SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

September 2016, Vol. 59 092204:1–092204:10
doi: 10.1007/s11432-015-5423-6

Controllability of probabilistic Boolean control networks with time-variant delays in states

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Received February 9, 2016; accepted April 6, 2016; published online August 23, 2016

Abstract This paper investigates the controllability of probabilistic Boolean control networks (PBCNs) with time-variant delays in states. By cutting the time sequence, we split the network into at most countably infinitely many subnetworks with no delays, where any one of the longest subnetworks is called a controllability constructed path (CCP). When the CCP is of infinite length, we prove that the network is controllable iff any CCP is controllable, and give an equivalent condition for the controllability of the network. When it is of finite length, we give a necessary condition and a sufficient condition for the controllability of the network, and show that the controllability of the network is not equivalent to the controllability of a CCP.

Keywords probabilistic Boolean control network, time delay, controllability constructed path, semi-tensor product of matrices, controllability

Citation Zhang K Z, Zhang L J. Controllability of probabilistic Boolean control networks with time-variant delays in states. Sci China Inf Sci, 2016, 59(9): 092204, doi: 10.1007/s11432-015-5423-6

1 Introduction

The concept of Boolean networks (BNs) was initiated by Kauffman [1] in 1969 to approximately model genetic regulatory networks (GRNs). A BN consists of a finite number of nodes, each of which is in state 0 or 1, and these nodes update according to the states of these nodes. Though simple, a BN still reflects local interaction of nodes. In recent years, it became a common focus of both biologists and control theorists. In 2007, Akutsu et al. [2] proposed the concept of controllability of Boolean control networks (BCNs, a BN with external nodes), and pointed out that "One of the major goals of systems biology is to develop a control theory for complex biological systems." Hence the study on control theory of BCNs has its intrinsic meaning in the control-theoretic field, and also has potential applications in biological systems. In 2009, Cheng et al. [3] built a control-theoretic framework for BCNs based on their semi-tensor product (STP) of matrices and solved how to determine the controllability of BCNs. Since then, the study on control theory of BCNs has drawn much attention (cf. [4–11] and the monograph [12]).

When the properties of basic BCNs have been revealed, researchers usually turn to investigate extended BCNs, and also try to find the differences between basic BCNs and extended BCNs, because extended

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BCNs may possess more complex and interesting properties than the basic ones. For example, ref. [4] studied the controllability of BCNs with time-invariant delays in states. Later in [6], we revealed the essential difference of controllability between BCNs and BCNs with time-variant delays in states. In this paper, we continue the topics of [6], and go on with the theoretical research on BCNs with time delays in states. Specifically, we try to reveal the difference of controllability between BCNs and probabilistic BCNs (PBCNs) both with time-variant delays in states. The concept of probabilistic BNs (PBNs) was proposed in [13] to estimate real large-scale GRNs more accurately and to describe the uncertain factors that affect the updates of GRNs. A PBN/PBCN consists of a finite number of BNs/BCNs, and at each time step, it updates according to one BN/BCN with a fixed probability. So a PBN/PBCN is a generalization of a BN/BCN. According to their definitions, a PBN/PBCN with delays can describe more systems than a BN/BCN does. To the best of the authors' knowledge, there is only one related literature [7] dealing with the controllability of PBCNs with time-invariant delays. In this paper, we extend the results in [7] from time-invariant delays to time-variant delays, and extend the results in our previous paper [6] from the deterministic version to the probabilistic version.

The result of this paper is based on the results given in [6, 8]. Ref. [8] gave a fundamental equivalent condition for the controllability of PBCNs using the Markov chain theory. Recently, we proposed a new approach, i.e., seeking for a controllability constructed path (CCP), to deal with the controllability of BCNs with time-variant delays in states [6]. The highlight of this approach is to identify the controllability of the network with that of its CCP, i.e., a BCN that can be seen as a subsystem of the network. Hence it reduces the computational complexity considerably. Specifically, in [6], we showed that the controllability of the network is equivalent to that of its CCP, no matter the CCP is of finite or infinite length. In this paper, we will show that for probabilistic case, this equivalence does not always hold. Particularly, when the CCP is of infinite length, this equivalence still holds. Note that usually in the practical case time delay is bounded. Bounded time delay implies that the CCP is of infinite length, but the converse is not true (shown in Remark 1). So the main results of this paper are beyond practical use. For the practical case (i.e., bounded time delay), our method is much more effective than the existing method (shown in [6]). The other advantage of our method is that only our method can be used to deal with unbounded time delay.

