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Feedback control for a class of second order hyperbolic distributed parameter systems

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Abstract This paper deals with the problem of state feedback control for a class of the distributed parameter systems with the disturbance term. And the considered distributed parameter systems are composed of the second order hyperbolic partial differential equations. Two different classes of restrictions on the disturbance term are given, one is that the disturbance term satisfies the linear growth constraint condition to the state variables of the system, and the other is that the disturbance term obeys the bound constraint under the significance of L_2. Based on a variable structure method, the state feedback controllers are obtained by means of constructing appropriate Lyapunov functional. The closed-loop systems are globally asymptotically stable on $W^{1,2}(0,1) \times L_2(0,1)$ space under the effect of the state feedback control laws. Simulation results illustrate the effectiveness of the proposed method.

Keywords state feedback, globally asymptotically stable, second order hyperbolic distributed parameter systems, variable structure method, Lyapunov functional

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1 Introduction

Since many practical problems can be described by the DPSs (distributed parameter systems) governed by the PDEs (partial differential equations), the applications of DPSs have been involved in many fields in the last few years, and a series of the research achievements have been obtained [1–9]. In the field of the control for DPSs, hitherto, two methods are often used: one is the boundary control [1–4], the other is the distributed control [5–9]. This paper deals with the distributed control problems of DPSs.

Recently, more attention has been paid to the control problem of DPSs based on the variable structure method. Ref. [4] proposed the sliding mode boundary control problem of a parabolic PDE system with parameter variations and boundary uncertainties. For a class of parabolic DPSs with the bounded disturbance, Ref. [7] constructed a discontinuous sliding-mode feedback controller to guarantee the asymptotic stability of the tracking errors on $L_2(0,1)\times L_2(0,1)$ space. Refs. [8,9] discussed the tracking control problem for the second order hyperbolic DPSs with the bounded disturbance, and obtained the asymptotic stability of the tracking errors on $W^{1,2}(0,1) \times L_2(0,1)$ space based on a discontinuous sliding-mode feedback controller. Because of the discontinuity of the controller, the conclusions obtained in [7–9] shared

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something in common: the asymptotic stability of the tracking errors was under the significance of the generalized solution, no matter whether the system contained the disturbance term or not.

Based on [9], this paper further studies the feedback control problem for second order hyperbolic DPSs. For the same system as that in [9], two different classes of restrictions on the disturbance term (one is discussed in [9], the other is not) are given. And by constructing Lyapunov functionals and controllers which are different from that in [9], two conclusions are obtained, one is that, when the disturbance term satisfies the linear growth constraint condition to the state variables of the system, which is not discussed in [9], the strong solution of the closed-loop system is globally asymptotically stable on $W^{1,2}(0,1)\times L_2(0,1)$ space; the other is that, when the disturbance term obeys the same bound constraint as that in [9], the generalized solution of the closed-loop system is globally asymptotically stable on $W^{1,2}(0,1)\times L_2(0,1)$ space. Furthermore, when the disturbance term vanishes, the asymptotic stability in this paper is under the significance of strong solution, which is different from [9]. Therefore, under the same assumptions, the results obtained in this paper are better than that in [9]. Since tracking problem can be converted into stabilization problem, for convenience, this paper focuses on the stabilization problem, i.e., set the given reference y_r in [9] to zero.

In this paper, we adopt the following notational conventions: for function $z(\varsigma)\in L_2(a,b)$, $a\leqslant b$, denote $\|z(\cdot)\|_2=\sqrt{\int_a^b z^2(\varsigma)\mathrm{d}\varsigma}$, and $W^{l,2}(a,b)$ stands for the Sobolev space of absolutely continuous scalar functions on $[a,b]$ with square integrable derivatives of the order $l\geqslant 1$.

2 Problem formulation

First of all, we give a brief description of the work in [9]. Consider the following second order hyperbolic DPS [9]:

$$y_{tt}(\xi,t)=v^2y_{\xi\xi}(\xi,t)+u(\xi,t)+\psi(\xi,t), \tag{1}$$

where $\xi\in[0,1]$ is the one-dimensional space variable, $t>0$ is the time variable, and $(y,y_t)\in L_2(0,1)\times L_2(0,1)$, $t\geqslant 0$, is the state vector. The coefficient $v^2\in R^+$ stands for elasticity, $u(\xi,t)$ is the distributed control input, and $\psi(\xi,t)$ represents a distributed uncertain disturbance source term.

