SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

September 2016, Vol. 59 092203:1–092203:24
doi: 10.1007/s11432-015-5488-2

Stochastic stability of cubature predictive filter

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Received August 31, 2015; accepted December 16, 2015; published online August 23, 2016

Abstract In this paper, the cubature predictive filter (CPF) is derived based on a third-degree spherical-radial cubature rule. It provides a set of cubature-points scaling linearly with the state-vector dimension, which makes it possible to numerically compute multivariate moment integrals encountered in the nonlinear predictive filter (PF). In order to facilitate the new method, the algorithm CPF is given firstly. Then, the theoretical analyses demonstrate that the estimated accuracy of the model error and system for the proposed CPF is higher than that of the traditional PF. Moreover, the authors analyze the stochastic boundedness and the error behavior of CPF for general nonlinear systems in a stochastic framework. In particular, the theoretical results present that the estimation error remains bounded and the covariance keeps stable if the system's initial estimation error, disturbing noise terms as well as the model error are small enough, which is the core part of the CPF theory. All of the results have been demonstrated by numerical simulations for a nonlinear example system.

Keywords predictive filter, cubature rule, nonlinear filter, stochastic system

Citation Cao L, Chen X Q. Stochastic stability of cubature predictive filter. Sci China Inf Sci, 2016, 59(9): 092203, doi: 10.1007/s11432-015-5488-2

1 Introduction

The real-time predictive filter is a widely used method in many application areas, such as spacecraft attitude determination [1–5], chaotic synchronization system [6], inertial alignment system, etc. It provides a way of determining optimal state estimates in the presence of significant error in the assumed (nominal) model, which is first introduced and developed by Lu [7,8], and Crassidis et al. [9,10]. Because the model errors are not restricted only to Gaussian noise, it is manifest that the PF is more general than other nonlinear estimated methods, such as the unscented Kalman filter [11]. In essential, the PF owns the advantages of both the Kalman filter (i.e., a real-time estimator) and the minimum model error (MME) estimator (i.e., determines actual model error trajectories), which includes: (1) the model error and process noise are assumed unknown and are estimated as part of the solution; (2) the model error may take any form (even nonlinear); (3) the algorithm can be implemented online to measure filter noisy as well as to estimate state trajectories; (4) the algorithm is robust in the presence of high noise measurement. Hence, it has attracted a lot of attentions from researchers.

However, the PF also has some serious disadvantages that remain to be improved. According to the theoretical analyses of the PF, the estimated model error is only accurately to the 1st order of the posterior

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mean, which will introduce serious errors into the state estimate. Therefore, it will inevitably bring some problems, such as the poor estimate precision, slow rate of convergence, etc. If the estimated model error deviates largely from its actual value, the error will be propagated to amplify the error effect, resulting in filter divergence. With a view to tackle these shortcomings, the CPF is derived through the use of the deterministically chosen cubature-points based on third-degree spherical-radial cubature rule [12,13]. The estimated model error of the CPF is demonstrated to capture the posterior mean accurately to the 3rd order for nonlinear Gaussian system, with errors only introduced in the 4th and high orders. At the same time, the estimated model error can capture the posterior mean accurately to the 2nd order for any nonlinearity. This derivativeless, based on predictive filter consistently outperform the PF not only in terms of estimated accuracy, but also in filter robustness and easy implementation. Without loss of generality, a study of a more general nonlinear case in a stochastic framework would also be of some interest, which is the core of the CPF theory.

In light of the above factors, we obtain the stability results for general nonlinear estimation problems by analyzing the error behavior of the CPF. Particularly, it is proved that the estimated error of CPF remains bounded and the covariance keeps stable if the system's initial estimation error, the disturbing noise terms as well as the model errors are small enough, which consist of the conditions about the effectiveness of the proposed CPF. It is also the main contributions. The paper is organized as follows. In Section 2, the classical PF and the general algorithm flow of the CPF are shown. Then, in Section 3, the error analyses of the model error and system state are given, which shows that the estimated accuracy of the model error and system state are both higher than that of PF. Moreover, we review some auxiliary results from stochastic stability theory and obtain the stochastic boundedness of the CPF in Section 4. In Section 5, the stochastic stability of CPF is analyzed. Then, the state estimation error and covariance will remain bounded if certain conditions are satisfied. Section 6 contains numerical simulation results for an example system to illustrate the theoretical results of Sections 3 and 5. In Section 7, the conclusion is drawn.

2 The flow of CPF and analysis

In the nonlinear predictive filter, the state are obtained by propagating an equation of the plant dynamics, which are assumed to be of the discrete-time form:

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k) + \boldsymbol{g'}(\boldsymbol{x}_k)\boldsymbol{d'}_k, \quad \boldsymbol{y}_k = \boldsymbol{h'}(\boldsymbol{x}_k) + \boldsymbol{\nu_k}, \tag{1}$$

where $k \in N_0$ is the discrete time, $\boldsymbol{f}(\cdot) \in \mathbb{R}^n$ and is sufficiently differentiable vector fields and $\mathbb{R}^n$ is the real n-dimensional vector space, $\boldsymbol{x}(t) \in \mathbb{R}^n$ denotes the system state vector, $\boldsymbol{d'}(t) \in \mathbb{R}^l$ denotes the model error vector, and $\boldsymbol{g'}(\cdot) \in \mathbb{R}^{n\times l}$ denotes the model error distribution matrix, which determines how the model error is introduced to the plant dynamics. $\boldsymbol{y_k} \in \mathbb{R}^n$ denotes the measurement, $\boldsymbol{h'}(\cdot) \in \mathbb{R}^m$ denotes sufficiently differentiable, $\boldsymbol{v}_k \in \mathbb{R}^m$ reprents the measurement noise vector, which is assumed to be a zero-mean, Gaussian white-noise distributed process with

$$\mathrm{E}(\boldsymbol{\nu}_k) = 0, \quad \mathrm{E}\left\{\boldsymbol{\nu}_k\boldsymbol{\nu}_l^{\mathrm{T}}\right\} = \boldsymbol{R}\delta_{kl}, \tag{2}$$

where $\boldsymbol{R} \in \mathbb{R}^{m\times m}$ is a positive-definite measurement covariance matrix. For ease of the following research, $\boldsymbol{g'}(\boldsymbol{x}_k)\boldsymbol{d'}_k = \boldsymbol{d}_k$ is defined as the total model error. Hence, Eq. (1) can be rewritten as

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k) + \boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{d}_k, \quad \boldsymbol{y}_k = \boldsymbol{h'}(\boldsymbol{x}_k) + \boldsymbol{\nu}_k. \tag{3}$$

For simplicity and clarity, we consider that $\boldsymbol{g}(\boldsymbol{x}_k) = \boldsymbol{I} \in \mathbb{R}^{n\times n}$ is well defined; therefore, Eqs. (3) and (1) are mathematical of consistency. It is also assumed that the state and output estimates are given by a preliminary model and a to-be-determined model error vector, given by

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{f}(\hat{\boldsymbol{x}}_k) + \boldsymbol{g}(\hat{\boldsymbol{x}}_k)\boldsymbol{d}_k, \quad \hat{\boldsymbol{y}}_k = \boldsymbol{h'}(\hat{\boldsymbol{x}}_k). \tag{4}$$

According to the PF theoretical results, based on the estimate principle of MME, a cost function consisting of the weighted sum square of the measurement-minus-estimate residuals plus the weighted sum square of the model correction term is minimized, such that

$$\boldsymbol{J}\left[\boldsymbol{d}_k\right]=\frac{1}{2}[\boldsymbol{y}_{k+1}-\hat{\boldsymbol{y}}_{k+1}]^{\mathrm{T}}\cdot\boldsymbol{R}^{-1}\cdot[\boldsymbol{y}_{k+1}-\hat{\boldsymbol{y}}_{k+1}]+\frac{1}{2}\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{W}\boldsymbol{d}_k, \tag{5}$$

where $\boldsymbol{W}\in\mathbb{R}^{l\times l}$ is a positive semidefinite weighting matrix.