The result of this paper is in the framework of an intuitive algebraic form of PBCNs based on the STP of matrices. The rest of this paper is organized as follows. Section 2 introduces some preliminaries about STP, PBCNs with time-variant delays in states, and their algebraic forms. Section 3 shows the main results on the controllability of PBCNs with time-variant delays in states: (i) an equivalent condition when the CCP is of infinite length, (ii) a sufficient condition and a necessary condition when the CCP is of finite length, and (iii) the essential difference of controllability between PBCNs with time-variant delays in states and BCNs with time-variant delays in states. A brief conclusion ends this paper in Section 4.

2 Preliminaries

Since the framework of STP is used in this paper, some notations about logic and STP are introduced.

- $\mathcal{R}$, the set of all real numbers
- $\mathcal{Z}$, the set of all integers
- $\mathcal{N}$, the set of all non-negative integers
- $\mathcal{D}$, the set $\{0, 1\}$
- δ_n^i, the i-th column of the identity matrix I_n
- $\mathbf{1}_k$, $\sum_{i=1}^{k} \delta_k^i$
- Δ_n, the set $\{\delta_n^1, \delta_n^2, \ldots, \delta_n^n\}$ ($\Delta := \Delta_2$)
- $\delta_n[i_1, \ldots, i_s]$, the matrix $[\delta_n^{i_1}, \ldots, \delta_n^{i_s}]$ ($i_1, \ldots, i_s \in \{1, ..., n\}$)
- $\mathcal{L}_{n\times s}$, the set of all $n \times s$ logical matrices, i.e., $\{\delta_n[i_1, \ldots, i_s] : i_1, \ldots, i_s \in \{1, 2, \ldots, n\}\}$
- $\mathcal{B}_{n\times s}$, the set of all $n \times s$ Boolean matrices, i.e., the set of all $n \times s$ matrices with all entries in $\mathcal{D}$
- $A > 0$, each entry of matrix A is positive

Definition 1 ([12]). Let $A \in \mathcal{R}_{m\times n}$, $B \in \mathcal{R}_{p\times q}$, and $\alpha = \mathrm{lcm}(n,p)$ be the least common multiple of n and p. The STP of A and B is defined as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}), \tag{1}$$

where $\otimes$ denotes the Kronecker product.

From this definition, one sees that the conventional product of matrices is a particular case of STP. Since STP keeps almost all properties of the conventional product [12], e.g., the associative law, we omit the symbol "$\ltimes$" of STP hereinafter.

In order to state our main results conveniently, we also introduce some basic notions about Boolean algebra briefly.

First, Boolean addition $\oplus$ and Boolean product $\odot$ of two Boolean variables $a, b \in \mathcal{D}$ are defined as

$$a \oplus b := a \vee b \in \mathcal{D} \tag{2}$$

and

$$a \odot b := a \wedge b \in \mathcal{D}, \tag{3}$$

respectively, where $\vee$ and $\wedge$ denote disjunction and conjunction, respectively.

Second, Boolean addition $\oplus$ and Boolean product $\odot$ of two Boolean matrices $A = (a_{ij}), B = (b_{ij}) \in \mathcal{B}_{n\times n}$ are defined as

$$A \oplus B := (a_{ij} \vee b_{ij}) \in \mathcal{B}_{n\times n} \tag{4}$$

and

$$A \odot B := (c_{ij}) \in \mathcal{B}_{n\times n}, \tag{5}$$

where $c_{ij} = \oplus_{k=1}^{n}(a_{ik} \odot b_{kj})$, respectively.

In this paper, we investigate the following PBCNs with n state nodes, m input nodes and time-variant time delays in states (PBCNTD):

$$x(t+1) = f(t, u(t), x(t-\tau(t))), \tag{6}$$

where $x \in \mathcal{D}^n$; $u \in \mathcal{D}^m$; $t_0 \in \mathcal{Z}$; $t = t_0, t_0+1, \ldots$; $\tau : \{t \in \mathcal{Z} : t \geqslant t_0\} \to \mathcal{N}$ is a mapping, called the time delay function. Throughout this paper, without loss of generality, we assume that for all $t \geqslant t_0$, $t - \tau(t) \geqslant t_0 - \tau(t_0)$ to assure that system (6) has a starting point. For each time step $t \geqslant t_0$, each updating function $f(t)$ is chosen randomly from M distinct logical functions from $\mathcal{D}^{n+m}$ to $\mathcal{D}^n$:

$$\{f_1, f_2, \ldots, f_M\}$$

with the fixed probabilities

$$\Pr(f(t) = f_j) = P_j > 0, \quad j = 1, 2, \ldots, M,$$

where $\sum_{j=1}^{M} P_j = 1$.