The system (1) satisfies the following homogeneous Neumann BCs:

$$y_\xi(0,t)=y_\xi(1,t)=0, \tag{2}$$

or Dirichlet BCs:

$$y(0,t)=y(1,t)=0. \tag{3}$$

The initial conditions (ICs)

$$y(\xi,0)=\varphi_0(\xi)\in W^{2,2}(0,1),\quad y_t(\xi,0)=\varphi_1(\xi)\in W^{2,2}(0,1) \tag{4}$$

are assumed to meet the boundary conditions (BCs) imposed on the system (1).

Assume the disturbance term ψ satisfy the following condition:

$$\|\psi\|_2=\sqrt{\int_0^1\psi^2(\xi,t)\mathrm{d}\xi}\leqslant M,\ \forall t\geqslant 0, \tag{5}$$

where M is a prior known constant. By constructing the following variable structure distributed controller:

$$u(y,y_t,\xi,t)=-\lambda_1\frac{y(\xi,t)}{\|y(\cdot,t)\|_2}-\lambda_2\frac{y_t(\xi,t)}{\|y_t(\cdot,t)\|_2}, \tag{6}$$

and the two different Lyapunov functionals:

$$V(t)=\lambda_1\sqrt{\int_0^1 y^2(\xi,t)\mathrm{d}\xi}+\frac{1}{2}\int_0^1 y_t^2(\xi,t)\mathrm{d}\xi+\frac{1}{2}v^2\int_0^1 y_\xi^2(\xi,t)\mathrm{d}\xi, \tag{7}$$

$$V_R(t) = V(t) + K_R \int_0^1 y(\xi,t) y_t(\xi,t) \mathrm{d}\xi, \tag{8}$$

Ref. [9] obtained that: when $\lambda_2 > M$, $\lambda_1 > \lambda_2 + M$, the generalized solution of the system (1) is globally asymptotically stable on $W^{1,2}(0,1) \times L_2(0,1)$ space (see Theorem 1 in [9], where $y_r = 0$). Where Eq. (7) is used to prove the stability, Eq. (8) is used to prove the asymptotic convergence, Eq. (6) is referred to as the distributed twisting controller (see [9]), where "$\frac{y(\xi,t)}{\|y(\cdot,t)\|_2}$" and "$\frac{y_t(\xi,t)}{\|y_t(\cdot,t)\|_2}$" are called as the unit feedback signals (see [10]), whose norms are 1 everywhere with the exception of the discontinuity manifold, $\|y(\cdot,t)\|_2 = \sqrt{\int_0^1 y^2(\xi,t)\mathrm{d}\xi}$, and $\|y_t(\cdot,t)\|_2 = \sqrt{\int_0^1 y_t^2(\xi,t)\mathrm{d}\xi}$.

As mentioned in Remark 1 of [9], generally speaking, the disturbance term ψ may depend on the state variables y and y_t. When ψ satisfies the linear growth constraint condition to y and y_t, that is

$$|\psi(y(\xi,t), y_t(\xi,t), t)| \leqslant M_1 |y(\xi,t)| + M_2 |y_t(\xi,t)|, \tag{9}$$

how to design the feedback controller of the system (1) is not discussed in [9].

This paper studies the feedback control problem of the system (1) under the ICs (4) and the homogeneous BCs (2) (or (3)). By constructing the Lyapunov functional which is different from (7), (8) and the feedback controller which is different from (6), we conclude that when the disturbance term ψ satisfies the linear growth constraint condition (9), the strong solution of the closed-loop system is globally asymptotically stable, and when the disturbance term ψ satisfies the bound constraint (5), the generalized solution of the closed-loop system is globally asymptotically stable. For the definitions of strong solution and generalized solution, see [9].