To satisfy the covariance constraint condition of the PF, the estimated model error is given by

$$\begin{aligned}&\hat{\boldsymbol{d}}_k=-\left(\boldsymbol{Z}\left[\hat{\boldsymbol{x}}_k,\Delta t\right]+\boldsymbol{h}\left(\hat{\boldsymbol{x}}_k,t\right)\right),\\&\boldsymbol{M}\left(\hat{\boldsymbol{x}}_k\right)=\left\{\left[\boldsymbol{\Lambda}\left(\Delta t\right)\boldsymbol{U}\left(\hat{\boldsymbol{x}}_k\right)\right]^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{\Lambda}\left(\Delta t\right)\boldsymbol{U}\left(\hat{\boldsymbol{x}}_k\right)+\boldsymbol{W}\right\}^{-1}\cdot\left[\boldsymbol{\Lambda}\left(\Delta t\right)\boldsymbol{U}\left(\hat{\boldsymbol{x}}_k\right)\right]^{\mathrm{T}}\boldsymbol{R}^{-1},\\&\boldsymbol{Z}\left[\hat{\boldsymbol{x}}_k,\Delta t\right]=\boldsymbol{M}\left(\hat{\boldsymbol{x}}_k\right)\boldsymbol{Z'}\left[\hat{\boldsymbol{x}}_k,\Delta t\right],\quad\boldsymbol{h}\left(\hat{\boldsymbol{x}}_k\right)=\boldsymbol{M}\left(\hat{\boldsymbol{x}}_k\right)\left(\boldsymbol{h'}\left(\hat{\boldsymbol{x}}_k\right)-\boldsymbol{y}_{k+1}\right).\end{aligned} \tag{6}$$

Similarly, the real model error is given by

$$\begin{aligned}&\boldsymbol{d}_k=-\left(\boldsymbol{Z}\left[\boldsymbol{x}_k,\Delta t\right]+\boldsymbol{h}\left(\boldsymbol{x}_k\right)\right),\\&\boldsymbol{M}\left(\boldsymbol{x}_k\right)=\left\{\left[\boldsymbol{\Lambda}\left(\Delta t\right)\boldsymbol{U}\left(\boldsymbol{x}_k\right)\right]^{\mathrm{T}}\boldsymbol{R}^{-1}\boldsymbol{\Lambda}\left(\Delta t\right)\boldsymbol{U}\left(\boldsymbol{x}_k\right)+\boldsymbol{W}\right\}^{-1}\cdot\left[\boldsymbol{\Lambda}\left(\Delta t\right)\boldsymbol{U}\left(\boldsymbol{x}_k\right)\right]^{\mathrm{T}}\boldsymbol{R}^{-1},\\&\boldsymbol{Z}\left[\boldsymbol{x}_k,\Delta t\right]=\boldsymbol{M}\left(\boldsymbol{x}_k\right)\boldsymbol{Z'}\left[\boldsymbol{x}_k,\Delta t\right],\quad\boldsymbol{h}\left(\boldsymbol{x}_k\right)=\boldsymbol{M}\left(\boldsymbol{x}_k\right)\left(\boldsymbol{h'}\left(\boldsymbol{x}_k\right)+\boldsymbol{\nu}_k-\boldsymbol{y}_{k+1}\right).\end{aligned} \tag{7}$$

The ith row of $\boldsymbol{Z'}\left[\hat{\boldsymbol{x}}_k,\Delta t\right]\in\mathbb{R}^m$ is given by

$$Z_i'\left[\hat{\boldsymbol{x}}_k,\Delta t\right]=\sum_{k=1}^{p_i}\frac{\Delta t^k}{k!}L_f^k\left({h'}_i\right),\ i=1,2,\ldots,m, \tag{8}$$

where is the lowest order of the derivative of $h_i'\left(\boldsymbol{x}\left(t_k\right)\right)$ in which any component of the model error $\boldsymbol{d}_k$ first appears. $L_f^k\left({h'}_i\right)$ is a kth order Lie dervative.

$\boldsymbol{\Lambda}\left(\Delta t\right)\in\mathbb{R}^{m\times m}$ is a diagonal matrix with elements given by

$$\lambda_{ii}=\Delta t^{p_i}/p_i!,\ i=1,2,\ldots,m. \tag{9}$$

$\boldsymbol{U}\left[\hat{\boldsymbol{x}}_k\right]\in\mathbb{R}^{m\times l}$ is a matrix with each ith row given by

$$\boldsymbol{u}_i=\left\{L_{g_1}\left[L_f^{p_i-1}\left({h'}_i\right)\right],\ldots,L_{g_l}\left[L_f^{p_i-1}\left({h'}_i\right)\right]\right\},\ i=1,2,\ldots,m, \tag{10}$$

where the Lie derivative with respect to L_{g_l} in Eq. (10) is defined by

$$L_{g_j}\left[L_f^{p_i-1}\left({h'}_i\right)\right]=\frac{\partial L_f^{p_i-1}}{\partial\hat{\boldsymbol{x}}}g_j,\ j=1,2,\ldots,l. \tag{11}$$

Eq. (11) is in essence a generalized sensitivity matrix for nonlinear systems. Now, we expand Eq. (7) using a multi-dimensional Taylor series expansion around, and yields

$$\boldsymbol{d}_k=-\left(\boldsymbol{Z}\left[\hat{\boldsymbol{x}}_k,\Delta t\right]+\boldsymbol{D}_{\Delta x}\boldsymbol{Z}+\frac{\boldsymbol{D}_{\Delta x}^2\boldsymbol{Z}}{2}+\cdots+\boldsymbol{h}\left(\hat{\boldsymbol{x}}_k\right)+\boldsymbol{D}_{\Delta x}\boldsymbol{h}+\frac{\boldsymbol{D}_{\Delta x}^2\boldsymbol{h}}{2}+\cdots\right), \tag{12}$$

where $\Delta\boldsymbol{x}=\boldsymbol{x}_k-\hat{\boldsymbol{x}}_k$, and $\Delta\boldsymbol{x}$ is a zero-mean random variable with the same covariance $\boldsymbol{P}_k$ as $\boldsymbol{x}_k$ apparently. $\boldsymbol{D}_{\Delta x}$ is an vector operator that evaluates the total differential of $\boldsymbol{h}\left(\cdot\right)$ when perturbed around a nominal value $\hat{\boldsymbol{x}}_k$ by $\Delta\boldsymbol{x}$, i.e.,

$$\boldsymbol{D}_{\Delta x}\boldsymbol{h}=\boldsymbol{G}_h\Delta\boldsymbol{x}, \tag{13}$$

where $\boldsymbol{G}_k$ is the Jacobian matrix. In addition, the operator $\boldsymbol{D}_{\Delta x}$ can be rewritten and interpreted as the scalar operator

$$\boldsymbol{D}_{\Delta x}=\left[\left(\Delta\boldsymbol{x}^{\mathrm{T}}\nabla\right)\left(\boldsymbol{h}\left(\boldsymbol{x}\right)\right)^{\mathrm{T}}\right]^{\mathrm{T}}\Big|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}=\sum_{j=1}^{n}\Delta x_j\frac{\partial}{\partial x_j}, \tag{14}$$

which acts on $\boldsymbol{h}(\cdot)$ on a component-by-component basis. Using this definition, the ith term in the Taylor series for $\boldsymbol{h}(\boldsymbol{x}_k)$ is thus given by

$$\frac{\boldsymbol{D}_{\Delta x}^{i}\boldsymbol{h}}{i!}=\frac{\boldsymbol{D}_{\Delta x}\left(\boldsymbol{D}_{\Delta x}^{i-1}\boldsymbol{h}\right)}{i!}=\frac{\left((\Delta\boldsymbol{x})^{\mathrm{T}}\nabla\right)^{i}\boldsymbol{h}(\boldsymbol{x})\Big|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}}{i!}=\frac{1}{i!}\left(\sum_{j=1}^{n}\Delta x_j\frac{\partial}{\partial x_j}\right)^{i}\boldsymbol{h}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k},\tag{15}$$

where $\Delta\boldsymbol{x}_j$ is the jth component of $\Delta\boldsymbol{x}$, $\frac{\partial}{\partial x_j}$ is the normal partial derivative operator with respect to x_j (the jth component of $\boldsymbol{x}_k$), and n is the dimension of $\boldsymbol{x}_k$).

In the same way, $\frac{\boldsymbol{D}_{\Delta x}^{i}\boldsymbol{Z}}{i!}$ has the same expansion. Comparing Eq. (7) with Eq. (12), the estimated model error $\hat{\boldsymbol{d}}_k$ is only accurately to the first-order accuracy of the real model error $\boldsymbol{d}_k$. The first order Taylor approximation provides an insufficiently accurate representation in many cases, and significant bias, or even convergence problems, are commonly encountered due to the overly crude approximation. Hence, the estimated accuracy of the model error plays a significant role in enhancing the PF's performance.

On the basis of the above mentioned, the CPF is presented, which employs the third-degree spherical-radial cubature rule to outperform the PF not only in terms of estimated accuracy of model error, but also in the estimated accuracy of system state. This is completed by using a set of deterministically chosen cubature-points. These cubature-points completely have the same mean and covariance to the system state distribution. When these points are propagated through the nonlinear model error functions, the cubature-points can capture the higher estimated accuracy of the model error. Finally, the CPF becomes the recursive filter to enhance the estimated accuracy of the system state so that the general algorithm flow is proposed.