Identifying $1 \sim \delta_2^1$, $0 \sim \delta_2^2$, using the STP of matrices, (6) can be represented equivalently as

$$x(t+1) = L(t)u(t)x(t-\tau(t)), \tag{7}$$

where $x \in \Delta_{2^n}$; $u \in \Delta_{2^m}$; t_0, t and $\tau(t)$ are the same as those in system (6); for each $t \geqslant t_0$,

$$\Pr(L(t) = L_\lambda) = P_\lambda.$$

Note that in (7), $x(t)$ is a 2^n-valued logical variable, and $L(1), L(2), \ldots$ form a sequence of independent identically distributed random logical matrices taking value L_λ with probability $P_\lambda, \lambda = 1, 2, \ldots, M$.

The details on the properties of STP, and how to transform a BCN into its equivalent algebraic form are referred to [12]. The transformation of (6) to (7) is similar to that of a BCN to its equivalent algebraic form. We omit the procedure.

3 Main results

In this section, we investigate the controllability of PBCNTD (7). Next the definition of controllability is given, which is a natural generalization of [2, Definition 1], [3, Definition 17], [4, Definition 3.1], [6, Definition 3.1], [7, Definition 3.1] and [8, Definition 6].

Definition 2. Consider PBCNTD (7). For any given initial time t_0, any given time delay function, any given initial state sequence $X_0 = (x(t_0 - \tau(t_0)), x(t_0 - \tau(t_0) + 1), \ldots, x(t_0)) \in (\Delta_{2^n})^{(\tau(t_0)+1)}$ and any given destination state $x_d \in \Delta_{2^n}$,

- The PBCNTD is said to be controllable from X_0 to x_d (with probability 1), if there exists a control sequence such that

$$\Pr(x(t) = x_d \text{ for some } t \geqslant t_0 + 1 | X(t_0) = X_0) = 1. \tag{8}$$

- The PBCNTD is said to be controllable from X_0, if it is controllable from X_0 to any $x_d \in \Delta_{2^n}$.
- The PBCNTD is said to be controllable, if it is controllable from any $X_0 \in (\Delta_{2^n})^{(\tau(t_0)+1)}$.

3.1 Controllability constructed path

In this section, we introduce the key tool to solve the controllability of PBCNTD (7), whose deterministic version was first proposed in [6] for system (7) with M equal to 1. We will prove that the tool proposed here has similar properties as its deterministic version.

Next we introduce some necessary basic concepts in graph theory. A directed graph is a 2-tuple (V, E), where V is the set of vertices, and $E \subset V \times V$ is the set of edges. In this paper, V is infinite. An edge from v_1 to v_2 is denoted by (v_1, v_2) or $v_1 \to v_2$, where $v_1, v_2 \in V$, $(v_1, v_2) \in E$. Then v_1 is called a parent of v_2, and similarly v_2 is called a child of v_1. A path is a sequence of vertices $v_1, \ldots, v_n$ such that for all $i = 1, \ldots, n-1$, $(v_i, v_{i+1}) \in E$, or a sequence of infinite vertices $v_1, v_2, \ldots$ such that for all $i = 1, 2, \ldots$, $(v_i, v_{i+1}) \in E$. A path $v_1, \ldots, v_n$ is called a cycle if $v_1 = v_n$. A forest is a directed graph that has no cycle when the direction of edges is omitted. A tree is a forest, in which there is a path between any two distinct vertices when the direction of edges is omitted. The vertex of a tree to which there is no edge is called the root of the tree, if it exists. For example, the tree $(\mathcal{N}, \{(i+1, i) : i \in \mathcal{N}\})$ has no roots.

Definition 3. A directed graph $G(V, E)$ is said to be the constructed forest of PBCNTD (7), if the vertex set $V = \{t \in \mathcal{Z} : t \geqslant t_0 - \tau(t_0)\}$, i.e., the time sequence of PBCNTD (7), and the edge set $E = \{(t', t'') : t' = t'' - 1 - \tau(t'' - 1)\} \subset V \times V$, where t_0 is shown in (7).

The constructed forest intuitively describes the time transition of PBCNTD (7). That is, $(t', t'') \in E$ iff the state at time step t'' is determined by the state at time step t', i.e., $x(t'') = L(t''-1)u(t''-1)x(t')$. If $\tau(t) \equiv 0$, then $E = \{(t, t+1) : t = t_0, t_0 + 1, \ldots\}$, a path.

We identify each vertex t with the state $x(t)$ of PBCNTD (7) hereinafter. The following lemma holds.