3 Main results

Theorem 1. Assume that the homogeneous BCs (2) (or (3)) and ICs (4) are satisfied and the disturbance term ψ satisfies the linear growth constraint condition (9). Construct the following state feedback controller for the system (1):

$$u(y, y_t) = -\lambda_1 y(\xi,t) - \lambda_2 y_t(\xi,t), \tag{10}$$

then the strong solution of the closed-loop system (1) is globally asymptotically stable on $W^{1,2}(0,1) \times L_2(0,1)$ space, i.e., $\lim_{t\to\infty} \int_0^1 (y^2(\xi,t) + y_\xi^2(\xi,t) + y_t^2(\xi,t))\mathrm{d}\xi = 0$, under the condition that

$$\lambda_1 > M_1 + \max\{M_1, M_2\}, \quad \lambda_2 > M_2 + 1 + \max\{M_1, M_2\}. \tag{11}$$

Proof. Construct the following Lyapunov functional:

$$V(t) = \frac{1}{2}\int_0^1 \left\{(\lambda_1+\lambda_2) y^2(\xi,t) + 2y(\xi,t)y_t(\xi,t) + y_t^2(\xi,t)\right\}\mathrm{d}\xi + \frac{v^2}{2}\int_0^1 y_\xi^2 \mathrm{d}\xi. \tag{12}$$

It follows from (11) that $\lambda_1 + \lambda_2 > 1$. So V is positive definite for $(y, y_t) \in W^{1,2}(0,1) \times L_2(0,1)$, i.e., there exists $\beta_2(\lambda_1, \lambda_2, v^2) \geqslant \beta_1(\lambda_1, \lambda_2, v^2) > 0$ such that

$$\beta_1 \int_0^1 (y^2 + y_\xi^2 + y_t^2)\mathrm{d}\xi \leqslant V(t) \leqslant \beta_2 \int_0^1 (y^2 + y_\xi^2 + y_t^2)\mathrm{d}\xi. \tag{13}$$

Taking the derivative of V, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \int_0^1 \left\{(\lambda_1+\lambda_2) y y_t + y_t^2 + y y_{tt} + y_t y_{tt}\right\}\mathrm{d}\xi + v^2 \int_0^1 y_\xi y_{t\xi} \mathrm{d}\xi.$$

Integrating by parts and combining with (1) and (10), it yields

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \int_0^1 \left\{(\lambda_1+\lambda_2) y y_t + y_t^2 + y(v^2 y_{\xi\xi} - \lambda_1 y - \lambda_2 y_t + \psi) + y_t(v^2 y_{\xi\xi} - \lambda_1 y - \lambda_2 y_t + \psi)\right\}\mathrm{d}\xi$$

$$+v^2\int_0^1 y_\xi y_{t\xi}\mathrm{d}\xi$$

$$=\int_0^1\left\{-\lambda_1 y^2-(\lambda_2-1)y_t^2+v^2yy_{\xi\xi}+v^2y_ty_{\xi\xi}+y\psi+y_t\psi\right\}\mathrm{d}\xi+v^2\int_0^1 y_\xi y_{t\xi}\mathrm{d}\xi$$

$$=\int_0^1\left\{-\lambda_1 y^2-(\lambda_2-1)y_t^2-v^2y_\xi^2-v^2y_{t\xi}y_\xi+y\psi+y_t\psi\right\}\mathrm{d}\xi+v^2yy_\xi\big|_0^1+v^2y_ty_\xi\big|_0^1$$

$$+v^2\int_0^1 y_\xi y_{t\xi}\mathrm{d}\xi$$

$$=\int_0^1\left\{-\lambda_1 y^2-(\lambda_2-1)y_t^2-v^2y_\xi^2+y\psi+y_t\psi\right\}\mathrm{d}\xi+v^2yy_\xi\big|_0^1+v^2y_ty_\xi\big|_0^1.$$

