noises is designed for stochastic dynamic systems with colored measurement noises, and its state estimation is optimal in the MMSE sense under Assumptions that one-step predictive PDFs of state and colored measurement are Gaussian [7]. It is interesting that the proposed GA filter is identical to existing GA filter with one-step randomly delayed measurements for stochastic dynamic systems with randomly delayed measurements and existing GA filter with colored measurement noises for stochastic dynamic systems with colored measurement noises. In other words, both existing GA filter with one-step randomly delayed measurements and existing GA filter with colored measurement noises are special cases of the proposed GA filter, which will be proved in the following Theorems 4 and 5.

Theorem 4. If correlation parameter $\boldsymbol{\Psi}_{k-1} = \boldsymbol{0}$, the proposed GA filter will degrade to existing GA filter with one-step randomly delayed measurements.

Proof. Substituting $\boldsymbol{\Psi}_{k-1} = \mathbf{0}$ in (21), (23) and (24), we can obtain

$$\hat{\boldsymbol{v}}_{k|k-1} = \mathbf{0}, \qquad \boldsymbol{P}^{vv}_{k|k-1} = \boldsymbol{R}_{k-1}, \qquad \boldsymbol{P}^{xv}_{k|k-1} = \mathbf{0}. \tag{66}$$

Substituting (66) in (37), (38), (41) and (42), we can obtain

$$\hat{\boldsymbol{z}}_{k|k-1} = \int_{\mathbb{R}^n} \boldsymbol{h}_k(\boldsymbol{x}_k) N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k-1}, \boldsymbol{P}_{k|k-1}) \mathrm{d}\boldsymbol{x}_k, \tag{67}$$

$$\boldsymbol{P}^{xz}_{k|k-1} = \int_{\mathbb{R}^n} \boldsymbol{x}_k \boldsymbol{h}^{\mathrm{T}}_k(\boldsymbol{x}_k) N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k-1}, \boldsymbol{P}_{k|k-1}) \mathrm{d}\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k-1} \hat{\boldsymbol{z}}^{\mathrm{T}}_{k|k-1}, \tag{68}$$

$$\hat{\boldsymbol{v}}_{x,k|k-1} = \mathbf{0}, \qquad \boldsymbol{\Omega}_{k|k-1} = \boldsymbol{R}_{k-1}. \tag{69}$$

Using (66) and (69) in (39) and (40), we have

$$\boldsymbol{P}^{zz}_{k|k-1} = \int_{\mathbb{R}^n} \boldsymbol{h}_k(\boldsymbol{x}_k) \boldsymbol{h}^{\mathrm{T}}_k(\boldsymbol{x}_k) N(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k-1}, \boldsymbol{P}_{k|k-1}) \mathrm{d}\boldsymbol{x}_k + \boldsymbol{R}_{k-1} - \hat{\boldsymbol{z}}_{k|k-1} \hat{\boldsymbol{z}}^{\mathrm{T}}_{k|k-1}, \tag{70}$$

$$\boldsymbol{P}^{vz}_{k|k-1} = \boldsymbol{R}_{k-1}. \tag{71}$$

Using (66) and $\boldsymbol{\Psi}_{k-1} = \mathbf{0}$ in (47), we can obtain

$$\boldsymbol{P}^{vz}_{k,k-1|k-1} = \mathbf{0}. \tag{72}$$

With (66) in (54), (56) and (57), we can obtain

$$\hat{\boldsymbol{v}}_{k|k} = \boldsymbol{K}^v_k (\boldsymbol{y}_k - \hat{\boldsymbol{y}}_{k|k-1}), \tag{73}$$

$$\boldsymbol{P}^{vv}_{k|k} = \boldsymbol{R}_{k-1} - \boldsymbol{K}^v_k \boldsymbol{P}^{yy}_{k|k-1} (\boldsymbol{K}^v_k)^{\mathrm{T}}, \tag{74}$$

$$\boldsymbol{P}^{xv}_{k|k} = -\boldsymbol{K}^x_k \boldsymbol{P}^{yy}_{k|k-1} (\boldsymbol{K}^v_k)^{\mathrm{T}}. \tag{75}$$

Eqs. (20), (22), (25), (33)–(36), (43)–(46), (51)–(53), (55), (67), (68) and (70)–(75) constitute existing GA filter with one-step randomly delayed measurements. Thus, existing GA filter with one-step randomly delayed measurements is a special case of the proposed GA filter when correlation parameter $\boldsymbol{\Psi}_{k-1} = \mathbf{0}$.

Theorem 5. If latency probability $p_k = 0$, the proposed GA filter will degrade to existing GA filter with colored measurement noises.

Proof. If $p_k = 0$, we have

$$\gamma_k = 0, \qquad \boldsymbol{y}_k = \boldsymbol{z}_k. \tag{76}$$

Substituting $p_k = 0$ in (33)–(36), we can obtain

$$\hat{\boldsymbol{y}}_{k|k-1} = \hat{\boldsymbol{z}}_{k|k-1} \quad \boldsymbol{P}^{yy}_{k|k-1} = \boldsymbol{P}^{zz}_{k|k-1} \quad \boldsymbol{P}^{xy}_{k|k-1} = \boldsymbol{P}^{xz}_{k|k-1} \quad \boldsymbol{P}^{vy}_{k|k-1} = \boldsymbol{P}^{vz}_{k|k-1}. \tag{77}$$

Substituting (76), (77) in (51)–(57), we can obtain

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}^x_k (\boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k|k-1}), \tag{78}$$

$$\boldsymbol{K}^x_k = \boldsymbol{P}^{xz}_{k|k-1} (\boldsymbol{P}^{zz}_{k|k-1})^{-1}, \tag{79}$$

$$\boldsymbol{P}_{k|k} = \boldsymbol{P}_{k|k-1} - \boldsymbol{K}^x_k \boldsymbol{P}^{zz}_{k|k-1} (\boldsymbol{K}^x_k)^{\mathrm{T}}, \tag{80}$$

$$\hat{\boldsymbol{v}}_{k|k} = \hat{\boldsymbol{v}}_{k|k-1} + \boldsymbol{K}^v_k (\boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k|k-1}), \tag{81}$$

$$\boldsymbol{K}^v_k = \boldsymbol{P}^{vz}_{k|k-1} (\boldsymbol{P}^{zz}_{k|k-1})^{-1}, \tag{82}$$

$$\boldsymbol{P}^{vv}_{k|k} = \boldsymbol{P}^{vv}_{k|k-1} - \boldsymbol{K}^v_k \boldsymbol{P}^{zz}_{k|k-1} (\boldsymbol{K}^v_k)^{\mathrm{T}}, \tag{83}$$

$$\boldsymbol{P}^{xv}_{k|k} = \boldsymbol{P}^{xv}_{k|k-1} - \boldsymbol{K}^x_k \boldsymbol{P}^{zz}_{k|k-1} (\boldsymbol{K}^v_k)^{\mathrm{T}}. \tag{84}$$

Eqs. (20)–(25), (37)–(42) and (78)–(84) constitute existing GA filter with colored measurement noises. Thus, existing GA filter with colored measurement noises is a special case of the proposed GA filter when latency probability $p_k = 0$.