To calculate the statistics of $\boldsymbol{M}_k(\cdot)|_{\boldsymbol{x}_k=\hat{\boldsymbol{x}}_{k-1}}\boldsymbol{Z}_k(\cdot)|_{\boldsymbol{x}_k=\hat{\boldsymbol{x}}_{k-1}}$ and $\boldsymbol{h}_k(\cdot)|_{\boldsymbol{x}_k=\hat{\boldsymbol{x}}_{k-1}}\boldsymbol{P}_k$ using the cubature rule based on the nonlinear system of Eq. (3), the filtering recursion formulas of the CPF can be obtained.

(1) Denote the statistical property of the initial state:

$$\begin{cases}\hat{\boldsymbol{x}}_0=\mathrm{E}(\boldsymbol{x}_0),\\ P_0=\mathrm{Var}(\boldsymbol{x}_0)=\mathrm{E}\left[(\boldsymbol{x}_0-\hat{\boldsymbol{x}}_0)(\boldsymbol{x}_0-\hat{\boldsymbol{x}}_0)^{\mathrm{T}}\right].\end{cases}\tag{16}$$

(2) Make out the cubature-point by using the cubature rule.

(3) Update the model error. According to the cubature-point selection scheme in Step (2), the cubature-point $\xi_{i,k-1}$ $(i=1,\ldots,2n)$ can be calculated through the use of $\hat{\boldsymbol{x}}_{k-1}$ and $\boldsymbol{P}_{k-1}$. Each cubature-point is propagated through the nonlinear functions of $\boldsymbol{h}_{k-1}(\cdot)$, $\boldsymbol{Z}_{k-1}(\cdot)$, $\boldsymbol{M}_{k-1}(\cdot)$ to generate $\gamma_{i,k-1}$, $\alpha_{i,k-1}$, $\beta_{i,k-1}$, which yields the model error $\hat{\boldsymbol{d}}_{i,k-1}$ corresponding to each cubature point and the total model error $\hat{\boldsymbol{d}}_{k-1}$ in response to the CPF algorithm,

$$\begin{aligned}&\beta_{i,k-1}=\boldsymbol{M}_{k-1}(\xi_{i,k-1}),\quad(i=1,\ldots,2n),\\&\gamma_{i,k-1}=\beta_{i,k-1}(\boldsymbol{h'}_{k-1}(\xi_{i,k-1})-\boldsymbol{y}(t+\Delta t)),\quad(i=1,\ldots,2n),\\&\alpha_{i,k-1}=\beta_{i,k-1}\boldsymbol{Z'}_{k-1}(\xi_{i,k-1}),\quad(i=1,\ldots,2n),\end{aligned}\tag{17}$$

$$\bar{\boldsymbol{h}}_{k-1}=\frac{1}{2n}\sum_{i=1}^{2n}\gamma_{i,k-1},\quad\bar{\boldsymbol{Z}}_k=\frac{1}{2n}\sum_{i=1}^{2n}\alpha_{i,k},\tag{18}$$

$$\hat{\boldsymbol{d}}_{i,k-1}=-(\alpha_{i,k-1}+\gamma_{i,k-1}),\quad\hat{\boldsymbol{d}}_{k-1}=-\left(\bar{\boldsymbol{Z}}_{k-1}+\bar{\boldsymbol{h}}_{k-1}\right),\tag{19}$$

(4) Estimate the state and update the covariance. Given the set of cubature-points, each point is instantiated through the nonlinear function $\boldsymbol{f}_{k-1}(\cdot)$ to generate $\delta_{i,\,k|k-1}$. The update of state estimation $\hat{\boldsymbol{x}}_k$ and covariance $\boldsymbol{P}_k$ are calculated from the set $\{\delta_{i,\,k|k-1},\hat{\boldsymbol{d}}_{i,k-1}\}$.

To update the state estimation:

$$\delta_{i,\,k|k-1}=\boldsymbol{f}_{k-1}(\xi_{i,k-1}),\quad(i=1,\ldots,2n),$$

$$\hat{\boldsymbol{x}}_{k|k-1}=\frac{1}{2n}\sum_{i=1}^{2n}\delta_{i,\,k|k-1},\ \ \hat{\boldsymbol{x}}_k=\hat{\boldsymbol{x}}_{k|k-1}+\boldsymbol{g}_{k-1}\hat{\boldsymbol{d}}_{k-1},\ \ \hat{\boldsymbol{d}}'_{k-1}=\boldsymbol{g}'(\hat{\boldsymbol{x}}_k)^{-1}\hat{\boldsymbol{d}}_{k-1}. \tag{20}$$

To update the covariance:

$$\chi_{i,k}=\delta_{i,\,k|k-1}+\boldsymbol{g}_k\hat{\boldsymbol{d}}_{i,k-1},\ \ \boldsymbol{P}_k=\frac{1}{2n}\chi_{i,k}\chi_{i,k}^{\mathrm{T}}-\hat{\boldsymbol{x}}_k\hat{\boldsymbol{x}}_k^{\mathrm{T}}. \tag{21}$$

Note that the estimate model error is obtained in the Step (3) of the CPF algorithm; therefore, $\hat{\boldsymbol{d}}_{k-1}$ is the constant and there $\boldsymbol{g}_{k-1}=\boldsymbol{I}$ exists in Eq. (21).

3 The analysis of model error and state error for CPF

3.1 Error analysis of model error by cubature rule appoximation

In this subsection, the cubature rule is used to approximate the posterior distribution of the model error, which will yield higher estimated accuracy of the model error. We provide a detailed analysis about the posterior distribution of the model error. The numerically efficient Cholesky factorization method is typically used to calculate the matrix square root of $\boldsymbol{P}_k$, such that

$$\boldsymbol{P}_k=S_kS_k^{\mathrm{T}}=\sum_{i=1}^{n}\sigma_{k_i}\sigma_{k_i}^{\mathrm{T}}, \tag{22}$$

where, σ_{k_i} is the ith column of the matrix S_k.

First, the set of cubature-points are deterministically chosen so that they completely capture the true mean and covariance of the prior random variable $\boldsymbol{x}_k$. The set of $2n$ cubature-points are given by [12]

$$\begin{cases}\xi_i=\hat{\boldsymbol{x}}_k+\sqrt{n}\sigma_{k_i}=\hat{\boldsymbol{x}}+\hat{\sigma}_{k_i},\\ \xi_{i+n}=\hat{\boldsymbol{x}}_k-\sqrt{n}\sigma_{k_i}=\hat{\boldsymbol{x}}-\hat{\sigma}_{k_i},\end{cases}\quad i=1,2,\ldots,n. \tag{23}$$

The cubature rule approximates an n-dimensional Gaussian weighted integral as follows:

$$\int_{\mathbb{R}^{n_x}}f(x)\cdot(x;\hat{x}_k,\boldsymbol{P}_k)\,\mathrm{d}x\approx\frac{1}{2n}\sum_{i=1}^{2n}f(\xi_i). \tag{24}$$

Each cubature-point $\xi_i\,(i=1,\ldots,2n)$ is now propagated through the nonlinear function to generate γ_i,

$$\begin{cases}\gamma_i=\boldsymbol{h}(\xi_i)=\boldsymbol{h}(\hat{\boldsymbol{x}}_k)+\boldsymbol{D}_{\hat{\sigma}_{k_i}}\boldsymbol{h}+\dfrac{\boldsymbol{D}^2_{\hat{\sigma}_{k_i}}\boldsymbol{h}}{2!}+\dfrac{\boldsymbol{D}^3_{\hat{\sigma}_{k_i}}\boldsymbol{h}}{3!}+\cdots,\\ \gamma_{i+n}=\boldsymbol{h}(\xi_{i+n})=\boldsymbol{h}(\hat{\boldsymbol{x}}_k)+\boldsymbol{D}_{-\hat{\sigma}_{k_i}}\boldsymbol{h}+\dfrac{\boldsymbol{D}^2_{-\hat{\sigma}_{k_i}}\boldsymbol{h}}{2!}+\dfrac{\boldsymbol{D}^3_{-\hat{\sigma}_{k_i}}\boldsymbol{h}}{3!}+\cdots,\end{cases}\quad i=1,2,\ldots,n,$$
$$\begin{cases}\alpha_i=\boldsymbol{Z}(\xi_i)=\boldsymbol{Z}(\hat{\boldsymbol{x}}_k)+\boldsymbol{D}_{\hat{\sigma}_{k_i}}\boldsymbol{Z}+\dfrac{\boldsymbol{D}^2_{\hat{\sigma}_{k_i}}\boldsymbol{Z}}{2!}+\dfrac{\boldsymbol{D}^3_{\hat{\sigma}_{k_i}}\boldsymbol{Z}}{3!}+\cdots,\\ \alpha_{i+n}=\boldsymbol{Z}(\xi_{i+n})=\boldsymbol{Z}(\hat{\boldsymbol{x}}_k)+\boldsymbol{D}_{-\hat{\sigma}_{k_i}}\boldsymbol{Z}+\dfrac{\boldsymbol{D}^2_{-\hat{\sigma}_{k_i}}\boldsymbol{Z}}{2!}+\dfrac{\boldsymbol{D}^3_{-\hat{\sigma}_{k_i}}\boldsymbol{Z}}{3!}+\cdots,\end{cases}\quad i=1,2,\ldots,n. \tag{25}$$