Taking into account the BCs (2) (or (3)), it yields

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\int_0^1\left\{-\lambda_1 y^2-(\lambda_2-1)y_t^2-v^2y_\xi^2+y\psi+y_t\psi\right\}\mathrm{d}\xi$$

$$=\int_0^1\left\{-\lambda_1 y^2-(\lambda_2-1)y_t^2+y\psi+y_t\psi\right\}\mathrm{d}\xi-v^2\int_0^1 y_\xi^2\mathrm{d}\xi$$

$$\leqslant\int_0^1\left\{-\lambda_1 y^2-(\lambda_2-1)y_t^2+|y|\,|\psi|+|y_t|\,|\psi|\right\}\mathrm{d}\xi-v^2\int_0^1 y_\xi^2\mathrm{d}\xi.$$

From the linear growth constraint condition (9), we have

$$\frac{\mathrm{d}V}{\mathrm{d}t}\leqslant\int_0^1\left\{-(\lambda_1-M_1)y^2+(M_1+M_2)\,|y|\,|y_t|-(\lambda_2-1-M_2)y_t^2\right\}\mathrm{d}\xi-v^2\int_0^1 y_\xi^2\mathrm{d}\xi. \tag{14}$$

It follows from (11) that

$$(M_1+M_2)^2\leqslant 4\left(\max\{M_1,M_2\}\right)^2<4(\lambda_1-M_1)(\lambda_2-1-M_2).$$

So the right-hand side term of (14) is negative definite for $(y,y_t)\in W^{1,2}(0,1)\times L_2(0,1)$, i.e., there exists $\gamma(\lambda_1,\lambda_2,M_1,M_2,v^2)>0$ such that

$$\int_0^1\left\{-(\lambda_1-M_1)y^2+(M_1+M_2)\,|y|\,|y_t|-(\lambda_2-1-M_2)y_t^2\right\}\mathrm{d}\xi-v^2\int_0^1 y_\xi^2\mathrm{d}\xi$$

$$\leqslant-\gamma\int_0^1(y^2+y_\xi^2+y_t^2)\mathrm{d}\xi. \tag{15}$$

From (13)–(15), we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t}\leqslant-\frac{\gamma}{\beta_2}V.$$

So

$$V(t)\leqslant \mathrm{e}^{-\frac{\gamma}{\beta_2}t}\mathrm{V}(0).$$

Combining with (13) and the above expression, we have

$$\int_0^1(y^2(\xi,t)+y_\xi^2(\xi,t)+y_t^2(\xi,t))\mathrm{d}\xi\leqslant\frac{1}{\beta_1}V(t)\leqslant\frac{1}{\beta_1}\mathrm{e}^{-\frac{\gamma}{\beta_2}t}\mathrm{V}(0)$$

$$\leqslant\frac{\beta_2}{\beta_1}\mathrm{e}^{-\frac{\gamma}{\beta_2}t}\int_0^1(\mathrm{y}^2(\xi,0)+\mathrm{y}_\xi^2(\xi,0)+\mathrm{y}_\mathrm{t}^2(\xi,0))\mathrm{d}\xi,$$

which implies that the strong solution of the closed-loop system (1) is globally asymptotically stable on $W^{1,2}(0,1)\times L_2(0,1)$ space. This completes the proof.

Remark 1. From [9], if the distributed control input $u(\xi,t)$ is sufficiently smooth, then the system (1) possesses a unique strong solution $y(\xi,t)$, and if the distributed control input $u(\xi,t)$ is discontinuous, then the solution of the system (1) is generalized.

Theorem 2. Assume that the homogeneous BCs (2) (or (3)) and ICs (4) are satisfied and the disturbance term ψ satisfies the bound constraint condition (5). Construct the following variable structure state feedback controller for the system (1):

$$u(y,y_t) = -\lambda_1 y(\xi,t) - \lambda_2 y_t(\xi,t) - 3M\frac{y(\xi,t)}{\|y(\cdot,t)\|_2} - M\frac{y_t(\xi,t)}{\|y_t(\cdot,t)\|_2}, \tag{16}$$

then the generalized solution of the closed-loop system (1) is globally asymptotically stable on $W^{1,2}(0,1)\times L_2(0,1)$ space, when

$$\lambda_1 > 0,\ \lambda_2 > 1.$$

Proof. It follows from $\lambda_1+\lambda_2 > 1$ that the Lyapunov functional V in (12) is positive definite for $(y,y_t)\in W^{1,2}(0,1)\times L_2(0,1)$. Therefore (13) still holds.