4 Simulation

In this section, the superior performance of the proposed method as compared with existing methods for a nonlinear system with one-step randomly delayed measurements and colored measurement noises is shown by a target tracking application. Target tracking has been widely used as a benchmark problem to validate the performances of nonlinear filters because of its practical application values. Its nonlinear process equation is formulated as follows [8, 15, 18]:

$$\boldsymbol{x}_k = \begin{bmatrix} 1 & \frac{\sin\Omega T_0}{\Omega} & 0 & \frac{\cos\Omega T_0-1}{\Omega} & 0 \\ 0 & \cos\Omega T_0 & 0 & -\sin\Omega T_0 & 0 \\ 0 & \frac{1-\cos\Omega T_0}{\Omega} & 1 & \frac{\sin\Omega T_0}{\Omega} & 0 \\ 0 & \sin\Omega T_0 & 0 & \cos\Omega T_0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}_{k-1} + \boldsymbol{w}_{k-1}, \tag{85}$$

where the state vector $\boldsymbol{x} = [\varsigma\ \dot{\varsigma}\ \eta\ \dot{\eta}\ \Omega]^{\mathrm{T}}$, ς and η denote positions, $\dot{\varsigma}$ and $\dot{\eta}$ denote velocities in the x and y directions respectively, Ω denotes constant but unknown turn rate. The time-interval between two consecutive measurements is $T_0 = 1$ s; the process noise vector $\boldsymbol{w}_k \sim N(\boldsymbol{0}, \boldsymbol{Q}_k)$ with a nonsingular covariance matrix $\boldsymbol{Q}_k = \mu \cdot \mathrm{diag}[q_1\boldsymbol{M}\ \ q_1\boldsymbol{M}\ \ q_2 T_0]$, $q_1 = 0.1\ \mathrm{m}^2 \cdot \mathrm{s}^{-3}$, $q_2 = 1.75 \times 10^{-4}\ \mathrm{s}^{-3}$, where μ is a parameter to control the uncertainty level of the system, and

$$\boldsymbol{M} = \begin{bmatrix} T_0^3/3 & T_0^2/2 \\ T_0^2/2 & T_0 \end{bmatrix}. \tag{86}$$

The measurement equation is given by

$$\boldsymbol{z}_k = \begin{bmatrix} r_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} \sqrt{\varsigma_k^2 + \eta_k^2} \\ \tan^{-1}(\frac{\eta_k}{\varsigma_k}) \end{bmatrix} + \boldsymbol{v}_k, \tag{87}$$

where $\boldsymbol{z}_k$ is the undelayed measurement vector at time k, the measurement noise vector $\boldsymbol{v}_k$ is colored as formulated in (3), the white noise vector $\boldsymbol{\xi}_k \sim N(\boldsymbol{0}, \boldsymbol{R}_k)$ with $\boldsymbol{R}_k = \tau \cdot \mathrm{diag}[\sigma_r^2\ \ \sigma_\theta^2]$ and $\sigma_r = 10$ m and $\sigma_\theta = \sqrt{10}$ mrad, where τ is a parameter to control the accuracy level of the measurement. The actual measurement vector $\boldsymbol{y}_k$ is related to ideal measurement vector $\boldsymbol{z}_k$ by (4). The true initial state vector is given by

$$\boldsymbol{x}_0 = \left[1000\ \mathrm{m}\ \ 300\ \mathrm{m}\cdot\mathrm{s}^{-1}\ \ 1000\ \mathrm{m}\ \ 0\ \mathrm{m}\cdot\mathrm{s}^{-1}\ \ \Omega_0 \right]^{\mathrm{T}}, \tag{88}$$

where Ω_0 is the initial turn rate, and the associated covariance matrix is given by

$$\boldsymbol{P}_{0|0} = \mu \cdot \mathrm{diag}\left[100\ \mathrm{m}^2\ \ 10\ \mathrm{m}^2\cdot\mathrm{s}^{-2}\ \ 100\ \mathrm{m}^2\ \ 10\ \mathrm{m}^2\cdot\mathrm{s}^{-2}\ \ 100\ \mathrm{mrad}^2\cdot\mathrm{s}^{-2} \right]. \tag{89}$$

In each run, the initial state estimation vector $\hat{\boldsymbol{x}}_{0|0}$ is chosen randomly from $N(\boldsymbol{x}_0, \boldsymbol{P}_{0|0})$ and all filters are initialized with the same condition and the simulation time is 150 s. For a fair comparison, we make 500 independent Monte Carlo runs. To compare the performances of these filters, we use the root-mean square error (RMSE) of the position, velocity and turn rate. We define the RMSE in position at time k as

$$\mathrm{RMSE}_{\mathrm{pos}}(k) = \sqrt{\frac{1}{N}\sum_{s=1}^{N}\left((\varsigma_k^s - \hat{\varsigma}_k^s)^2 + (\eta_k^s - \hat{\eta}_k^s)\right)^2}, \tag{90}$$

where $(\varsigma_k^s, \eta_k^s)$ and $(\hat{\varsigma}_k^s, \hat{\eta}_k^s)$ are the true and estimated positions at the n-th Monte Carlo run. Similar to the RMSE in position, we may also write formulas of the RMSE in velocity and turn rate. To assess the performance of the proposed method, we consider following two explanatory cases.

Case 1. In this case, we aim to show that the proposed GA filter outperforms existing methods for nonlinear systems with randomly delayed measurements and colored measurement noises, where

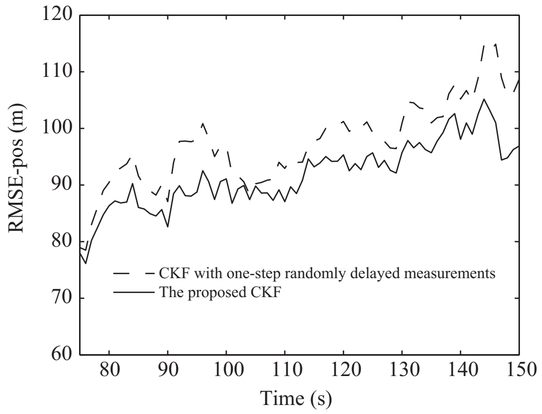

Figure 1 RMSEs of the position of the proposed method and existing methods.

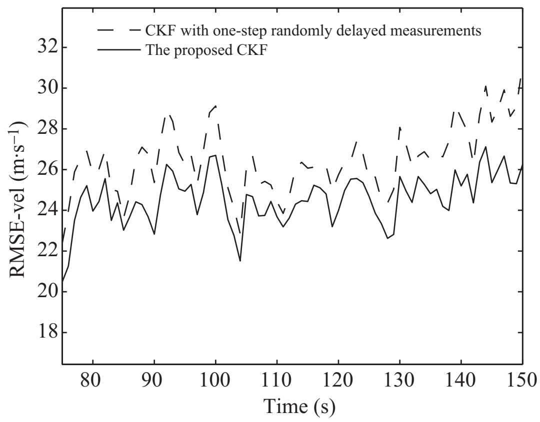

Figure 2 RMSEs of the velocity of the proposed method and existing methods.

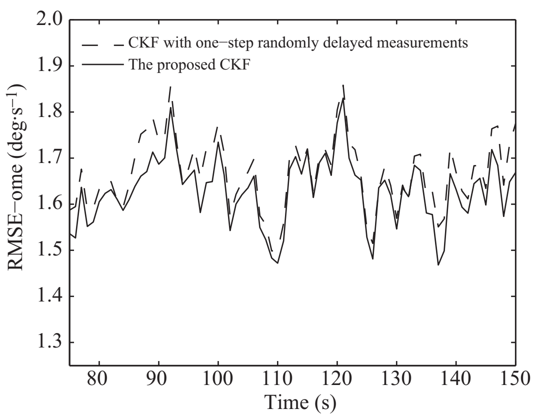

Figure 3 RMSEs of the turn rate of the proposed method and existing methods.

correlation parameter $\boldsymbol{\Psi}_{k-1} = \mathrm{diag}[0.8 \quad 0.8](k \geqslant 2)$, latency probability $p_k = 0.5$, $\mu = \tau = 1$ and $\Omega_0 = -3$ deg $\cdot$ s^{-1}. We choose the third-degree spherical-radial cubature rule [15] to implement the proposed GA filter, existing GA filter with one-step randomly delayed measurements [3], and existing GA filter with colored measurement noises [7], which leads to CKF with one-step randomly delayed measurements and colored measurement noises, CKF with one-step randomly delayed measurements and CKF with colored measurement noises respectively. The RMSE results are shown in Figures 1–3. CKF with colored measurement noises was often found to halt its operation in this case because it used incorrect measurement model, thus its simulation results are not shown in Figures 1–3. (Note that the proposed CKF is identical to the proposed UKF with $\kappa = 0$, as developed in Appendix D.)