Due to the vector differential operator Eq. (15), we get

$$\begin{cases}\boldsymbol{D}^m_{-\hat{\sigma}_{k_i}}\boldsymbol{h}=\left((-\hat{\sigma}_{k_i})^{\mathrm{T}}\nabla\right)^m\boldsymbol{h}(\boldsymbol{x})|_{x=\hat{x}_k}=-\left((\hat{\sigma}_{k_i})^{\mathrm{T}}\nabla\right)^m\boldsymbol{h}(\boldsymbol{x})|_{x=\hat{x}_k}=-\boldsymbol{D}^m_{\hat{\sigma}_{k_i}}\boldsymbol{h},\\ \boldsymbol{D}^p_{-\hat{\sigma}_{k_i}}\boldsymbol{h}=\left((-\hat{\sigma}_{k_i})^{\mathrm{T}}\nabla\right)^p\boldsymbol{h}(\boldsymbol{x})|_{x=\hat{x}_k}=\left((\hat{\sigma}_{k_i})^{\mathrm{T}}\nabla\right)^p\boldsymbol{h}(\boldsymbol{x})|_{x=\hat{x}_k}=\boldsymbol{D}^p_{\hat{\sigma}_{k_i}}\boldsymbol{h},\end{cases}$$
$$\begin{cases}\boldsymbol{D}^m_{-\hat{\sigma}_{k_i}}\boldsymbol{Z}=\left((-\hat{\sigma}_{k_i})^{\mathrm{T}}\nabla\right)^m\boldsymbol{Z}(\boldsymbol{x})|_{x=\hat{x}_k}=-\boldsymbol{D}^m_{\hat{\sigma}_{k_i}}\boldsymbol{Z},\\ \boldsymbol{D}^p_{-\hat{\sigma}_{x_i}}\boldsymbol{Z}=\left((-\hat{\sigma}_{x_i})^{\mathrm{T}}\nabla\right)^p\boldsymbol{Z}(\boldsymbol{x})|_{x=\hat{x}}=\boldsymbol{D}^p_{\hat{\sigma}_{x_i}}\boldsymbol{Z},\end{cases} \tag{26}$$

where m is odd number; p is even number.

Meanwhile, we can notice the following relationship exist:

$$\frac{1}{2n}\sum_{i=1}^{2n}\left[\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^2\boldsymbol{h}}{2!}\right]=\frac{1}{2n}\mathrm{E}\left[\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}\left(\boldsymbol{D}_{\hat{\sigma}_{k_i}}\boldsymbol{h}\right)}{2!}\right]$$
$$=\frac{1}{2n}\sum_{i=1}^{2n}\left[\frac{\left(\nabla^{\mathrm{T}}\hat{\sigma}_{k_i}(\hat{\sigma}_{k_i})^{\mathrm{T}}\nabla\right)\boldsymbol{h}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}}{2!}\right]=\frac{\left(\nabla^{\mathrm{T}}\boldsymbol{P}_k\nabla\right)\boldsymbol{h}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}}{2!}. \tag{27}$$

Using Eq. (15) we get

$$\frac{1}{2n}\sum_{i=1}^{2n}\left[\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^2\boldsymbol{Z}}{2!}\right]=\mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^2\boldsymbol{Z}}{2!}\right]. \tag{28}$$

After the apporximation, the posterior distributions of different nonlinear functions are represented by

$$\bar{\boldsymbol{h}}_{\mathrm{CPF}}=\frac{1}{2n}\sum_{i=1}^{n}(\gamma_i+\gamma_{i+n})=\boldsymbol{h}(\hat{\boldsymbol{x}}_k)+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^2\boldsymbol{h}}{2!}+\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^4\boldsymbol{h}}{4!}+\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^6\boldsymbol{h}}{6!}+\cdots\right]$$
$$=\boldsymbol{h}(\hat{\boldsymbol{x}}_k)+\mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^2\boldsymbol{h}}{2!}\right]+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^4\boldsymbol{h}}{4!}+\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^6\boldsymbol{h}}{6!}+\cdots\right],$$
$$\bar{\boldsymbol{Z}}_{\mathrm{CPF}}=\frac{1}{2n}\sum_{i=1}^{n}(\alpha_i+\alpha_{i+n})=\boldsymbol{Z}(\hat{\boldsymbol{x}}_k)+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^2\boldsymbol{Z}}{2!}+\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^4\boldsymbol{Z}}{4!}+\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^6\boldsymbol{Z}}{6!}+\cdots\right]$$
$$=\boldsymbol{Z}(\hat{\boldsymbol{x}}_k)+\mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^2\boldsymbol{Z}}{2!}\right]+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^4\boldsymbol{Z}}{4!}+\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^6\boldsymbol{Z}}{6!}+\cdots\right]. \tag{29}$$

With Eq. (7) we can obtain the minimal estimated model error after the cubature rule as follows:

$$\hat{\boldsymbol{d}}_{\mathrm{CPF}}=-\left(\boldsymbol{h}(\hat{\boldsymbol{x}}_k)+\mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^2\boldsymbol{h}}{2!}\right]+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^4\boldsymbol{h}}{4!}+\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^6\boldsymbol{h}}{6!}+\cdots\right]\right.$$
$$\left.+\boldsymbol{Z}(\hat{\boldsymbol{x}}_k)+\mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^2\boldsymbol{Z}}{2!}\right]+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^4\boldsymbol{Z}}{4!}+\frac{\boldsymbol{D}_{\hat{\sigma}_{k_i}}^6\boldsymbol{Z}}{6!}+\cdots\right]\right). \tag{30}$$

Comparing (30) with (12), the CPF can capture the posterior mean of the model error accurately to 3rd order for nonlinear Gaussian system through the use of the symmetrically cubature points, with errors only introduced in the 4th and higher order moments. Moreover, the CPF completely capture the posterior mean of the model error accurately to 2nd order for any nonlinear function. In comparison, the classical PF only calculates the posterior mean and covariance accurately to the 1st order with all higher order moments truncated. It is evident that the CPF will have higher estimated accuracy than classical PF in state estimation.

3.2 Error analysis of state and covariance for CPF

In this subsection, the research emphasis is the analysis of the state and covariance of CPF. From Eqs. (3), (4), (6) and (7), it is assumed that $-\boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{d}(\boldsymbol{x}_k)=\boldsymbol{\eta}(\boldsymbol{x}_k)-\boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{M}(\boldsymbol{x}_k)\boldsymbol{\nu}_k$ and $-\boldsymbol{g}(\hat{\boldsymbol{x}}_k)\boldsymbol{d}(\hat{\boldsymbol{x}}_k)=\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)$ exist and yield

$$\begin{aligned}\boldsymbol{x}_{k+1}&=\boldsymbol{x}_k+\Delta t\boldsymbol{f}(\boldsymbol{x}_k)+\Delta t\boldsymbol{\eta}(\boldsymbol{x}_k)-\Delta t\boldsymbol{g}(\boldsymbol{x}_k)\boldsymbol{M}(\boldsymbol{x}_k)\boldsymbol{\nu}_k+\mu(\boldsymbol{x}_k,\boldsymbol{d}_k),\\ \hat{\boldsymbol{x}}_{k+1}&=\hat{\boldsymbol{x}}_k+\Delta t\boldsymbol{f}(\hat{\boldsymbol{x}}_k)+\Delta t\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)+\mu(\hat{\boldsymbol{x}}_k,\hat{\boldsymbol{d}}_k),\end{aligned} \tag{31}$$

where $\mu(\boldsymbol{x}_k,\boldsymbol{d}_k)$ and $\mu(\hat{\boldsymbol{x}}_k,\hat{\boldsymbol{d}}_k)$ are the higher-order linearization and discretization errors. Hence, with Eq. (31) we can get the state estimated error $\tilde{\boldsymbol{x}}_{k+1}=\boldsymbol{x}_{k+1}-\hat{\boldsymbol{x}}_{k+1}$ is given by