Taking the derivative of V, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \int_0^1 \left\{(\lambda_1+\lambda_2)yy_t + y_t^2 + yy_{tt} + y_ty_{tt}\right\}\mathrm{d}\xi + v^2\int_0^1 y_\xi y_{t\xi}\mathrm{d}\xi.$$

Integrating by parts and combining with (1) and (16), we obtain

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \int_0^1 \left\{(\lambda_1+\lambda_2)yy_t + y_t^2 + y\left(v^2y_{\xi\xi} - \lambda_1 y - \lambda_2 y_t - 3M\frac{y}{\|y\|_2} - M\frac{y_t}{\|y_t\|_2} + \psi\right)\right\}\mathrm{d}\xi \\ &\quad + \int_0^1 \left\{y_t\left(v^2y_{\xi\xi} - \lambda_1 y - \lambda_2 y_t - 3M\frac{y}{\|y\|_2} - M\frac{y_t}{\|y_t\|_2} + \psi\right)\right\}\mathrm{d}\xi + v^2\int_0^1 y_\xi y_{t\xi}\mathrm{d}\xi \\ &= \int_0^1 \left\{-\lambda_1y^2 - (\lambda_2-1)y_t^2 + v^2yy_{\xi\xi} + v^2y_ty_{\xi\xi} + y\psi + y_t\psi\right\}\mathrm{d}\xi + v^2\int_0^1 y_\xi y_{t\xi}\mathrm{d}\xi \\ &\quad -3M\int_0^1 \frac{y^2}{\|y\|_2}\mathrm{d}\xi - M\int_0^1 \frac{yy_t}{\|y_t\|_2}\mathrm{d}\xi - 3M\int_0^1 \frac{yy_t}{\|y\|_2}\mathrm{d}\xi - M\int_0^1 \frac{y_t^2}{\|y_t\|_2}\mathrm{d}\xi \\ &= \int_0^1 \left\{-\lambda_1y^2 - (\lambda_2-1)y_t^2 - v^2y_\xi^2 - v^2y_{t\xi}y_\xi + y\psi + y_t\psi\right\}\mathrm{d}\xi + v^2yy_\xi\big|_0^1 + v^2y_ty_\xi\big|_0^1 \\ &\quad +v^2\int_0^1 y_\xi y_{t\xi}\mathrm{d}\xi - 3M\int_0^1 \frac{y^2}{\|y\|_2}\mathrm{d}\xi - M\int_0^1 \frac{yy_t}{\|y_t\|_2}\mathrm{d}\xi - 3M\int_0^1 \frac{yy_t}{\|y\|_2}\mathrm{d}\xi - M\int_0^1 \frac{y_t^2}{\|y_t\|_2}\mathrm{d}\xi.\end{aligned}$$

Taking into account the BCs (2) (or (3)), it yields

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \int_0^1 \left\{-\lambda_1y^2 - (\lambda_2-1)y_t^2 - v^2y_\xi^2 + y\psi + y_t\psi\right\}\mathrm{d}\xi \\ &\quad -3M\|y\|_2 - M\int_0^1 \frac{yy_t}{\|y_t\|_2}\mathrm{d}\xi - 3M\int_0^1 \frac{yy_t}{\|y\|_2}\mathrm{d}\xi - M\|y_t\|_2.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we have

$$\int_0^1 y\psi\mathrm{d}\xi \leqslant \|y\|_2\|\psi\|_2, \quad \int_0^1 y_t\psi\mathrm{d}\xi \leqslant \|y_t\|_2\|\psi\|_2, \quad \left|\int_0^1 yy_t\mathrm{d}\xi\right| \leqslant \int_0^1 |y|\,|y_t|\mathrm{d}\xi \leqslant \|y\|_2\|y_t\|_2.$$