It is clear to see from Figures 1–3 that the proposed CKF has higher estimation accuracy than CKF with one-step randomly delayed measurements when measurement noises are colored and measurements are randomly delayed one sampling time. Thus, the proposed method is more suitable for achieving the state estimation of nonlinear systems with randomly delayed measurements and colored measurement noises than existing methods.

Case 2. In this case, we aim to compare the performance of existing standard EKF [10], UKF with the recommended free parameter $\kappa = -2$ [13], CKF [15], the proposed EKF, the proposed UKF with the recommended free parameter $\kappa = -2$, and the proposed CKF, where correlation parameter $\boldsymbol{\Psi}_{k-1} = \mathrm{diag}[0.8 \quad 0.8](k \geqslant 2)$, latency probability $p_k = 0.5$, $\mu = 0.15$, $\tau = 0.05$ and $\Omega_0 = -1$ deg $\cdot$ s^{-1}. The RMSE results are shown in Figures 4–6. The proposed UKF with the recommended free parameter $\kappa = -2$ was often found to halt its operation in this case due to its numerical instability, thus its simulation results are not shown in Figures 4–6. Moreover, existing standards EKF, UKF, CKF were

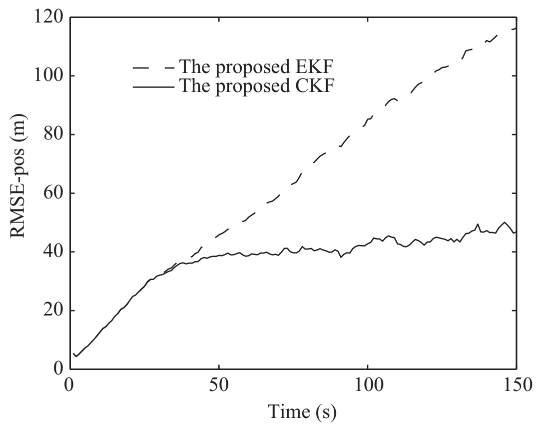

Figure 4 RMSEs of the position of the proposed methods.

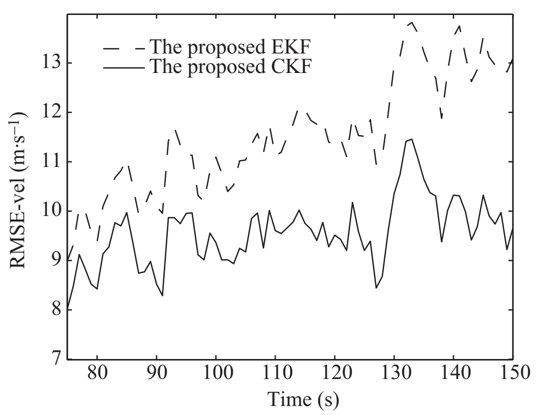

Figure 5 RMSEs of the velocity of the proposed methods.

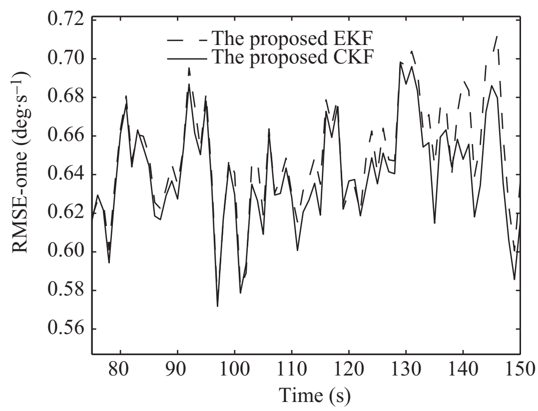

Figure 6 RMSEs of the turn rate of the proposed methods.

all often found to halt their operation in this case because they used incorrect measurement model, thus their simulation results are also not shown in Figures 4–6.

It can be seen from Figures 4–6 that the proposed CKF has higher estimation accuracy than the proposed EKF. Thus, the third-degree spherical-radial cubature rule is more suitable for implementing the proposed GA filter as compared with the first Taylor series expansion and UT with the recommended free parameter, which is consistent with the fact that the third-degree spherical-radial cubature rule has higher numerical accuracy than the first Taylor series expansion and better numerical stability than UT with the recommended free parameter. Besides, the proposed method is more suitable for achieving the state estimation of nonlinear systems with randomly delayed measurements and colored measurement noises as compared with existing standard GA filtering methods.

5 Conclusion

A new GA filter is derived for both linear and nonlinear stochastic dynamic systems with one-step randomly delayed measurements and colored measurement noises. For nonlinear systems, the solution is recursively calculated by analytical computations and Gaussian weighted integrals, and different GA filters can be obtained by utilizing different numerical methods to compute such Gaussian weighted integrals. Existing GA filter with one-step randomly delayed measurements and existing GA filter with colored measurement noises are special cases of the proposed GA filter. Simulation results show the effectiveness and superior performance of the proposed method as compared with existing methods. The approach can be potentially used in many applications where one-step randomly delayed measurements and colored measurement noises both exist.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 61201409, 61371173), China Postdoctoral Science Foundation (Grant Nos. 2013M530147, 2014T70309), Heilongjiang Postdoctoral Fund (Grant Nos. LBH-Z13052, LBH-TZ0505), and Fundamental Research Funds for the Central Universities of Harbin Engineering University (Grant No. HEUCFQ20150407).

Conflict of interest The authors declare that they have no conflict of interest.