$$\tilde{\boldsymbol{x}}_{k+1}=\tilde{\boldsymbol{x}}_k+\Delta t(\boldsymbol{f}(\boldsymbol{x}_k)-\boldsymbol{f}(\hat{\boldsymbol{x}}_k))+\Delta t(\boldsymbol{\eta}(\boldsymbol{x}_k)-\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k))-\Delta t\bar{\boldsymbol{B}}(\hat{\boldsymbol{x}}_k)\nu_k+\boldsymbol{\gamma}(\boldsymbol{x}_k,\boldsymbol{d}_k,\hat{\boldsymbol{x}}_k,\hat{\boldsymbol{d}}_k), \tag{32}$$

where $\bar{\boldsymbol{B}}(\hat{\boldsymbol{x}}_k) = \boldsymbol{g}(\hat{\boldsymbol{x}}_k)\boldsymbol{M}(\hat{\boldsymbol{x}}_k)$, $\varepsilon\left(\bar{\boldsymbol{B}}(\hat{\boldsymbol{x}}_k)\right)$ is the high-order discretization error, and $\boldsymbol{\gamma}(\boldsymbol{x}_k, \boldsymbol{d}_k, \hat{\boldsymbol{x}}_k, \hat{\boldsymbol{d}}_k) = (\mu(\boldsymbol{x}_k, \boldsymbol{d}_k) - \mu(\hat{\boldsymbol{x}}_k, \hat{\boldsymbol{d}}_k) - \varepsilon(\bar{\boldsymbol{B}}(\hat{\boldsymbol{x}}_k))\nu_k)$.

Substituting Eq. (32) into the covariance $\boldsymbol{P}_{k+1} = \mathrm{E}\left[\tilde{\boldsymbol{x}}_{k+1}\tilde{\boldsymbol{x}}_{k+1}^{\mathrm{T}}\right]$ yields

$$\begin{aligned}\boldsymbol{P}_{k+1} = \boldsymbol{P}_k &+ \Delta t\mathrm{E}\{\tilde{\boldsymbol{x}}_k[\boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)]^{\mathrm{T}} + \tilde{\boldsymbol{x}}_k[\boldsymbol{\eta}(\boldsymbol{x}_k) - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)]^{\mathrm{T}} + [\boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)]\tilde{\boldsymbol{x}}_k^{\mathrm{T}} \\ &+ [\boldsymbol{\eta}(\boldsymbol{x}_k) - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)]\tilde{\boldsymbol{x}}_k^{\mathrm{T}}\} + (\Delta t)^2\bar{\boldsymbol{B}}(\hat{\boldsymbol{x}}_k)R\bar{\boldsymbol{B}}(\hat{\boldsymbol{x}}_k)^{\mathrm{T}} + \boldsymbol{\psi}(\nu_k) \\ &+ (\Delta t)^2\mathrm{E}\{[\boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)][\boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)]^{\mathrm{T}} + [\boldsymbol{\eta}(\boldsymbol{x}_k) - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)][\boldsymbol{\eta}(\boldsymbol{x}_k) - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)]^{\mathrm{T}} \\ &+ [\boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)][\boldsymbol{\eta}(\boldsymbol{x}_k) - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)]^{\mathrm{T}} + [\boldsymbol{\eta}(\boldsymbol{x}_k) - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)][\boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)]^{\mathrm{T}}\} \\ &+ \boldsymbol{o}\left(\boldsymbol{x}_k, \boldsymbol{d}_k, \hat{\boldsymbol{x}}_k, \hat{\boldsymbol{d}}_k\right), \end{aligned} \tag{33}$$

where $\boldsymbol{o}(\boldsymbol{x}_k, \boldsymbol{d}_k, \hat{\boldsymbol{x}}_k, \hat{\boldsymbol{d}}_k)$ is the polynomial sum of $\boldsymbol{\gamma}(\boldsymbol{x}_k, \boldsymbol{d}_k, \hat{\boldsymbol{x}}_k, \hat{\boldsymbol{d}}_k)$ as well as $\boldsymbol{\psi}(\nu_k)$ is the polynomial sum of ν_k.

Rearranging the terms of Eq. (33) yields

$$\boldsymbol{P}_{k+1} = \boldsymbol{A}_k\boldsymbol{P}_k\boldsymbol{A}_k^{\mathrm{T}} + (\Delta t)^2\bar{\boldsymbol{B}}_k\boldsymbol{R}\bar{\boldsymbol{B}}_k^{\mathrm{T}} + \tilde{\boldsymbol{Q}}_k, \tag{34}$$

where

$$\begin{aligned} &\boldsymbol{A}_k = (\boldsymbol{I} + \Delta t\boldsymbol{G}_f + \Delta t\boldsymbol{G}_\eta), \ \tilde{\boldsymbol{Q}}_k = \Delta t\bar{\boldsymbol{\Sigma}}_1 + (\Delta t)^2\bar{\boldsymbol{\Sigma}}_2 + \boldsymbol{\psi}(\nu_k) + \boldsymbol{o}_k, \\ &\bar{\boldsymbol{\Sigma}}_1 = \sum_{i=1}^{\infty}\frac{1}{(2i+1)!}\mathrm{E}\left\{\tilde{\boldsymbol{x}}_k\left[\boldsymbol{D}_{\tilde{\boldsymbol{x}}_k}^{2i+1}(\boldsymbol{f}+\boldsymbol{\eta})\right]^{\mathrm{T}} + \left[\boldsymbol{D}_{\tilde{\boldsymbol{x}}_k}^{2i+1}(\boldsymbol{f}+\boldsymbol{\eta})\right]\tilde{\boldsymbol{x}}_k^{\mathrm{T}}\right\}, \\ &\bar{\boldsymbol{\Sigma}}_2 = \mathrm{E}\left\{\underbrace{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i!j!}\left(\left[\boldsymbol{D}_{\tilde{\boldsymbol{x}}_k}^{i}(\boldsymbol{f}+\boldsymbol{\eta})\right]\left[\boldsymbol{D}_{\tilde{\boldsymbol{x}}_k}^{j}(\boldsymbol{f}+\boldsymbol{\eta})\right]^{\mathrm{T}}\right)}_{\text{condition 1}}\right\}, \end{aligned} \tag{35}$$

which is the covariance of traditional PF and condition 1: $i \cdot j > 1, i + j$ is odd number.

In this subsection, we will provide the detailed analysis about the estimated state and covariance. From Eqs. (22) and (23), the covariance $\boldsymbol{P}_k$ of the state variable $(\boldsymbol{x}_k)_{\mathrm{CPF}}$ is adopted cubature-points to calculate for any time. Meanwhile, the set of cubature-points $\xi_{k,i}$ $(i = 1, \ldots, 2n)$ is represented by

$$\begin{cases} \xi_{k,i} = (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \sqrt{n}\sigma_{x_{ki}} = (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \hat{\sigma}_{x_{ki}}, \\ \xi_{k,i+n} = (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} - \sqrt{n}\sigma_{x_{ki}} = (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} - \hat{\sigma}_{x_{ki}}, \end{cases} \quad i = 1, 2, \ldots, n. \tag{36}$$

Substituting these points into Eq. (31) to propagate through the nonlinear transformation and yields

$$\begin{aligned} &\begin{cases} \delta_{k,i} = \boldsymbol{f}(\xi_i) = \boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}\boldsymbol{f} + \dfrac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^2\boldsymbol{f}}{2!} + \dfrac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^3\boldsymbol{f}}{3!} + \cdots, \\ \delta_{k,i+n} = \boldsymbol{f}(\xi_{i+n}) = \boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \boldsymbol{D}_{-\hat{\sigma}_{x_{ki}}}\boldsymbol{f} + \dfrac{\boldsymbol{D}_{-\hat{\sigma}_{x_{ki}}}^2\boldsymbol{f}}{2!} + \dfrac{\boldsymbol{D}_{-\hat{\sigma}_{x_{ki}}}^3\boldsymbol{f}}{3!} + \cdots, \end{cases} \quad i = 1, 2, \ldots, n, \\ &\begin{cases} \rho_{k,i} = \boldsymbol{\eta}(\xi_i) = \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}\boldsymbol{\eta} + \dfrac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^2\boldsymbol{\eta}}{2!} + \dfrac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^3\boldsymbol{\eta}}{3!} + \cdots, \\ \rho_{k,i+n} = \boldsymbol{\eta}(\xi_{i+n}) = \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \boldsymbol{D}_{-\hat{\sigma}_{x_{ki}}}\boldsymbol{\eta} + \dfrac{\boldsymbol{D}_{-\hat{\sigma}_{x_{ki}}}^2\boldsymbol{\eta}}{2!} + \dfrac{\boldsymbol{D}_{-\hat{\sigma}_{x_{ki}}}^3\boldsymbol{\eta}}{3!} + \cdots, \end{cases} \quad i = 1, 2, \ldots, n, \end{aligned} \tag{37}$$