Combining with the bound constraint condition (5), it follows that

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leqslant \int_0^1 \left\{-\lambda_1y^2 - (\lambda_2-1)y_t^2 - v^2y_\xi^2\right\}\mathrm{d}\xi + \|y\|_2\|\psi\|_2 + \|y_t\|_2\|\psi\|_2$$

$$-3M\left\|y\right\|_2+\frac{M}{\left\|y_t\right\|_2}\left|\int_0^1 yy_t\mathrm{d}\xi\right|-3M\int_0^1\frac{yy_t}{\left\|y\right\|_2}\mathrm{d}\xi-M\left\|y_t\right\|_2$$

$$\leqslant\int_0^1\left\{-\lambda_1y^2-(\lambda_2-1)y_t^2-v^2y_\xi^2\right\}\mathrm{d}\xi+M\left\|y\right\|_2+M\left\|y_t\right\|_2$$

$$-3M\left\|y\right\|_2+M\left\|y\right\|_2-3M\int_0^1\frac{yy_t}{\left\|y\right\|_2}\mathrm{d}\xi-M\left\|y_t\right\|_2$$

$$=\int_0^1\left\{-\lambda_1y^2-(\lambda_2-1)y_t^2-v^2y_\xi^2\right\}\mathrm{d}\xi-3\frac{M}{\left\|y\right\|_2}\int_0^1 yy_t\mathrm{d}\xi-M\left\|y\right\|_2.$$

Since

$$\frac{1}{\left\|y\right\|_2}\int_0^1 yy_t\mathrm{d}\xi=\frac{1}{2\left\|y\right\|_2}\int_0^1\frac{\partial}{\partial t}\left\{y^2\right\}\mathrm{d}\xi=\frac{1}{2\sqrt{\int_0^1 y^2d\xi}}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^1 y^2\mathrm{d}\xi\right\}$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sqrt{\int_0^1 y^2\mathrm{d}\xi}\right\}=\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|y\right\|_2\right\},$$

we can obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t}\leqslant\int_0^1\left\{-\lambda_1y^2-(\lambda_2-1)y_t^2-v^2y_\xi^2\right\}\mathrm{d}\xi-3M\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|y\right\|_2\right\}-M\left\|y\right\|_2.$$

Denoting $\alpha=\min\left\{\lambda_1,\lambda_2-1,v^2\right\}$, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t}+3M\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|y\right\|_2\right\}\leqslant\int_0^1\left\{-\lambda_1y^2-(\lambda_2-1)y_t^2-v^2y_\xi^2\right\}\mathrm{d}\xi-M\left\|y\right\|_2$$

$$\leqslant-\alpha\int_0^1(y^2+y_\xi^2+y_t^2)\mathrm{d}\xi-M\left\|y\right\|_2\leqslant 0. \tag{17}$$

Firstly, we prove the stability. From (17) we know

$$V(t)+3M\left\|y(\cdot,t)\right\|_2\leqslant V(0)+3M\left\|y(\cdot,0)\right\|_2.$$

Denoting $E(t)=\int_0^1(y^2+y_\xi^2+y_t^2)\mathrm{d}\xi$ and combining with (13), it yields

$$\beta_1E(t)=\beta_1\int_0^1(y^2+y_\xi^2+y_t^2)\mathrm{d}\xi\leqslant V(t)\leqslant V(t)+3M\left\|y(\cdot,t)\right\|_2\leqslant V(0)+3M\left\|y(\cdot,0)\right\|_2$$

$$\leqslant\beta_2\int_0^1(y^2(\xi,0)+y_\xi^2(\xi,0)+y_t^2(\xi,0))\mathrm{d}\xi+3M\sqrt{\int_0^1 y^2(\xi,0)\mathrm{d}\xi}$$

$$\leqslant\beta_2\int_0^1(y^2(\xi,0)+y_\xi^2(\xi,0)+y_t^2(\xi,0))\mathrm{d}\xi+3M\sqrt{\int_0^1(y^2(\xi,0)+y_\xi^2(\xi,0)+y_t^2(\xi,0))\mathrm{d}\xi}$$