References

1 Lei M, van Wyk B J, Qi Y. Online estimation of the approximate posterior Cramer-Rao lower bound for discrete-time nonlinear filtering. IEEE Trans Aerosp Electron Syst, 2011, 47: 37–57
2 Huang Y L, Zhang Y G, Wang X X, et al. Gaussian filter for nonlinear systems with correlated noises at the same epoch. Automatica, 2015, 60: 122–126
3 Wang X X, Liang Y, Pan Q, et al. Gaussian filter for nonlinear systems with one-step randomly delayed measurements. Automatica, 2013, 49: 976–986
4 Wang X X, Pan Q, Liang Y, et al. Gaussian smoothers for nonlinear systems with one-step randomly delayed measurements. IEEE Trans Autom Control, 2013, 58: 1828–1835
5 Wang X X, Liang Y, Pan Q, et al. Design and implementation of Gaussian filter for nonlinear system with randomly delayed measurements and correlated noises. Appl Math Comput, 2014, 232: 1011–1024
6 Wang X X, Liang Y, Pan Q, et al. Gaussian/Gaussian-mixture filters for non-linear stochastic systems with delayed states. IET Control Theory Appl, 2014, 8: 996–1008
7 Wang X X, Pan Q. Nonlinear Gaussian filter with the colored measurement noise. In: Proceedings of the 17th International Conference on Information Fusion, Salamanca, 2014. 1–7
8 Wang X X, Liang Y, Pan Q, et al. Nonlinear Gaussian smoothers with colored measurement noise. IEEE Trans Autom Control, 2015, 60: 870–876
9 Wang X X, Liang Y, Pan Q, et al. A Gaussian approximation recursive filter for nonlinear systems with correlated noises. Automatica, 2012, 48: 2290–2297
10 Anderson B D O, Moore J B. Optimal Filtering, Englewood Cliffs. Upper Saddle River: Prentice Hall, 1979
11 Ito K, Xiong K. Gaussian filters for nonlinear filtering problems. IEEE Trans Autom Control, 2000, 45: 910–927
12 Nørgaard M, Poulsen N K, Ravn O. New developments in state estimation for nonlinear systems. Automatica, 2000, 36: 1627–1638
13 Julier S J, Uhlman J K. Unscented filtering and nonlinear estimation. Proc IEEE, 2004, 92: 401–422
14 Zhang Y G, Huang Y L, Wu Z M, et al. A high order unscented Kalman filtering method. Acta Automatica Sinica, 2014, 40: 838–848
15 Arasaratnam I, Haykin S. Cubature Kalman filter. IEEE Trans Autom Control, 2009, 54: 1254–1269
16 Jia B, Xin M, Cheng Y. Sparse-grid quadrature nonlinear filtering. Automatica, 2012, 48: 327–341
17 Chang L B, Hu B Q, Li A, et al. Transformed unscented Kalman filter. IEEE Trans Autom Control, 2013, 58: 252–257
18 Jia B, Xin M, Cheng Y. High-degree cubature Kalman filter. Automatica, 2013, 49: 510–518
19 Duník J, Straka O, Šimandl M. Stochastic integration filter. IEEE Trans Autom Control, 2013, 58: 1561–1566
20 Bhaumik S, Swati. Cubature quadrature Kalman filter. IET Signal Process, 2013, 7: 533–541
21 Zhang X C. A novel cubature Kalman filter for nonlinear state estimation. In: Proceedings of the 52nd IEEE Conference on Decision and Control, Florence, 2013. 7797–7802
22 Zhang Y G, Huang Y L, Li N, et al. Embedded cubature Kalman filter with adaptive setting of free parameter. Signal Process, 2015, 114: 112–116
23 Zhang Y G, Huang Y L, Wu Z M, et al. Quasi-stochastic integration filter for nonlinear estimation. Math Problems Eng, 2014, 2014: 1–10
24 Wang S Y, Feng J C, Tse C K. Spherical simplex-radial cubature Kalman filter. IEEE Signal Process Lett, 2014, 21: 43–46
25 Zhang X C, Teng Y L. A new derivation of the cubature Kalman filters. Asian J Control, 2015, 17: 1–10
26 Zhang Y G, Huang Y L, Li N, et al. Interpolatory cubature Kalman filters. IET Control Theory Appl, 2015, 9: 1731–1739
27 Hu J, Wang Z D, Shen B, et al. Gain-constrained recursive filtering with stochastic nonlinearities and probabilistic sensor delays. IEEE Trans Signal Process, 2013, 61: 1230–1238
28 Wang X X, Liang Y, Pan Q, et al. Unscented Kalman filter for nonlinear systems with colored measurement noise. Acta Automatica Sinica, 2012, 38: 986–998
29 Wen R, Chang D C. Maneuvering target tracking with colored noise. IEEE Trans Aerosp Electron Syst, 1996, 32: 1311–1320
30 Hermoso-Carazo A, Linares-Pérez J. Extended and unscented filtering algorithms using one-step randomly delayed observations. Appl Math Comput, 2007, 190: 1375–1393

31 Hermoso-Carazo A, Linares-Pérez J. Unscented filtering algorithm using two-step randomly delayed obserations in nonlinear systems. Appl Math Model, 2009, 33: 3705–3717
32 Zhang Y G, Huang Y L, Zhao L. A general framework solution to Gaussian filter with multiple-step randomly-delayed measurements. Acta Automatica Sinica, 2015, 41: 122–135
33 Zhang Y G, Huang Y L, Li N, et al. Conditional posterior Cramér-Rao lower bound for nonlinear sequential Bayesian estimation with one-step randomly delayed measurements. Acta Automatica Sinica, 2015, 41: 559–574
34 Yuan G N, Xie Y J, Song Y, et al. Multipath parameters estimation of weak GPS signal based on new colored noise unscented Kalman filter. In: Proceedings of IEEE International Conference on Information and Automation, Harbin, 2010. 1852–1856
35 Titterton D H, Weston J L. Strapdown Inertial Navigation Technology. London: Peter Peregrinus Ltd, 1997
36 Jamoos A, Grivel E, Bobillet W, et al. Errors-in-variables based approach for the identification of AR time-varying fading channels. IEEE Signal Process Lett, 2007, 14: 793–796
37 Mahmoudi A, Karimi M, Amindavar H. Parameter estimation of autoregressive signals in presence of colored AR(1) noise as a quadratic eigenvalue problem. Signal Process, 2012, 92: 1151–1156
38 Srinivasan S, Aichner R, Kleijn W B, et al. Multi-channel parametric speech enhancement. IEEE Signal Process Lett, 2006, 13: 304–307
39 Ray A. Output feedback control under randomly varying distributed delays. J Guidance Contr Dyn, 1994, 17: 701–711
40 Wang Z D, Ho W C, Liu X. Robust filtering under randomly varying sensor delay with variance constraints. IEEE Trans Circuits Syst II Exp Briefs, 2004, 51: 320–326
41 Su C L, Lu C N. Interconnected network state estimation using randomly delayed measurements. IEEE Trans Power Syst, 2001, 16: 870–878
42 Duník J, Straka O, Šimandl M. Nonlinearity and nongaussianity measures for stochastic dynamic systems. In: Proceedings of the 16th International Conference on Information Fusion, Istanbul, 2013. 204–211

Appendix A Marginal integral formula

If the joint PDF of random vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is Gaussian, i.e.

$$p(\boldsymbol{a},\boldsymbol{b}) = N\left(\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix};\begin{bmatrix}\boldsymbol{\mu}_a\\ \boldsymbol{u}_b\end{bmatrix},\begin{bmatrix}\boldsymbol{P}_{aa} & \boldsymbol{P}_{ab}\\ (\boldsymbol{P}_{ab})^{\mathrm{T}} & \boldsymbol{P}_{bb}\end{bmatrix}\right) = N(\boldsymbol{\zeta};\boldsymbol{\mu}_\zeta,\boldsymbol{P}_{\zeta\zeta}), \tag{A1}$$

then we have

$$\mathrm{E}[\boldsymbol{b}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})] = \int \hat{\boldsymbol{b}}_a\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})N(\boldsymbol{a};\boldsymbol{u}_a,\boldsymbol{P}_{aa})\mathrm{d}\boldsymbol{a}, \tag{A2}$$

where $\boldsymbol{\zeta}$, $\boldsymbol{\mu}_\zeta$ and $\boldsymbol{P}_{\zeta\zeta}$ are defined in (A1), $\boldsymbol{\mu}_a$ and $\boldsymbol{P}_{aa}$ denote the mean vector and covariance matrix of $\boldsymbol{a}$, $\boldsymbol{\mu}_b$ and $\boldsymbol{P}_{bb}$ denote the mean vector and covariance matrix of $\boldsymbol{b}$, and $\boldsymbol{P}_{ab}$ denotes the cross-covariance of $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\hat{\boldsymbol{b}}_a$ is given as follows:

$$\hat{\boldsymbol{b}}_a = \boldsymbol{\mu}_b + (\boldsymbol{P}_{ab})^{\mathrm{T}}(\boldsymbol{P}_{aa})^{-1}(a-\boldsymbol{\mu}_a). \tag{A3}$$