Lemma 1. The constructed forest G consists of $\tau(t_0) + 1$ trees, where the roots are $t_0 - \tau(t_0), t_0 - \tau(t_0) + 1, \ldots, t_0$.

The constructed forest can be obtained easily according to the following procedure: first, draw the roots $t_0 - \tau(t_0), t_0 - \tau(t_0) + 1, \ldots, t_0$; second, draw the vertex $t_0 + 1$ and the edge pointing to it; third, draw the vertex $t_0 + 2$ and the edge pointing to it; and so on ... (see Figure 1).

For the sake of discussion, some notations are presented.

- Let $\{T_{t_0-\tau(t_0)}, T_{t_0-\tau(t_0)+1}, \ldots, T_{t_0}\}$ be the constructed forest $G(V, E)$, where $T_{t_0-\tau(t_0)+i}$ denotes a tree, $i = 0, 1, \ldots, \tau(t_0)$.
- Let $P_{t_0-\tau(t_0)+i}$ and N_i be any one given longest path of the tree $T_{t_0-\tau(t_0)+i}$ and the length of $P_{t_0-\tau(t_0)+i}$, respectively, $i = 0, 1, \ldots, \tau(t_0)$ (If $P_{t_0-\tau(t_0)+j}$ has an infinite number of vertices, set $N_j = +\infty$, $j = 0, 1, \ldots, \tau(t_0)$.).
- Let P_c and N_c be any one given longest path in

$$\{P_{t_0-\tau(t_0)}, P_{t_0-\tau(t_0)+1}, \ldots, P_{t_0}\} \tag{9}$$

and the length of P_c, respectively.

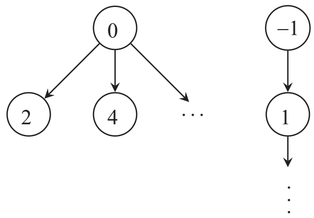

Figure 1 The constructed forest generated by the time delay function (10), where the number in each circle denotes the time step.

• Let P_o and N_o be any one given shortest path in $\{P_{t_0-\tau(t_0)}, P_{t_0-\tau(t_0)+1}, \dots, P_{t_0}\}$ and the length of P_o, respectively.

We call P_c a CCP, since it can be used to characterize the controllability of PBCNTD (7), as will be shown later.

Remark 1. A CCP has either a finite number of vertices or a countably infinite number of vertices. In particular, if $\tau(t)$ is bounded, then $N_c = +\infty$. In fact, if it is not true, there must exist one vertex that has a countably infinite number of children, which infers that $\tau(t)$ is unbounded. However, if $\tau(t)$ is unbounded, N_c still can be $+\infty$. Such an example is the following time delay function (10):

$$\tau^*(t) = \begin{cases} 1, & \text{if } t \geqslant t_0 = 0 \text{ is even}, \\ t, & \text{if } t \geqslant t_0 = 0 \text{ is odd}, \end{cases} \tag{10}$$

where $\tau^*(t)$ is obviously unbounded, but the length of the unique CCP $\{-1, 1, 3, 5, \dots\}$ equals $+\infty$ (Figure 1).

According to the constructed forest of PBCNTD (7), a path of the constructed forest is called a subsystem of PBCNTD (7), if its root is one of the roots of the forest, and it has either a vertex from which there is no edge in the forest or infinitely many vertices. According to this definition, every $P_{t_0-\tau(t_0)+i}$ $(0 \leqslant i \leqslant \tau(t_0))$ is a subsystem.

Denoted by

$$\{\boldsymbol{t}_0, t_1, \dots, t_N\} (\text{or } \{\boldsymbol{t}_0, t_1, \dots\}) \subset \mathcal{Z}, \tag{11}$$

the time sequence of a subsystem of PBCNTD (7), where $t_0 - \tau(t_0) \leqslant \boldsymbol{t}_0 \leqslant t_0$, $t_{i+1} > t_i$ for all $i \geqslant 0$. Then, each subsystem is in the form of (12):

$$x(\boldsymbol{t}_{k+1}) = L(\boldsymbol{t}_{k+1} - 1) u(\boldsymbol{t}_{k+1} - 1) x(\boldsymbol{t}_k), \tag{12}$$

where $k \in \{0, 1, \dots, N-1\}$ if (11) is of a finite length N, and $k \in \mathcal{N}$ if (11) is of length $+\infty$; $\boldsymbol{t}_k = t_k$ shown in (11), $k = 1, 2, \dots, N-1$ or $1, 2, \dots$; and $x \in \Delta_{2^n}$; $u \in \Delta_{2^m}$; $L(t)$ are the same as those in PBCNTD (7).