$$=\beta_2E(0)+3M\sqrt{E(0)}.$$

So

$$E(t)\leqslant\frac{\beta_2}{\beta_1}E(0)+\frac{3M}{\beta_1}\sqrt{E(0)}.$$

For any given $\varepsilon>0$, taking

$$\int_0^1(y^2(\xi,0)+y_\xi^2(\xi,0)+y_t^2(\xi,0))\mathrm{d}\xi=E(0)<\delta(\varepsilon)=\left(\frac{-\frac{3M}{\beta_1}+\sqrt{\left(\frac{3M}{\beta_1}\right)^2+4\frac{\beta_2}{\beta_1}\varepsilon}}{2\frac{\beta_2}{\beta_1}}\right)^2,$$

then we have

$$\int_0^1 (y^2(\xi,t) + y_\xi^2(\xi,t) + y_t^2(\xi,t))\mathrm{d}\xi = E(t) \leqslant \frac{\beta_2}{\beta_1}E(0) + \frac{3M}{\beta_1}\sqrt{E(0)} < \varepsilon, \quad \forall t \geqslant 0.$$

Therefore the generalized solution of the closed-loop system (1) is stable on $W^{1,2}(0,1) \times L_2(0,1)$ space.

Next, we prove the global asymptotic convergence. From (13) and (17), we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} + 3M\frac{\mathrm{d}}{\mathrm{d}t}\{\|y\|_2\} \leqslant -\alpha\int_0^1 (y^2 + y_\xi^2 + y_t^2)\mathrm{d}\xi - M\|y\|_2 \leqslant -\frac{\alpha}{\beta_2}V - M\|y\|_2$$

$$= -\frac{\alpha}{\beta_2}V - \frac{1}{3}3M\|y\|_2 \leqslant -\theta(V + 3M\|y\|_2),$$

where $\theta = \min\left\{\frac{\alpha}{\beta_2}, \frac{1}{3}\right\}$. Denoting $\bar{V} = V + 3M\|y\|_2$, it yields

$$\frac{\mathrm{d}\bar{V}}{\mathrm{d}t} \leqslant -\theta\bar{V}.$$

Further we have

$$\bar{V}(t) \leqslant \bar{V}(0)\mathrm{e}^{-\theta t}.$$

So

$$\lim_{t\to\infty} \bar{V}(t) = 0.$$

From $\bar{V} = V + 3M\|y\|_2$, we have

$$\lim_{t\to\infty} V(t) = 0. \tag{18}$$

It follows from (13) that

$$\frac{1}{\beta_2}V(t) \leqslant \int_0^1 (y^2 + y_\xi^2 + y_t^2)\mathrm{d}\xi \leqslant \frac{1}{\beta_1}V(t),$$

which together with (18) implies

$$\lim_{t\to\infty}\int_0^1 (y^2(\xi,t) + y_t^2(\xi,t) + y_\xi^2(\xi,t))\mathrm{d}\xi = 0.$$

Combining with the stability and the global asymptotic convergence above, we conclude that the generalized solution of the closed-loop system (1) is globally asymptotically stable on $W^{1,2}(0,1)\times L_2(0,1)$ space. This completes the proof.

Remark 2. For the same system, Theorem 2 acquires the same conclusion with that in Theorem 1 of [9] under the same conditions. For the differences of both the controllers ((16) and (6)) and the Lyapunov functionals ((12) and (7), (8)), the proof of Theorem 2 is different from that of Theorem 1 in [9]. When the disturbance term vanishes (M=0), the controller (16) is transformed into

$$u(y, y_t) = -\lambda_1 y(\xi,t) - \lambda_2 y_t(\xi,t),$$

and it is continuous differentiable. So the asymptotic stability in Theorem 2 is under the significance of strong solution. From the controller (6), we know that it is discontinuous even if the disturbance term vanishes (M=0). So the asymptotic stability in Theorem 1 of [9] is always under the significance of generalized solution. Therefore, it is obvious that the result obtained in Theorem 2 is better than that in Theorem 1 of [9].

Combining with Theorems 1 and 2, we have the following corollary.