Proof. By using (A1), we can obtain the square root matrix of $\boldsymbol{P}_{\zeta\zeta}$ as follows:

$$\boldsymbol{S}_{\zeta\zeta} = \begin{bmatrix}\boldsymbol{\Sigma} & \boldsymbol{0}\\ \boldsymbol{\varphi} & \boldsymbol{\Delta}\end{bmatrix}, \tag{A4}$$

where

$$\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathrm{T}} = \boldsymbol{P}_{aa}, \qquad \boldsymbol{\varphi} = (\boldsymbol{P}_{ab})^{\mathrm{T}}(\boldsymbol{P}_{aa})^{-1}\boldsymbol{\Sigma}, \qquad \boldsymbol{\Delta}\boldsymbol{\Delta}^{\mathrm{T}} = \boldsymbol{P}_{bb} - (\boldsymbol{P}_{ab})^{\mathrm{T}}(\boldsymbol{P}_{aa})^{-1}\boldsymbol{P}_{ab}. \tag{A5}$$

By using (A1), $\mathrm{E}[\boldsymbol{b}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})]$ can be computed as

$$\mathrm{E}[\boldsymbol{b}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})] = \int \boldsymbol{b}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})N(\boldsymbol{\zeta};\boldsymbol{\mu}_\zeta,\boldsymbol{P}_{\zeta\zeta})\mathrm{d}\boldsymbol{\zeta} = \int \boldsymbol{g}(\boldsymbol{\zeta})N(\boldsymbol{\zeta};\boldsymbol{\mu}_\zeta,\boldsymbol{P}_{\zeta\zeta})\mathrm{d}\boldsymbol{\zeta} = \int \boldsymbol{g}(\boldsymbol{S}_{\zeta\zeta}\boldsymbol{\zeta}+\boldsymbol{\mu}_\zeta)N(\boldsymbol{\zeta};\boldsymbol{0},\boldsymbol{I})\mathrm{d}\boldsymbol{\zeta}, \tag{A6}$$

where $\boldsymbol{I}$ denotes a identity matrix with the same dimension as $\boldsymbol{\zeta}$ and

$$\boldsymbol{g}(\boldsymbol{\zeta}) = \boldsymbol{b}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a}). \tag{A7}$$

Substituting (A4) and (A7) into (A6), $\mathrm{E}[\boldsymbol{b}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})]$ can be reformulated as

$$\mathrm{E}[\boldsymbol{b}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})] = \int (\boldsymbol{\varphi}\boldsymbol{a}+\boldsymbol{\Delta}\boldsymbol{b}+\boldsymbol{\mu}_b)\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\mu}_a+\boldsymbol{\Sigma}\boldsymbol{a})N(\boldsymbol{\zeta};\boldsymbol{0},\boldsymbol{I})\mathrm{d}\boldsymbol{\zeta} = \int (\boldsymbol{\varphi}\boldsymbol{a}+\boldsymbol{\mu}_b)\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{\mu}_a+\boldsymbol{\Sigma}\boldsymbol{a})N(\boldsymbol{\zeta};\boldsymbol{0},\boldsymbol{I})\mathrm{d}\boldsymbol{a}. \tag{A8}$$

By using (A5), we can obtain

$$\mathrm{E}[\boldsymbol{b}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})] = \int (\boldsymbol{\varphi}\boldsymbol{\Sigma}^{-1}(\boldsymbol{a}-\boldsymbol{\mu}_a)+\boldsymbol{\mu}_b)\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})N(\boldsymbol{a};\boldsymbol{u}_a,\boldsymbol{P}_{aa})\mathrm{d}\boldsymbol{a} = \int \hat{\boldsymbol{b}}_a\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{a})N(\boldsymbol{a};\boldsymbol{u}_a,\boldsymbol{P}_{aa})\mathrm{d}\boldsymbol{a}. \tag{A9}$$

Appendix B Gaussian update rule

If the joint PDF of $\boldsymbol{\xi}_j$ and $\boldsymbol{y}_l$ conditioned on Y_{l-1} i.e. $p(\boldsymbol{\xi}_j, \boldsymbol{y}_l|\boldsymbol{Y}_{l-1})$ is Gaussian, $p(\boldsymbol{\xi}_j|\boldsymbol{Y}_l)$ can be computed as Gaussian with mean $\hat{\boldsymbol{\xi}}_{j|l}$ and corresponding covariance matrix $\boldsymbol{P}^{\xi\xi}_{j,j|l}$ as the unified form:

$$\hat{\boldsymbol{\xi}}_{j|l} = \hat{\boldsymbol{\xi}}_{j|l-1} + \boldsymbol{K}^{\xi}_j(\boldsymbol{y}_l - \hat{\boldsymbol{y}}_{l|l-1}), \tag{B1}$$

$$\boldsymbol{P}^{\xi\xi}_{j,j|l} = \boldsymbol{P}^{\xi\xi}_{j,j|l-1} - \boldsymbol{K}^{\xi}_j\boldsymbol{P}^{yy}_{l,l|l-1}(\boldsymbol{K}^{\xi}_j)^{\mathrm{T}}, \tag{B2}$$

$$\boldsymbol{K}^{\xi}_j = \boldsymbol{P}^{\xi y}_{j,l|l-1}(\boldsymbol{P}^{yy}_{l,l|l-1})^{-1}. \tag{B3}$$

Proof. Since the joint PDF of $\boldsymbol{\xi}_j$ and $\boldsymbol{y}_l$ conditioned on Y_{l-1} is Gaussian, thus the PDF of $\boldsymbol{y}_l$ conditioned on Y_{l-1} is also Gaussian. Then $p(\boldsymbol{\xi}_j, \boldsymbol{y}_l|\boldsymbol{Y}_{l-1})$ and $p(\boldsymbol{y}_l|\boldsymbol{Y}_{l-1})$ can be formulated as

$$p(\boldsymbol{\xi}_j, \boldsymbol{y}_l|\boldsymbol{Y}_{l-1}) = N\left(\begin{bmatrix}\boldsymbol{\xi}_j\\ \boldsymbol{y}_l\end{bmatrix}; \begin{bmatrix}\hat{\boldsymbol{\xi}}_{j|l-1}\\ \hat{\boldsymbol{y}}_{l|l-1}\end{bmatrix}, \begin{bmatrix}\boldsymbol{P}^{\xi\xi}_{j,j|l-1} & \boldsymbol{P}^{\xi y}_{j,l|l-1}\\ (\boldsymbol{P}^{\xi y}_{j,l|l-1})^{\mathrm{T}} & \boldsymbol{P}^{yy}_{l,l|l-1}\end{bmatrix}\right), \tag{B4}$$

$$p(\boldsymbol{y}_l|\boldsymbol{Y}_{l-1}) = N(\boldsymbol{y}_l; \hat{\boldsymbol{y}}_{l|l-1}, \boldsymbol{P}^{yy}_{l,l|l-1}). \tag{B5}$$

According to the Bayesian rule, we have

$$p(\boldsymbol{\xi}_j|Y_l) = \frac{p(\boldsymbol{\xi}_j, \boldsymbol{y}_l|\boldsymbol{Y}_{l-1})}{p(\boldsymbol{y}_l|\boldsymbol{Y}_{l-1})}. \tag{B6}$$

Let

$$\Sigma = \begin{bmatrix}\boldsymbol{P}^{\xi\xi}_{j,j|l-1} & \boldsymbol{P}^{\xi y}_{j,l|l-1}\\ (\boldsymbol{P}^{\xi y}_{j,l|l-1})^{\mathrm{T}} & \boldsymbol{P}^{yy}_{l,l|l-1}\end{bmatrix}. \tag{B7}$$