Note that system (12) is a system with no time delays if the subscript of $\boldsymbol{t}$ is regarded as its time step.

It was proved in [6] that PBCNTD (7) with M equal to 1 is controllable iff any subsystem P_c is controllable, and is observable iff any subsystem P_o is observable.

3.2 Controllability

In order to state our main results, we introduce some Boolean matrices which reflect the controllability of PBCNTD (7), which will be used repeatedly hereinafter.

For each submodel of PBCNTD (7), i.e. $x(t+1) = L_\lambda u(t) x(t - \tau(t))$, with the corresponding structure matrix $L_\lambda \in \mathcal{L}_{2^n \times 2^{n+m}}$, $\lambda = 1, 2, \dots, M$, denote

• $L_\lambda := [\mathrm{Blk}_1(L_\lambda), \mathrm{Blk}_2(L_\lambda), \dots, \mathrm{Blk}_{2^m}(L_\lambda)]$, where each $\mathrm{Blk}_i(L_\lambda) \in \mathcal{L}_{2^n \times 2^n}$ denotes the i-th square block of L_λ.

• $R_P := \oplus_{s=1}^{\min\{2^n, N_c\}} (\odot_{j=1}^{s} (\oplus_{\lambda=1}^{M} (\oplus_{i=1}^{2^m} \mathrm{Blk}_i(L_\lambda)))) \in \mathcal{B}_{2^n \times 2^n}$,

where N_c is the length of the CCP.

From [6, Theorem 5], the following theorem holds.

Theorem 1. Consider PBCNTD (7). Assume $M = 1$. Denote $L_1 := [\mathrm{Blk}_1(L_1), \mathrm{Blk}_2(L_1), \dots, \mathrm{Blk}_{2^m}(L_1)]$, where each $\mathrm{Blk}_i(L_1) \in \mathcal{L}_{2^n \times 2^n}$, and define Boolean matrices

$$C_s := \begin{cases} \oplus_{k=1}^{\min\{N_s, 2^n\}} (\odot_{j=1}^{k} (\oplus_{i=1}^{2^m} \mathrm{Blk}_i(L_1))), & \text{if } N_s > 0, \\ \mathbf{0}_{2^n \times 2^n}, & \text{if } N_s = 0, \end{cases}$$

where N_s is the length of subsystem P_s, $s = 0, 1, \dots, \tau(t_0)$.

(1) The system is controllable from $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \dots, \delta_{2^n}^{i_{\tau(t_0)}})$ to $\delta_{2^n}^{i}$ iff $\oplus_{l=0}^{\tau(t_0)} (C_l)_{i,i_l} > 0$.

(2) The system is controllable from $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \dots, \delta_{2^n}^{i_{\tau(t_0)}})$ iff $\oplus_{l=0}^{\tau(t_0)} (C_l)_{i,i_l} > 0$ for all $i = 1, 2, \dots, 2^n$.

(3) The system is controllable iff subsystem P_c is controllable.

(4) The system is controllable iff $\oplus_{k=1}^{\min\{N_c, 2^n\}} \odot_{j=1}^{k} (\oplus_{i=1}^{2^m} \mathrm{Blk}_i(L_1)) > 0$ 1).

From [8, Theorem 4], the following theorem holds.

Theorem 2. Assume that the length of system (12) is equal to $+\infty$. Then system (12) is controllable iff $R_P > 0$.

Next we give the first main result of this paper.

Theorem 3. Consider PBCNTD (7). Assume that $N_c = +\infty$. Then PBCNTD (7) is controllable, iff any system P_c (in the form of (12)) is controllable, iff $R_P > 0$.

Proof. The equivalence of any system P_c being controllable and $R_P > 0$ can be proved directly using Theorem 2. Next we prove the other part.

First we assume any system P_c in the form of (12) is controllable. Given any initial state sequence $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \dots, \delta_{2^n}^{i_{\tau(t_0)}}) \in (\Delta_{2^n})^{(\tau(t_0)+1)}$ and any destination state $\delta_{2^n}^{j} \in \Delta_{2^n}$. Without loss of generality, we assume that $\delta_{2^n}^{i_0}$ is the root of P_c. The time sequence of P_c is denoted by $t_0, t_1, \dots$. From the definition of controllability, there exists a control sequence $\{\bar{u}(t_1 - 1), \bar{u}(t_2 - 1), \dots\} := \mathcal{U}_{P_c} \subset \Delta_{2^m}$ such that for system P_c,

$$\Pr(x(t_k) = \delta_{2^n}^{j} \text{ for some } k > 0 | x(t_0) = \delta_{2^n}^{i_0}) = 1.$$