inserting into Eq. (31) and rearranging the terms, the error analysis of cubature-points is given by

$$\begin{aligned} \xi_{k+1,i} &= \xi_{k,i} + \Delta t\delta_{k,i} + \Delta t\rho_{k,i} + \mu_{k,i}, \\ (\hat{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}} &= \frac{1}{2n}\sum_{i=1}^{2n}\xi_{k+1,i} = \frac{1}{2n}\sum_{i=1}^{2n}(\xi_{k,i} + \Delta t\delta_{k,i} + \Delta t\rho_{k,i} + \mu_{k,i}) \\ &= (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \Delta t\boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \Delta t\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \frac{1}{2n}\sum_{i=1}^{2n}\mu_{k,i} \end{aligned}$$

$$+\frac{\Delta t}{n}\left\{\sum_{i=1}^{n}\left[\frac{\boldsymbol{D}^2_{\hat{\sigma}_{x_{ki}}}\boldsymbol{f}}{2!}+\frac{\boldsymbol{D}^4_{\hat{\sigma}_{x_{ki}}}\boldsymbol{f}}{4!}+\cdots\right]+\sum_{i=1}^{n}\left[\frac{\boldsymbol{D}^2_{\hat{\sigma}_{x_{ki}}}\boldsymbol{\eta}}{2!}+\frac{\boldsymbol{D}^4_{\hat{\sigma}_{x_{ki}}}\boldsymbol{\eta}}{4!}+\cdots\right]\right\}. \tag{38}$$

Taking Eqs. (32) and (38) into account, the estimated error is defined by

$$\begin{aligned}(\tilde{\boldsymbol{x}}_{k+1})_{\rm CPF} &= \boldsymbol{x}_{k+1}-(\hat{\boldsymbol{x}}_{k+1})_{\rm CPF}\\ &=(\tilde{\boldsymbol{x}}_k)_{\rm CPF}+\Delta t\left(\boldsymbol{G}_f(\tilde{\boldsymbol{x}}_k)_{\rm CPF}+\boldsymbol{G}_\eta(\tilde{\boldsymbol{x}}_k)_{\rm CPF}\right)+\bar{\boldsymbol{\mu}}_k\left(\boldsymbol{x}_k,\boldsymbol{d}_k,(\hat{\boldsymbol{x}}_k)_{\rm CPF},\left(\hat{\boldsymbol{d}}_k\right)_{\rm CPF}\right)+\boldsymbol{s}_k\\ &=(\boldsymbol{I}+\Delta t\boldsymbol{G}_f+\Delta t\boldsymbol{G}_\eta)(\tilde{\boldsymbol{x}}_k)_{\rm CPF}+\bar{\boldsymbol{\mu}}_k\left(\boldsymbol{x}_k,\boldsymbol{d}_k,(\hat{\boldsymbol{x}}_k)_{\rm CPF},\left(\hat{\boldsymbol{d}}_k\right)_{\rm CPF}\right)+\boldsymbol{s}_k\\ &=\boldsymbol{A}_k(\tilde{\boldsymbol{x}}_k)_{\rm CPF}+\bar{\boldsymbol{\mu}}_k\left(\boldsymbol{x}_k,\boldsymbol{d}_k,(\hat{\boldsymbol{x}}_k)_{\rm CPF},\left(\hat{\boldsymbol{d}}_k\right)_{\rm CPF}\right)+\boldsymbol{s}_k,\end{aligned} \tag{39}$$

where

$$\begin{aligned}&\boldsymbol{A}_k=(\boldsymbol{I}+\Delta t\boldsymbol{G}_f+\Delta t\boldsymbol{G}_\eta),\ \ \boldsymbol{s}_k=-\Delta t\bar{\boldsymbol{B}}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\nu_k=-\boldsymbol{B}_k\nu_k,\\ &\bar{\boldsymbol{\mu}}_k\left(\boldsymbol{x}_k,\boldsymbol{d}_k,(\hat{\boldsymbol{x}}_k)_{\rm CPF},\left(\hat{\boldsymbol{d}}_k\right)_{\rm CPF}\right)=\left(\mu(\boldsymbol{x}_k,\boldsymbol{d}_k)\nu_k-\frac{1}{2n}\sum_{i=1}^{2n}\mu_{k,i}-\varepsilon\left(\bar{\boldsymbol{B}}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\nu_k\right.\\ &\qquad\left.+\Delta t\left(\Delta\boldsymbol{f}\left(\boldsymbol{x}_k,(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)+\Delta\boldsymbol{\eta}\left(\boldsymbol{x}_k,\boldsymbol{d}_k,(\hat{\boldsymbol{x}}_k)_{\rm CPF},\left(\hat{\boldsymbol{d}}_k\right)_{\rm CPF}\right)\right)\right),\\ &\Delta\boldsymbol{f}\left(\boldsymbol{x}_k,(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)=\sum_{i=2}^{\infty}\frac{1}{i!}\boldsymbol{D}^i_{\hat{\mathbf{x}}_k}\boldsymbol{f}-\frac{\Delta t}{n}\sum_{j=1}^{n}\left[\frac{\boldsymbol{D}^2_{\hat{\sigma}_{x_{kj}}}\boldsymbol{f}}{2!}+\frac{\boldsymbol{D}^4_{\hat{\sigma}_{x_{kj}}}\boldsymbol{f}}{4!}+\cdots\right],\\ &\Delta\boldsymbol{\eta}\left(\boldsymbol{x}_k,\boldsymbol{d}_k,(\hat{\boldsymbol{x}}_k)_{\rm CPF},\left(\hat{\boldsymbol{d}}_k\right)_{\rm CPF}\right)=\sum_{i=2}^{\infty}\frac{1}{i!}\boldsymbol{D}^i_{\hat{\mathbf{x}}_k}\boldsymbol{\eta}-\frac{\Delta t}{n}\sum_{j=1}^{n}\left[\frac{\boldsymbol{D}^2_{\hat{\sigma}_{x_{kj}}}\boldsymbol{\eta}}{2!}+\frac{\boldsymbol{D}^4_{\hat{\sigma}_{x_{kj}}}\boldsymbol{\eta}}{4!}+\cdots\right].\end{aligned} \tag{40}$$

The error transition Eq. (39) is now in a recursive form. According to the definition of the covariance, the covariance can be estimated as follows:

$$\begin{aligned}&(\boldsymbol{P}_{k+1})_{\rm CPF}=\frac{1}{2n}\sum_{i=1}^{2n}\left(\xi_{k+1,i}-(\hat{\boldsymbol{x}}_{k+1})_{\rm CPF}\right)\left(\xi_{k+1,i}-(\hat{\boldsymbol{x}}_{k+1})_{\rm CPF}\right)^{\rm T}=(\boldsymbol{P}_k)_{\rm CPF}\\ &+\Delta t\left\{\frac{1}{2n}\sum_{i=1}^{2n}\left(\xi_{k,i}-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\left(\delta_{k,i}-\boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)^{\rm T}+\frac{1}{2n}\sum_{i=1}^{2n}\left(\xi_{k,i}-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\left(\rho_{k,i}-\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)^{\rm T}\right.\\ &\left.+\frac{1}{2n}\sum_{i=1}^{2n}\left(\delta_{k,i}-\boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\left(\xi_{k,i}-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)^{\rm T}+\frac{1}{2n}\sum_{i=1}^{2n}\left(\rho_{k,i}-\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\left(\xi_{k,i}-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)^{\rm T}\right\}+\bar{\boldsymbol{\psi}}(\nu_k)\\ &+\Delta t^2\left\{\frac{1}{2n}\sum_{i=1}^{2n}\left(\delta_{k,i}-\boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\left(\delta_{k,i}-\boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)^{\rm T}+\frac{1}{2n}\sum_{i=1}^{2n}\left(\rho_{k,i}-\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\left(\rho_{k,i}-\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)^{\rm T}\right.\\ &+\frac{1}{2n}\sum_{i=1}^{2n}\left(\delta_{k,i}-\boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\left(\rho_{k,i}-\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)^{\rm T}\\ &\left.+\frac{1}{2n}\sum_{i=1}^{2n}\left(\rho_{k,i}-\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\left(\delta_{k,i}-\boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)^{\rm T}\right\}+\bar{\boldsymbol{o}}(\delta_{k,i},\mu_{k,i}),\end{aligned} \tag{41}$$

where $\bar{\boldsymbol{o}}(\delta_{k,i},\mu_{k,i})$ is the sum of the polynomial of $(\mu_{k,i}-\frac{1}{2n}\sum_{i=i}^{2n}\mu_{k,i})$ and other polynomials, and $\bar{\boldsymbol{\psi}}(\nu_k)$ is the sum of the polynomial of ν_k.