Rearranging (B4) yields

$$p(\boldsymbol{\xi}_j, \boldsymbol{y}_l|\boldsymbol{Y}_{l-1}) = \frac{1}{\sqrt{|2\pi\Sigma|}}\exp\left(-\frac{1}{2}\begin{bmatrix}\tilde{\boldsymbol{\xi}}^{\mathrm{T}}_{j|l-1} & \tilde{\boldsymbol{y}}^{\mathrm{T}}_{l|l-1}\end{bmatrix}\Sigma^{-1}\begin{bmatrix}\tilde{\boldsymbol{\xi}}_{j|l-1}\\ \tilde{\boldsymbol{y}}_{l|l-1}\end{bmatrix}\right). \tag{B8}$$

Firstly, according to (B7), we rewrite Σ as follows:

$$\boldsymbol{\Sigma} = \begin{bmatrix}\boldsymbol{I}_L & \boldsymbol{K}^{\xi}_j\\ \boldsymbol{0}_{m\times L} & \boldsymbol{I}_m\end{bmatrix}\begin{bmatrix}\boldsymbol{P}^{\xi\xi}_{j,j|l} & \boldsymbol{0}_{L\times m}\\ \boldsymbol{0}_{m\times L} & \boldsymbol{P}^{yy}_{l,l|l-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}_L & \boldsymbol{0}_{L\times m}\\ (\boldsymbol{K}^{\xi}_j)^{\mathrm{T}} & \boldsymbol{I}_m\end{bmatrix}, \tag{B9}$$

and

$$|\Sigma| = \left|\boldsymbol{P}^{\xi\xi}_{j,j|l}\right|\left|\boldsymbol{P}^{yy}_{l,l|l-1}\right|, \tag{B10}$$

where $|\cdot|$ denotes matrix determinant, and L and m are the dimensions of vectors $\boldsymbol{\xi}_j$ and $\boldsymbol{y}_l$ respectively, and $\boldsymbol{K}^{\xi}_j$ and $\boldsymbol{P}^{\xi\xi}_{j,j|l}$ are given by (B1)–(B3). Then Σ^{-1} can be obtained as follows:

$$\Sigma^{-1} = \begin{bmatrix}\boldsymbol{I}_L & \boldsymbol{0}_{L\times m}\\ -(\boldsymbol{K}^{\xi}_j)^{\mathrm{T}} & \boldsymbol{I}_m\end{bmatrix}\begin{bmatrix}(\boldsymbol{P}^{\xi\xi}_{j,j|l})^{-1} & \boldsymbol{0}_{L\times m}\\ \boldsymbol{0}_{m\times L} & (\boldsymbol{P}^{yy}_{l,l|l-1})^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{I}_L & -\boldsymbol{K}^{\xi}_j\\ \boldsymbol{0}_{m\times L} & \boldsymbol{I}_m\end{bmatrix}. \tag{B11}$$

Further, by substituting (B10) and (B11) into (B8), and with the predicted measurement PDF $p(\boldsymbol{y}_l|\boldsymbol{Y}_{l-1})$ formulated in (B5), we obtain

$$\begin{aligned} p(\boldsymbol{\xi}_j, \boldsymbol{y}_l|\boldsymbol{Y}_{l-1}) &= \frac{1}{\sqrt{\left|2\pi\boldsymbol{P}^{\xi\xi}_{j,j|l}\right|\left|2\pi\boldsymbol{P}^{yy}_{l,l|l-1}\right|}}\\ &\quad\times\exp\left\{-\frac{1}{2}(\tilde{\boldsymbol{\xi}}_{j|l-1} - \boldsymbol{K}^{\xi}_j\tilde{\boldsymbol{y}}_{l|l-1})^{\mathrm{T}}(\boldsymbol{P}^{\xi\xi}_{j,j|l})^{-1}(\tilde{\boldsymbol{\xi}}_{j|l-1} - \boldsymbol{K}^{\xi}_j\tilde{\boldsymbol{y}}_{l|l-1}) - \frac{1}{2}(\tilde{\boldsymbol{y}}_{l|l-1})^{\mathrm{T}}(\boldsymbol{P}^{yy}_{l,l|l-1})^{-1}\tilde{\boldsymbol{y}}_{l|l-1}\right\}\\ &= N(\boldsymbol{\xi}_j; \hat{\boldsymbol{\xi}}_{j|l}, \boldsymbol{P}^{\xi\xi}_{j,j|l})p(\boldsymbol{y}_l|\boldsymbol{Y}_{l-1}), \end{aligned} \tag{B12}$$

where $\hat{\boldsymbol{\xi}}_{j|l}$ is given by (B1). By substituting (B12) into (B6), we finally obtain

$$p(\boldsymbol{\xi}_j|\boldsymbol{Y}_l) = N(\boldsymbol{\xi}_j; \hat{\boldsymbol{\xi}}_{j|l}, \boldsymbol{P}^{\xi\xi}_{j,j|l}), \tag{B13}$$

and mean $\hat{\boldsymbol{\xi}}_{j|l}$ and covariance matrix $\boldsymbol{P}^{\xi\xi}_{j,j|l}$ are formulated in (B1)–(B3).

Appendix C The proposed EKF with randomly delayed measurements and colored measurement noises

Given $\hat{\boldsymbol{x}}_{k-1|k-1}$, the first-order linearizations of $\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})$ and $\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1})$ about $\boldsymbol{x}_{k-1} = \hat{\boldsymbol{x}}_{k-1|k-1}$ can be formulated as

$$\boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1}) = \boldsymbol{f}_{k-1}(\hat{\boldsymbol{x}}_{k-1|k-1}) + \boldsymbol{\Phi}_{k-1|k-1}(\boldsymbol{x}_{k-1} - \hat{\boldsymbol{x}}_{k-1|k-1}), \tag{C1}$$

$$\boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1}) = \boldsymbol{h}_{k-1}(\hat{\boldsymbol{x}}_{k-1|k-1}) + \boldsymbol{H}_{k-1|k-1}(\boldsymbol{x}_{k-1} - \hat{\boldsymbol{x}}_{k-1|k-1}), \tag{C2}$$

where $\boldsymbol{\Phi}_{k-1|k-1}=\frac{\partial \boldsymbol{f}_{k-1}(\boldsymbol{x}_{k-1})}{\partial \boldsymbol{x}_{k-1}}|_{\boldsymbol{x}_{k-1}=\hat{\boldsymbol{x}}_{k-1|k-1}}$ and $\boldsymbol{H}_{k-1|k-1}=\frac{\partial \boldsymbol{h}_{k-1}(\boldsymbol{x}_{k-1})}{\partial \boldsymbol{x}_{k-1}}|_{\boldsymbol{x}_{k-1}=\hat{\boldsymbol{x}}_{k-1|k-1}}$. Substituting (C1) in (20), (22) and (24) and using (21), we can obtain

$$\hat{\boldsymbol{x}}_{k|k-1}=\boldsymbol{f}_{k-1}(\hat{\boldsymbol{x}}_{k-1|k-1}), \tag{C3}$$

$$\boldsymbol{P}_{k|k-1}=\boldsymbol{\Phi}_{k-1|k-1}\boldsymbol{P}_{k-1|k-1}\boldsymbol{\Phi}^{\mathrm{T}}_{k-1|k-1}+\boldsymbol{Q}_{k-1}, \tag{C4}$$