Hence for system P_c, under the control sequence $\mathcal{U}_{P_c}$,

$$\begin{aligned} &\Pr(x(t_k) \neq \delta_{2^n}^{j} \forall k > 0 | x(t_0) = \delta_{2^n}^{i_0}) \\ &\quad = 1 - \Pr(x(t_k) = \delta_{2^n}^{j} \text{ for some } k > 0 | x(t_0) = \delta_{2^n}^{i_0}) \\ &\quad = 0. \end{aligned} \tag{13}$$

Now design a control sequence $\{u(t)\}_{t=t_0, t_0+1, \dots} := \mathcal{U} \subset \Delta_{2^m}$ for PBCNTD (7) as follows:

$$u(t) := \begin{cases} \bar{u}(t_k - 1), & \text{if } t = t_k - 1 \text{ for some } k \geqslant 1, \\ \delta_{2^m}^{1}, & \text{otherwise.} \end{cases}$$

Note that PBCNTD (7) consists of a finite number of or a countably infinite number of subsystems, denoted by $\{S_\sigma : \sigma \in \Sigma\}$, in the form of (12) that are pairwise independent, where Σ is the index set of the subsystems, so Σ is a finite or a countably infinite set. For the σ-th subsystem with the initial state $\delta_{2^n}^{i_l}$, $l \in \{0, 1, \dots, \tau(t_0)\}$, under the control sequence $\mathcal{U}$, denote

$$\Pr(x(\boldsymbol{t}_k) \neq \delta_{2^n}^{j} \forall k > 0 | x(\boldsymbol{t}_0) = \delta_{2^n}^{i_l}) := P_{S_\sigma},$$

where $\boldsymbol{t}_0, \boldsymbol{t}_1, \dots$ denote the time sequence of the σ-th subsystem.

By the independence of the subsystems and (13), for PBCNTD (7), under the control sequence $\mathcal{U}$,

$$\Pr(x(t) \neq \delta_{2^n}^{j} \forall t \geqslant t_0 + 1 | X(t_0) = X_0) = \prod_{\sigma \in \Sigma} P_{S_\sigma} = 0.$$

1) In [6], it is not 2^n but $2^{n+m} - 1$. Since there are totally 2^n states and control is free, 2^n is essentially large enough.

Then for PBCNTD (7), under the control sequence $\mathcal{U}$,

$$\Pr(x(t) = \delta_{2^n}^j \text{ for some } t \geqslant t_0 + 1 | X(t_0) = X_0) = 1.$$

That is to say, PBCNTD (7) is controllable from X_0 to $x_{2^n}^j$. Since X_0 and $\delta_{2^n}^j$ are arbitrary, PBCNTD (7) is controllable.

Second, we assume one system P_c in the form of (12) is not controllable. Then from Theorem 2, there exist $1 \leqslant u, v \leqslant 2^n$ such that $(R_P)_{uv} = 0$. Further from Theorem 1, no matter what control sequence is chosen for system P_c, $\delta_{2^n}^u$ is not reachable from $\delta_{2^n}^v$. That is, system P_c is not controllable from $\delta_{2^n}^v$ to $\delta_{2^n}^u$.

Besides, every subsystem of PBCNTD (7) (in the form of (12)) has the same structure matrices $L(t)$ shown in (7) except that the length may be finite, and system P_c has the biggest length. Hence none of subsystems is controllable from $\delta_{2^n}^v$ to $\delta_{2^n}^u$. Then PBCNTD (7) is not controllable from $(\delta_{2^n}^v, \delta_{2^n}^v, \ldots, \delta_{2^n}^v) \in (\Delta_{2^n})^{(\tau(t_0)+1)}$to $\delta_{2^n}^u$. By the definition of controllability, PBCNTD (7) is not controllable.

Next, we discuss the controllability of PBCNTD (7) when $N_c < +\infty$. In this case, PBCNTD (7) consists of a countably infinite number of subsystems in the form of (12) that are of finite length and pairwise independent. Because none of the subsystems is of infinite length, we cannot use Definition 2 to define their controllability. As a result, we cannot use the CCP to investigate its controllability. However, in the deterministic version, we still can use Definition 2 to define their controllability, which implies Theorem 1 holds. On the other hand, no matter N_c equals $+\infty$ or not, we can use Definition 2 to define the controllability of PBCNTD (7).

In Theorem 3, because $N_c = +\infty$, we can use Theorem 2 to prove that the controllability of any system P_c is equivalent to that of PBCNTD (7). However, when $N_c < +\infty$, every subsystem is of finite length, we cannot use Theorem 2. Besides, we should testify the controllability of every subsystem to testify that of PBCNTD (7).