The set of cubature points are used to expand the terms of Eq. (41), yield

$$\begin{aligned}\frac{1}{2n}\sum_{i=1}^{2n}&\left(\xi_{k,i}-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\left(\delta_{k,i}-\boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)^{\rm T}=\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\xi_{k,i}-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\delta_i^{\rm T}+\left(\xi_{k,i+n}-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\right)\delta_{i+n}^{\rm T}\right]\\ &=\frac{1}{n}\sum_{i=1}^{n}\left\{\hat{\sigma}_{x_{ki}}\left[\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}\boldsymbol{f}+\frac{\boldsymbol{D}^3_{\hat{\sigma}_{x_{ki}}}\boldsymbol{f}}{3!}+\frac{\boldsymbol{D}^5_{\hat{\sigma}_{x_{ki}}}\boldsymbol{f}}{5!}+\cdots\right]^{\rm T}\right\}\end{aligned}$$

$$= \boldsymbol{P}_k \boldsymbol{G}_f^{\mathrm{T}} + \sum_{i=1}^{n} \sum_{j=1}^{\infty} \frac{1}{n\,(2j+1)!} \hat{\sigma}_{x_{ki}} \boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^{2j+1} \boldsymbol{f},$$

$$\frac{1}{2n} \sum_{i=1}^{2n} (\delta_{k,i} - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) \cdot (\delta_{k,i} - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}})^{\mathrm{T}} = \frac{1}{2n} \sum_{i=1}^{2n} \delta_{k,i} \delta_{k,i}^{\mathrm{T}} - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}} (\boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}})^{\mathrm{T}}$$
$$= \frac{1}{n} \sum_{i=1}^{n} \left\{ \left[\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}} \boldsymbol{f} + \frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^3 \boldsymbol{f}}{3!} + \cdots \right] \left[\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}} \boldsymbol{f} + \frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^3 \boldsymbol{f}}{3!} + \cdots \right]^{\mathrm{T}} \right.$$
$$\left. + \left[\frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^2 \boldsymbol{f}}{2!} + \frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^4 \boldsymbol{f}}{4!} + \cdots \right] \left[\frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^2 \boldsymbol{f}}{2!} + \frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^4 \boldsymbol{f}}{4!} + \cdots \right]^{\mathrm{T}} \right\}$$
$$- \frac{1}{n^2} \left\{ \sum_{i=1}^{n} \left[\frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^2 \boldsymbol{f}}{2!} + \frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^4 \boldsymbol{f}}{4!} + \cdots \right] \right\} \cdot \left\{ \sum_{i=1}^{n} \left[\frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^2 \boldsymbol{f}}{2!} + \frac{\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^4 \boldsymbol{f}}{4!} + \cdots \right] \right\}^{\mathrm{T}}$$
$$= \boldsymbol{G}_f \boldsymbol{P}_k \boldsymbol{G}_f^{\mathrm{T}} - \mathrm{E} \left[\frac{\boldsymbol{D}_{\Delta x}^2 \boldsymbol{f}}{2!} \right] E \left[\frac{\boldsymbol{D}_{\Delta x}^2 \boldsymbol{f}}{2!} \right]^{\mathrm{T}} + \frac{1}{n} \sum_{l=1}^{n} \left[\underbrace{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i!j!} \boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{i} \boldsymbol{f} \left(\boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{j} \boldsymbol{f} \right)^{\mathrm{T}}}_{\text{condition 1}} \right]$$
$$- \left[\underbrace{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(2i)!\,(2j)!n^2} \sum_{p=1}^{n} \sum_{m=1}^{n} \boldsymbol{D}_{\hat{\sigma}_{x_{kp}}}^{2i} \boldsymbol{f} \left(\boldsymbol{D}_{\hat{\sigma}_{x_{km}}}^{2j} \boldsymbol{f} \right)^{\mathrm{T}}}_{\text{condition 2}} \right], \tag{42}$$

where condition 1: $i \cdot j > 1, i + j$ is odd number; condition 2: $i \cdot j > 1$.

Similarly,

$$\frac{1}{2n} \sum_{i=1}^{2n} (\xi_{k,i} - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) \cdot (\rho_{k,i} - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}})^{\mathrm{T}} = \boldsymbol{P}_k \boldsymbol{G}_\eta^{\mathrm{T}} + \sum_{i=1}^{n} \sum_{j=1}^{\infty} \frac{1}{n\,(2j+1)!} \hat{\sigma}_{x_{ki}} \boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^{2j+1} \boldsymbol{\eta},$$
$$\frac{1}{2n} \sum_{i=1}^{2n} (\delta_{k,i} - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) \cdot (\xi_{k,i} - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}})^{\mathrm{T}} = \boldsymbol{G}_f \boldsymbol{P}_k + \sum_{i=1}^{n} \sum_{j=1}^{\infty} \frac{1}{n\,(2j+1)!} \boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^{2j+1} \boldsymbol{f} \hat{\sigma}_{x_{ki}},$$
$$\frac{1}{2n} \sum_{i=1}^{2n} (\rho_{k,i} - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) \cdot (\xi_{k,i} - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}})^{\mathrm{T}} = \boldsymbol{G}_\eta \boldsymbol{P}_k + \sum_{i=1}^{n} \sum_{j=1}^{\infty} \frac{1}{n\,(2j+1)!} \boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^{2j+1} \boldsymbol{\eta} \hat{\sigma}_{x_{ki}}, \tag{43}$$

$$\frac{1}{2n} \sum_{i=1}^{2n} (\rho_{k,i} - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) \cdot (\rho_{k,i} - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}})^{\mathrm{T}} = \boldsymbol{G}_\eta \boldsymbol{P}_k \boldsymbol{G}_\eta^{\mathrm{T}} + \frac{1}{n} \sum_{l=1}^{n} \left[\underbrace{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i!j!} \boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{i} \boldsymbol{\eta} \left(\boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{j} \boldsymbol{\eta} \right)^{\mathrm{T}}}_{\text{condition 1}} \right]$$
$$- \mathrm{E} \left[\frac{\boldsymbol{D}_{\Delta x}^2 \boldsymbol{\eta}}{2!} \right] E \left[\frac{\boldsymbol{D}_{\Delta x}^2 \boldsymbol{\eta}}{2!} \right]^{\mathrm{T}} - \left[\underbrace{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(2i)!\,(2j)!n^2} \sum_{p=1}^{n} \sum_{m=1}^{n} \boldsymbol{D}_{\hat{\sigma}_{x_{kp}}}^{2i} \boldsymbol{\eta} \left(\boldsymbol{D}_{\hat{\sigma}_{x_{km}}}^{2j} \boldsymbol{\eta} \right)^{\mathrm{T}}}_{\text{condition 2}} \right],$$

$$\frac{1}{2n} \sum_{i=1}^{2n} (\delta_{k,i} - \boldsymbol{f}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) \cdot (\rho_{k,i} - \boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}})^{\mathrm{T}} = \boldsymbol{G}_f \boldsymbol{P}_k \boldsymbol{G}_\eta^{\mathrm{T}} + \frac{1}{n} \sum_{l=1}^{n} \left[\underbrace{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i!j!} \boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{i} \boldsymbol{f} \left(\boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{j} \boldsymbol{\eta} \right)^{\mathrm{T}}}_{\text{condition 1}} \right]$$

$$- \mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^{2}\boldsymbol{f}}{2!}\right] E\left[\frac{\boldsymbol{D}_{\Delta x}^{2}\boldsymbol{\eta}}{2!}\right]^{\mathrm{T}} - \left[\underbrace{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(2i)!\,(2j)!n^{2}}\sum_{p=1}^{n}\sum_{m=1}^{n}\boldsymbol{D}_{\hat{\sigma}_{x_{kp}}}^{2i}\boldsymbol{f}\left(\boldsymbol{D}_{\hat{\sigma}_{x_{km}}}^{2j}\boldsymbol{\eta}\right)^{\mathrm{T}}}_{\text{condition 2}}\right],$$

$$\frac{1}{2n}\sum_{i=1}^{2n}\left(\rho_{k,i}-\boldsymbol{\eta}(\hat{\boldsymbol{x}}_{k})_{\mathrm{CPF}}\right)\cdot\left(\delta_{k,i}-\boldsymbol{f}(\hat{\boldsymbol{x}}_{k})_{\mathrm{CPF}}\right)^{\mathrm{T}} = \boldsymbol{G}_{\eta}\boldsymbol{P}_{k}\boldsymbol{G}_{f}^{\mathrm{T}} + \frac{1}{n}\sum_{l=1}^{n}\left[\underbrace{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i!j!}\boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{i}\boldsymbol{\eta}\left(\boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{j}\boldsymbol{f}\right)^{\mathrm{T}}}_{\text{condition 1}}\right]$$

$$- \mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^{2}\boldsymbol{\eta}}{2!}\right] E\left[\frac{\boldsymbol{D}_{\Delta x}^{2}\boldsymbol{f}}{2!}\right]^{\mathrm{T}} - \left[\underbrace{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(2i)!\,(2j)!n^{2}}\sum_{p=1}^{n}\sum_{m=1}^{n}\boldsymbol{D}_{\hat{\sigma}_{x_{kp}}}^{2i}\boldsymbol{\eta}\left(\boldsymbol{D}_{\hat{\sigma}_{x_{km}}}^{2j}\boldsymbol{f}\right)^{\mathrm{T}}}_{\text{condition 2}}\right], \tag{44}$$