$$\begin{aligned}\boldsymbol{P}^{xv}_{k|k-1}&=\int_{\mathbb{R}^n}[\boldsymbol{f}_{k-1}(\hat{\boldsymbol{x}}_{k-1|k-1})+\boldsymbol{\Phi}_{k-1|k-1}(\boldsymbol{x}_{k-1}-\hat{\boldsymbol{x}}_{k-1|k-1})][\hat{\boldsymbol{v}}_{k-1|k-1}\\&\quad+(\boldsymbol{P}^{xv}_{k-1|k-1})^{\mathrm{T}}\boldsymbol{P}^{-1}_{k-1|k-1}(\boldsymbol{x}_{k-1}-\hat{\boldsymbol{x}}_{k-1|k-1})]^{\mathrm{T}}N(\boldsymbol{x}_{k-1};\hat{\boldsymbol{x}}_{k-1|k-1},\boldsymbol{P}_{k-1|k-1})\mathrm{d}\boldsymbol{x}_{k-1}\boldsymbol{\Psi}^{\mathrm{T}}_{k-1}-\hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{v}}^{\mathrm{T}}_{k|k-1}\\&=\boldsymbol{f}_{k-1}(\hat{\boldsymbol{x}}_{k-1|k-1})(\boldsymbol{\Psi}_{k-1}\hat{\boldsymbol{v}}_{k-1|k-1})^{\mathrm{T}}+\boldsymbol{\Phi}_{k-1|k-1}\boldsymbol{P}^{xv}_{k-1|k-1}\boldsymbol{\Psi}^{\mathrm{T}}_{k-1}-\hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{v}}^{\mathrm{T}}_{k|k-1}\\&=\boldsymbol{\Phi}_{k-1|k-1}\boldsymbol{P}^{xv}_{k-1|k-1}\boldsymbol{\Psi}^{\mathrm{T}}_{k-1}.\end{aligned} \tag{C5}$$

Given $\hat{\boldsymbol{x}}_{k|k-1}$, the first-order linearization of $\boldsymbol{h}_k(\boldsymbol{x}_k)$ about $\boldsymbol{x}_k=\hat{\boldsymbol{x}}_{k|k-1}$ can be written as

$$\boldsymbol{h}_k(\boldsymbol{x}_k)=\boldsymbol{h}_k(\hat{\boldsymbol{x}}_{k|k-1})+\boldsymbol{H}_{k|k-1}(\boldsymbol{x}_k-\hat{\boldsymbol{x}}_{k|k-1}), \tag{C6}$$

where $\boldsymbol{H}_{k|k-1}=\frac{\partial \boldsymbol{h}_k(\boldsymbol{x}_k)}{\partial \boldsymbol{x}_k}|_{\boldsymbol{x}_k=\hat{\boldsymbol{x}}_{k|k-1}}$. Substituting (C6) in (37)–(40), we can obtain

$$\hat{\boldsymbol{z}}_{k|k-1}=\boldsymbol{h}_k(\hat{\boldsymbol{x}}_{k|k-1})+\hat{\boldsymbol{v}}_{k|k-1}, \tag{C7}$$

$$\boldsymbol{P}^{xz}_{k|k-1}=\boldsymbol{P}_{k|k-1}\boldsymbol{H}^{\mathrm{T}}_{k|k-1}+\boldsymbol{P}^{xv}_{k|k-1}, \tag{C8}$$

$$\boldsymbol{P}^{zz}_{k|k-1}=\boldsymbol{H}_{k|k-1}\boldsymbol{P}_{k|k-1}\boldsymbol{H}^{\mathrm{T}}_{k|k-1}+\boldsymbol{H}_{k|k-1}\boldsymbol{P}^{xv}_{k|k-1}+(\boldsymbol{H}_{k|k-1}\boldsymbol{P}^{xv}_{k|k-1})^{\mathrm{T}}+\boldsymbol{P}^{vv}_{k|k-1}, \tag{C9}$$

$$\boldsymbol{P}^{vz}_{k|k-1}=(\boldsymbol{H}_{k|k-1}\boldsymbol{P}^{xv}_{k|k-1})^{\mathrm{T}}+\boldsymbol{P}^{vv}_{k|k-1}. \tag{C10}$$

Using (C1) and (C2) in (43), (44), (46) and (47), we can obtain

$$\hat{\boldsymbol{z}}_{k-1|k-1}=\boldsymbol{h}_{k-1}(\hat{\boldsymbol{x}}_{k-1|k-1})+\hat{\boldsymbol{v}}_{k-1|k-1}, \tag{C11}$$

$$\boldsymbol{P}^{zz}_{k-1|k-1}=\boldsymbol{H}_{k-1|k-1}\boldsymbol{P}_{k-1|k-1}\boldsymbol{H}^{\mathrm{T}}_{k-1|k-1}+\boldsymbol{H}_{k-1|k-1}\boldsymbol{P}^{xv}_{k-1|k-1}+(\boldsymbol{H}_{k-1|k-1}\boldsymbol{P}^{xv}_{k-1|k-1})^{\mathrm{T}}+\boldsymbol{P}^{vv}_{k-1|k-1}, \tag{C12}$$

$$\boldsymbol{P}^{xz}_{k,k-1|k-1}=\boldsymbol{\Phi}_{k-1|k-1}\boldsymbol{P}_{k-1|k-1}\boldsymbol{H}^{\mathrm{T}}_{k-1|k-1}+\boldsymbol{\Phi}_{k-1|k-1}\boldsymbol{P}^{xv}_{k-1|k-1}, \tag{C13}$$

$$\boldsymbol{P}^{vz}_{k,k-1|k-1}=\boldsymbol{\Psi}_{k-1}(\boldsymbol{H}_{k-1|k-1}\boldsymbol{P}^{xv}_{k-1|k-1})^{\mathrm{T}}+\boldsymbol{\Psi}_{k-1}\boldsymbol{P}^{vv}_{k-1|k-1}. \tag{C14}$$

The proposed EKF with randomly delayed measurements and colored measurement noises includes one-step predictions of state and measurement noise in (C3)–(C5), (21) and (23), one-step prediction of delayed measurement in (33)–(36) and (C7)–(C14), and filtering updates of state and measurement noise in (51)–(57).

Appendix D The proposed UKF with randomly delayed measurements and colored measurement noises

Given $\hat{\boldsymbol{x}}_{k-1|k-1}$ and $\boldsymbol{P}_{k-1|k-1}$, we construct the sigma points $\boldsymbol{x}_{i,k-1|k-1}$ according to UT [13]:

$$\begin{cases}\boldsymbol{x}_{0,k-1|k-1}=\hat{\boldsymbol{x}}_{k-1|k-1}, & \omega_0=\kappa/(n+\kappa),\\ \boldsymbol{x}_{i,k-1|k-1}=\hat{\boldsymbol{x}}_{k-1|k-1}+\sqrt{n+\kappa}\boldsymbol{S}_{k-1|k-1}\boldsymbol{e}_i, & \omega_i=0.5/(n+\kappa),\\ \boldsymbol{x}_{i+n,k-1|k-1}=\hat{\boldsymbol{x}}_{k-1|k-1}-\sqrt{n+\kappa}\boldsymbol{S}_{k-1|k-1}\boldsymbol{e}_i, & \omega_{i+n}=0.5/(n+\kappa),\end{cases} \tag{D1}$$

where $i=1,2,\ldots,n$, w_i are the weights of $\boldsymbol{x}_{i,k-1|k-1}$, $\boldsymbol{e}_i$ denotes the i-th column of a unit matrix, and $\boldsymbol{S}_{k-1|k-1}$ is the square-root matrix of $\boldsymbol{P}_{k-1|k-1}$, i.e. $\boldsymbol{P}_{k-1|k-1}=\boldsymbol{S}_{k-1|k-1}\boldsymbol{S}^{\mathrm{T}}_{k-1|k-1}$. Compute the propagated sigma points