Theorem 1 shows that for the deterministic version of PBCNTD (7), no matter $N_c = +\infty$ or $N_c < +\infty$, the controllability of any system P_c is equivalent to that of PBCNTD (7). However, for PBCNTD (7), if $N_c < +\infty$, it is not true. Nevertheless, we still can use Theorem 1 to obtain some conditions for the controllability of PBCNTD (7).

Next, we give a necessary condition and a sufficient condition for the controllability of PBCNTD (7) when $N_c < +\infty$, respectively.

Theorem 4. Consider PBCNTD (7). Assume that $N_c < +\infty$. If the system is controllable, then $R_P > 0$.

Proof. Similar to Theorem 3, $R_P > 0$ can be proved using contradiction. Since the procedure is the same, we omit it. Hence no matter $N_c = +\infty$ or $N_c < +\infty$, we can use Theorem 1 to prove the necessary condition for controllability.

Theorem 5. Consider PBCNTD (7). Assume that $N_c < +\infty$. Denoted by S_μ, $\mu = 0, 1, \ldots, N_c$, the set of subsystems that are of length μ in the form of (12). If there exists $0 \leqslant \mu' \leqslant N_c$ such that $S_{\mu'}$ is an infinite set,

$$\oplus_{s=1}^{\min\{2^n,\mu'\}} (\odot_{j=1}^s (\oplus_{\lambda=1}^M (\oplus_{i=1}^{2^m} \mathrm{Blk}_i(L_\lambda)))) := R_{\mu'} \in \mathcal{B}_{2^n \times 2^n} > 0,$$

then PBCNTD (7) is controllable.

Proof. Note that the cardinality of each S_μ, $|S_\mu|$, satisfies $0 \leqslant |S_\mu| \leqslant +\infty$ and

$$\sum_{\mu=0}^{N_c} |S_\mu| = +\infty.$$

Hence there exists at least one $S_{\mu'}$ such that $|S_{\mu'}| = +\infty$. By Lemma 1, every μ' that satisfies $|S_{\mu'}| = +\infty$ is positive. Obviously $R_{\mu'} > 0$ implies $R_P > 0$. Hence the necessary condition shown in Theorem 4 holds.

Given an initial state sequence $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \ldots, \delta_{2^n}^{i_{\tau(t_0)}}) \in (\Delta_{2^n})^{(\tau(t_0)+1)}$ and a destination state $\delta_{2^n}^j \in \Delta_{2^n}$.

There exists an infinite subset $S'_{\mu'}$ of $S_{\mu'}$ such that the root of each subsystem of $S'_{\mu'}$ is the root of $T_{t_0-\tau(t_0)+k}$ for the same $0 \leqslant k \leqslant \tau(t_0)$. This is because there are totally a finite number of roots in $S_{\mu'}$ and $|S_{\mu'}| = +\infty$. Then, the initial state of each subsystem of $S'_{\mu'}$ is $\delta_{2^n}^{i_k}$.

Since $R_{\mu'} > 0$, for each subsystem of $S'_{\mu'}$, there exists a control sequence $\{u(t_0), u(t_1), \ldots, u(t_{\mu'})\} \subset \Delta_{2^m}$ such that

$$\Pr(x(t_l) = \delta_{2^n}^{j} \text{ for some } 0 < l \leqslant \mu' | x(t_0) = \delta_{2^n}^{i_k}) > 0,$$

then

$$\Pr(x(t_l) \neq \delta_{2^n}^{j} \forall 0 < l \leqslant \mu' | x(t_0) = \delta_{2^n}^{i_k}) \equiv p_{\mu'} < 1.$$

Since different subsystems have different time steps of control sequences, merging all above control sequences of subsystems of $S'_{\mu'}$ and adding arbitrary new control inputs at time steps that these control sequences do not cover, we obtain a new control sequence $\{u(t)\}_{t=t_0,t_0+1,\ldots} := \mathcal{U} \subset \Delta_{2^m}$. Under the new control sequence, for PBCNTD (7),

$$\Pr(x(t) \neq \delta_{2^n}^{j} \forall t > t_0 | X(t_0) = X_0) \leqslant p_{\mu'}^{+\infty} = 0,$$

since subsystems of PBCNTD (7) are pairwise independent. Then

$$\Pr(x(t) = \delta_{2^n}^{j} \text{ for some } t > t_0 | X(t_0) = X_0) = 1.$$

That is, PBCNTD (7) is controllable.