Corollary 1. Assume that the homogeneous BCs (2) (or (3)) and the ICs (4) are satisfied. And the disturbance term ψ satisfies the following constraint:

$$|\psi(y(\xi,t), y_t(\xi,t), t)| \leqslant M_1|y(\xi,t)| + M_2|y_t(\xi,t)| + M.$$

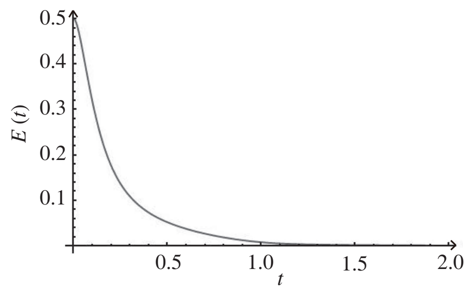

Figure 1 Trajectory of $E(t)$, $M=0$.

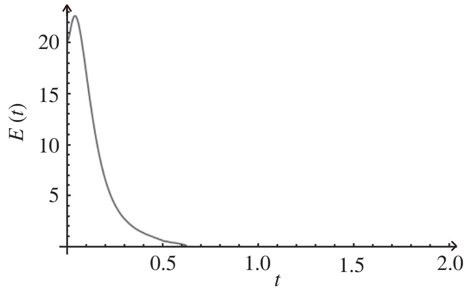

Figure 2 Trajectory of $E(t)$, $M=20$.

Constructing the following state feedback controller for the system (1):

$$u(y,y_t)=-\lambda_1 y(\xi,t)-\lambda_2 y_t(\xi,t)-3M\frac{y(\xi,t)}{\|y(\cdot,t)\|_2}-M\frac{y_t(\xi,t)}{\|y_t(\cdot,t)\|_2}.$$

We can conclude that the strong solution is globally asymptotically stable on $W^{1,2}(0,1)\times L_2(0,1)$ space when $M=0$, and the generalized solution is globally asymptotically stable on $W^{1,2}(0,1)\times L_2(0,1)$ space when $M>0$, under the condition that

$$\lambda_1>M_1+\max\{M_1,M_2\},\quad \lambda_2>M_2+1+\max\{M_1,M_2\}.$$

4 Simulation examples

Construct the following system:

$$y_{tt}(\xi,t)=y_{\xi\xi}(\xi,t)+u(\xi,t)+30\sin(2\pi\xi)y(\xi,t)+10\cos(2\pi t)y_t(\xi,t)+M\sin(2\pi\xi)\sin(2\pi t),$$

with the homogeneous Dirichlet BCs (3). Set the ICs

$$y(\xi,0)=\sin(2\pi\xi),\ y_t(\xi,0)=0.$$

According to Corollary 1, take $\lambda_1=61$, $\lambda_2=42$ and construct the state feedback controller as follows:

$$u(y,y_t)=-61y(\xi,t)-42y_t(\xi,t)-3M\frac{y(\xi,t)}{\|y(\cdot,t)\|_2}-M\frac{y_t(\xi,t)}{\|y_t(\cdot,t)\|_2}.$$

By using the mathematical software Mathematica, we have

Case 1. $M=0$. The simulation result of $E(t)=\int_0^1(y^2+y_t^2+y_\xi^2)\mathrm{d}\xi$ with the change of time is shown in Figure 1.

Case 2. $M=20$. The simulation result of $E(t)=\int_0^1(y^2+y_t^2+y_\xi^2)\mathrm{d}\xi$ with the change of time is shown in Figure 2.

5 Conclusion

This paper studies the feedback control problem of a class of uncertain distributed parameter systems. These systems are governed by second order hyperbolic partial differential equations and have well-posed ICs and BCs (first BCs or second BCs). Under the effect of the feedback control laws, we conclude that the strong solution of the closed-loop system is globally asymptotically stable on $W^{1,2}(0,1)\times L_2(0,1)$ space when the disturbance term ψ satisfies the linear growth constrain condition, and the generalized solution of the closed-loop system is globally asymptotically stable on $W^{1,2}(0,1)\times L_2(0,1)$ space when the disturbance term ψ satisfies the bound constraint condition. This paper extends the conclusion in [9] and the results obtained are better than that in [9], and it is verified by simulations.

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Conflict of interest The authors declare that they have no conflict of interest.

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