$$\boldsymbol{X}_{i,k|k-1}=\boldsymbol{f}_{k-1}(\boldsymbol{x}_{i,k-1|k-1}),\qquad \boldsymbol{\eta}_{i,k-1|k-1}=\boldsymbol{h}_{k-1}(\boldsymbol{x}_{i,k-1|k-1}), \tag{D2}$$

$$\hat{\boldsymbol{v}}_{i,x,k-1|k-1}=\hat{\boldsymbol{v}}_{k-1|k-1}+(\boldsymbol{P}^{xv}_{k-1|k-1})^{\mathrm{T}}\boldsymbol{P}^{-1}_{k-1|k-1}(\boldsymbol{x}_{i,k-1|k-1}-\hat{\boldsymbol{x}}_{k-1|k-1}), \tag{D3}$$

where $i=0,1,\ldots,2n$. Then, (20), (22), (24), (43), (44), (46) and (47) are approximated as follows:

$$\hat{\boldsymbol{x}}_{k|k-1}=\sum_{i=0}^{2n}w_i\boldsymbol{X}_{i,k|k-1}, \tag{D4}$$

$$\boldsymbol{P}_{k|k-1}=\sum_{i=0}^{2n}w_i\boldsymbol{X}_{i,k|k-1}\boldsymbol{X}^{\mathrm{T}}_{i,k|k-1}-\hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{x}}^{\mathrm{T}}_{k|k-1}+\boldsymbol{Q}_{k-1}, \tag{D5}$$

$$\boldsymbol{P}^{xv}_{k|k-1}=\left[\sum_{i=0}^{2n}w_i\boldsymbol{X}_{i,k|k-1}\hat{\boldsymbol{v}}^{\mathrm{T}}_{i,x,k-1|k-1}\right]\boldsymbol{\Psi}^{\mathrm{T}}_{k-1}-\hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{v}}^{\mathrm{T}}_{k|k-1}, \tag{D6}$$

$$\hat{\boldsymbol{z}}_{k-1|k-1} = \sum_{i=0}^{2n} w_i \boldsymbol{\eta}_{i,k-1|k-1} + \hat{\boldsymbol{v}}_{k-1|k-1}, \tag{D7}$$

$$\boldsymbol{P}^{zz}_{k-1|k-1} = \sum_{i=0}^{2n} w_i (\boldsymbol{\eta}_{i,k-1|k-1} + \hat{\boldsymbol{v}}_{i,x,k-1|k-1})(\boldsymbol{\eta}_{i,k-1|k-1} + \hat{\boldsymbol{v}}_{i,x,k-1|k-1})^{\mathrm{T}} + \boldsymbol{\Omega}_{k-1|k-1} - \hat{\boldsymbol{z}}_{k-1|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1}, \tag{D8}$$

$$\boldsymbol{P}^{xz}_{k,k-1|k-1} = \sum_{i=0}^{2n} w_i \boldsymbol{X}_{i,k|k-1}(\boldsymbol{\eta}_{i,k-1|k-1} + \hat{\boldsymbol{v}}_{i,x,k-1|k-1})^{\mathrm{T}} - \hat{\boldsymbol{x}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k-1|k-1}, \tag{D9}$$

$$\boldsymbol{P}^{vz}_{k,k-1|k-1} = \boldsymbol{\Psi}_{k-1} \sum_{i=0}^{2n} w_i \hat{\boldsymbol{v}}_{i,x,k-1|k-1} \boldsymbol{\eta}^{\mathrm{T}}_{i,k-1|k-1} + \boldsymbol{\Psi}_{k-1}\boldsymbol{P}^{vv}_{k-1|k-1} - \hat{\boldsymbol{v}}_{k|k-1}(\hat{\boldsymbol{z}}_{k-1|k-1} - \hat{\boldsymbol{v}}_{k-1|k-1})^{\mathrm{T}}. \tag{D10}$$

Given $\hat{\boldsymbol{x}}_{k|k-1}$ and $\boldsymbol{P}_{k|k-1}$ by (D4) and (D5), we construct the sigma points $\boldsymbol{x}_{i,k|k-1}$ according to UT.

$$\begin{cases} \boldsymbol{x}_{0,k|k-1} = \hat{\boldsymbol{x}}_{k-1|k-1}, \\ \boldsymbol{x}_{i,k|k-1} = \hat{\boldsymbol{x}}_{k-1|k-1} + \sqrt{n+\kappa}\boldsymbol{S}_{k-1|k-1}\boldsymbol{e}_i, \\ \boldsymbol{x}_{i+n,k|k-1} = \hat{\boldsymbol{x}}_{k-1|k-1} - \sqrt{n+\kappa}\boldsymbol{S}_{k-1|k-1}\boldsymbol{e}_i, \end{cases} \tag{D11}$$

where $i = 1, 2, \ldots, n$. Compute the propagated sigma points:

$$\boldsymbol{Z}_{i,k|k-1} = \boldsymbol{h}_k(\boldsymbol{x}_{i,k|k-1}), \qquad \hat{\boldsymbol{v}}_{i,x,k|k-1} = \hat{\boldsymbol{v}}_{k|k-1} + (\boldsymbol{P}^{xv}_{k|k-1})^{\mathrm{T}}\boldsymbol{P}^{-1}_{k|k-1}(\boldsymbol{x}_{i,k|k-1} - \hat{\boldsymbol{x}}_{k|k-1}), \tag{D12}$$

where $i = 0, 1, \ldots, 2n$. With (D12), (37)–(40) are approximated as follows:

$$\hat{\boldsymbol{z}}_{k|k-1} = \sum_{i=0}^{2n} w_i \boldsymbol{Z}_{i,k|k-1} + \hat{\boldsymbol{v}}_{k|k-1}, \tag{D13}$$

$$\boldsymbol{P}^{xz}_{k|k-1} = \sum_{i=0}^{2n} w_i \boldsymbol{x}_{i,k|k-1}\boldsymbol{Z}^{\mathrm{T}}_{i,k|k-1} + \boldsymbol{P}^{xv}_{k|k-1} - \hat{\boldsymbol{x}}_{k|k-1}(\hat{\boldsymbol{z}}_{k|k-1} - \hat{\boldsymbol{v}}_{k|k-1})^{\mathrm{T}}, \tag{D14}$$

$$\boldsymbol{P}^{zz}_{k|k-1} = \sum_{i=0}^{2n} w_i (\boldsymbol{Z}_{i,k|k-1} + \hat{\boldsymbol{v}}_{i,x,k|k-1})(\boldsymbol{Z}_{i,k|k-1} + \hat{\boldsymbol{v}}_{i,x,k|k-1})^{\mathrm{T}} + \boldsymbol{\Omega}_{k|k-1} - \hat{\boldsymbol{z}}_{k|k-1}\hat{\boldsymbol{z}}^{\mathrm{T}}_{k|k-1}, \tag{D15}$$

$$\boldsymbol{P}^{vz}_{k|k-1} = \sum_{i=0}^{2n} w_i \hat{\boldsymbol{v}}_{i,x,k|k-1}\boldsymbol{Z}^{\mathrm{T}}_{i,k|k-1} + \boldsymbol{P}^{vv}_{k|k-1} - \hat{\boldsymbol{v}}_{k|k-1}(\hat{\boldsymbol{z}}_{k|k-1} - \hat{\boldsymbol{v}}_{k|k-1})^{\mathrm{T}}. \tag{D16}$$

The proposed UKF with randomly delayed measurements and colored measurement noises includes one-step predictions of state and measurement noise in (D4)–(D6), (21) and (23), one-step prediction of delayed measurement in (33)–(36), (42), (45), (D7)–(D10) and (D13)–(D16), and filtering updates of state and measurement noise in (51)–(57).