3.3 An illustrative example

Example 1. Consider the following PBCN with time-variant delays in states

$$x(t+1) = L(t)u(t)x(t-\tau(t)), \tag{14}$$

where $t = 0, 1, 2, \ldots$; $u \in \Delta$; $x \in \Delta_4$; for all $t \geqslant 0$, $L(t)$ is chosen randomly from the following 6 logical matrices:

$$\begin{aligned} &L_1 = \delta_4[2,2,4,2,3,3,3,3],\ L_2 = \delta_4[2,1,4,1,4,3,4,3],\\ &L_3 = \delta_4[1,1,4,2,4,4,3,3],\ L_4 = \delta_4[4,4,4,4,3,3,3,3],\\ &L_5 = \delta_4[4,3,4,3,4,3,4,3],\ L_6 = \delta_4[3,3,4,4,4,4,3,3], \end{aligned}$$

with fixed positive probabilities $P_1, \ldots, P_6$, where $\sum_{i=1}^{6} P_i = 1$.

$$\oplus_{\lambda=1}^{6} \oplus_{i=1}^{2} \mathrm{Blk}_i(L_\lambda) = \begin{bmatrix} 1&1&0&1\\ 1&1&0&1\\ 1&1&1&1\\ 1&1&1&1 \end{bmatrix} := S,$$

$$\oplus_{s=1}^{2} (\odot_{j=1}^{s} (\oplus_{\lambda=1}^{6} (\oplus_{i=1}^{2} \mathrm{Blk}_i(L_\lambda)))) = \begin{bmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{bmatrix} := R.$$

If we choose the time delay function (10), $N_c = +\infty$,

$$R_P = R = \begin{bmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{bmatrix} > 0.$$

From Theorem 3, PBCNTD (14) with the time delay function (10) is controllable.

If we choose the following time delay function:

$$\tau(t)=\begin{cases} t+1, & \text{if } t\geqslant 0 \text{ is even},\\ t, & \text{if } t\geqslant 0 \text{ is odd},\end{cases} \tag{15}$$

$N_c=1$,

$$R_P=S=\begin{bmatrix}1&1&0&1\\1&1&0&1\\1&1&1&1\\1&1&1&1\end{bmatrix}.$$

From Theorem 4, PBCNTD (14) with the time delay function (15) is not controllable.

If we choose the following time delay function:

$$\tau(t)=\begin{cases} 0, & \text{if } t=0,1,\\ t, & \text{if } t\geqslant 2,\end{cases} \tag{16}$$

we claim that PBCNTD (14) is not controllable either.

In this case, the unique CCP is $\{0,1,2\}$. Hence $N_c=2$. From the matrix R, one sees that this BCN satisfies the necessary condition in Theorem 4. But it does not satisfy the sufficient condition in Theorem 5 from the matrix S and the fact that all other subsystems have length 1.

Arbitrarily choose a control sequence $\{u(0),u(1),\dots\}\subset\Delta$. From the matrix S and the pairwise independence of subsystems, we have for all $t\geqslant 1$ and $t\neq 2$,

$$\Pr(x(t)\neq\delta_4^1|x(0)=\delta_4^3)=1.$$

Consider the CCP, if we choose submodels L_6,L_6 in order, then there is no path from δ_4^3 to δ_4^1. Hence we have

$$\Pr(x(2)\neq\delta_4^1|x(0)=\delta_4^3):=p_c>0.$$

Then we have

$$\Pr(x(t)\neq\delta_4^1\forall t\geqslant 1|x(0)=\delta_4^3)=p_c\cdot 1^{+\infty}=p_c>0.$$

That is, this BCN is not controllable.

Remark 2. From Example 1, one sees that the controllability of PBCNTD (7) is closely related to its time delay function. From Theorems 3, 4 and 5 one sees that the more submodels PBCNTD (7) has and the longer the CCPs of PBCNTD (7) are, the more likely PBCNTD (7) is to be controllable.

On the other hand, this example also shows an essential difference between PBCN (7) and its deterministic case. Take PBCN (14) with the time delay function (16) for example, if it would be a deterministic BCN, then $N_c=2$ and the matrix R would show that it is controllable from Theorem 1. However, it is not controllable.

4 Conclusion

This paper proposed new concepts of the constructed forest and controllability constructed path to deal with the controllability of PBCNs with time-variant delays in states, and revealed the essential difference of controllability between BCNs and PBCNs both with time-variant delays in states.

Acknowledgements This work was partly supported by National Natural Science Foundation of China (Grant No. 61573288) and Program for New Century Excellent Talents in University of Ministry of Education of China.

Conflict of interest The authors declare that they have no conflict of interest.

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