Substituting Eqs. (42)–(44) into Eq. (41) yields

$$\begin{aligned}
&(\boldsymbol{P}_{k+1})_{\mathrm{CPF}} = \boldsymbol{A}_{k}(\boldsymbol{P}_{k})_{\mathrm{CPF}}\boldsymbol{A}_{k}^{\mathrm{T}} - (\Delta t)^{2}\left\{\mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^{2}\boldsymbol{f}}{2!}\right]+\mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^{2}\boldsymbol{\eta}}{2!}\right]\right\}^{2} + \tilde{\boldsymbol{\Sigma}}_{1}+\tilde{\boldsymbol{\Sigma}}_{2}+\bar{\boldsymbol{o}}_{k}+\bar{\boldsymbol{\psi}}_{k},\\
&\tilde{\boldsymbol{\Sigma}}_{1} = \left\{\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{\Delta t}{n\,(2j+1)!}\left(\hat{\sigma}_{x_{ki}}\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^{2j+1}(\boldsymbol{f}+\boldsymbol{\eta})+\boldsymbol{D}_{\hat{\sigma}_{x_{ki}}}^{2j+1}(\boldsymbol{f}+\boldsymbol{\eta})\,\hat{\sigma}_{x_{ki}}\right)\right\},\\
&\tilde{\boldsymbol{\Sigma}}_{2} = \frac{(\Delta t)^{2}}{n}\sum_{l=1}^{n}\left(\underbrace{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i!j!}\left[\boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{i}(\boldsymbol{f}+\boldsymbol{\eta})\right]\left[\boldsymbol{D}_{\hat{\sigma}_{x_{kl}}}^{j}(\boldsymbol{f}+\boldsymbol{\eta})\right]^{\mathrm{T}}}_{\text{condition 1}}\right)\\
&\qquad \underbrace{-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(\Delta t)^{2}}{(2i)!\,(2j)!n^{2}}\sum_{p=1}^{n}\sum_{m=1}^{n}\left[\boldsymbol{D}_{\hat{\sigma}_{x_{kp}}}^{2i}(\boldsymbol{f}+\boldsymbol{\eta})\right]\left[\boldsymbol{D}_{\hat{\sigma}_{x_{km}}}^{2j}(\boldsymbol{f}+\boldsymbol{\eta})\right]^{\mathrm{T}}}_{\text{condition 2}}.
\end{aligned}\tag{45}$$

Rearranging the terms of Eq. (45) yields

$$\begin{aligned}
&(\boldsymbol{P}_{k+1})_{\mathrm{CPF}} = \boldsymbol{A}_{k}(\boldsymbol{P}_{k})_{\mathrm{CPF}}\boldsymbol{A}_{k}^{\mathrm{T}} - \boldsymbol{K}_{k}\boldsymbol{K}_{k}^{\mathrm{T}} + \boldsymbol{Q}_{k},\\
&\boldsymbol{A}_{k} = (\boldsymbol{I}+\Delta t\boldsymbol{G}_{f}+\Delta t\boldsymbol{G}_{\eta}), \qquad \boldsymbol{Q}_{k} = \tilde{\boldsymbol{\Sigma}}_{1}+\tilde{\boldsymbol{\Sigma}}_{2}+\bar{\boldsymbol{o}}_{k}+\bar{\boldsymbol{\psi}}_{k},\\
&\boldsymbol{K}_{k} = (\Delta t)\left\{\left[\frac{\left(\nabla^{\mathrm{T}}\boldsymbol{P}_{k}\nabla\right)\boldsymbol{f}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_{k}}}{2!}\right]+\left[\frac{\left(\nabla^{\mathrm{T}}\boldsymbol{P}_{k}\nabla\right)\eta(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_{k}}}{2!}\right]\right\}\\
&\quad\;\; = (\Delta t)\left\{\mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^{2}\boldsymbol{f}}{2!}\right]+\mathrm{E}\left[\frac{\boldsymbol{D}_{\Delta x}^{2}\boldsymbol{\eta}}{2!}\right]\right\}.
\end{aligned}\tag{46}$$

As discussed above, $\boldsymbol{Q}_k$ is a matrix specially introduced in this paper. If the value of $\boldsymbol{Q}_k$ approximates the sum of the latter four terms of Eq. (45) and the initial value satisfies $(\boldsymbol{P}_0)_{\mathrm{CPF}} = \mathrm{E}\left(\tilde{\boldsymbol{x}}_0\tilde{\boldsymbol{x}}_0^{\mathrm{T}}\right)$, then $(\boldsymbol{P}_k)_{\mathrm{CPF}}$ calculated from Eq. (45) will approximate the actual value, that is, $(\boldsymbol{P}_k)_{\mathrm{CPF}} \approx \mathrm{E}\left(\tilde{\boldsymbol{x}}_k\tilde{\boldsymbol{x}}_k^{\mathrm{T}}\right)$. Therefore, Eq. (46) can be used to estimate the covariance of the state estimation error. In comparison of Eqs. (34) and (46), if the initial value satisfies $(\boldsymbol{P}_0)_{\mathrm{CPF}} = \boldsymbol{P}_0$, and $\boldsymbol{Q}_k$, $\tilde{\boldsymbol{Q}}_k$ are both the high-order small quantities. The inequality $(\boldsymbol{P}_k)_{\mathrm{CPF}} \leqslant \boldsymbol{P}_k$ exists for every $k \geqslant 0$. Hence, the covariance of the CPF has faster convergent speed than the classical PF, which will bring some better filtering performances.

4 State estimation and stochastic boundedness analysis of covariance

For the analyses of the error boundedness and stability, we make use of the following two concepts for the boundedness of stochastic processes [14].

Definition 1. The stochastic process ζ_k is said to be exponentially bounded in mean square, if there are real numbers $\eta, \upsilon > 0$ and $0 < \vartheta < 1$ such that

$$\mathrm{E}\left\{\|\zeta_k\|^2\right\} \leqslant \eta\|\zeta_0\|^2\vartheta^k + \upsilon \tag{47}$$

holds for every $k \geqslant 0$

Definition 2. The stochastic process is said to be bounded with probability one, if

$$\sup_{k\geqslant 0}\|\zeta_k\| < \infty \tag{48}$$

holds with probability one.

For later use we recall some standard results about the boundedness of stochastic processes [15].

Lemma 1 Assume there is a stochastic process $\underline{\upsilon}$, $\overline{\upsilon}$, $V_k(\zeta_k)$ as well as real numbers $\mu > 0$ and $0 < \alpha < 1$ such that

$$\underline{\upsilon}\|\zeta_k\|^2 \leqslant V_k(\zeta_k) \leqslant \bar{\upsilon}\|\zeta_k\|^2 \tag{49}$$

and

$$\mathrm{E}\left\{V_{k+1}(\zeta_{k+1})|\,\zeta_k\right\} - V_k(\zeta_k) \leqslant \mu - \alpha V_k(\zeta_k) \tag{50}$$

are fulfilled for every stochastic process. Then the stochastic process is exponentially bounded in mean square, i.e., we have

$$\mathrm{E}\left\{\|\zeta_k\|^2\right\} \leqslant \frac{\bar{\upsilon}}{\underline{\upsilon}}\mathrm{E}\left\{\|\zeta_0\|^2\right\}(1-\alpha)^k + \frac{\mu}{\underline{\upsilon}}\sum_{i=1}^{k-1}(1-\alpha)^i, \tag{51}$$

for every $k \geqslant 0$. Moreover, the stochastic process is bounded with probability one.

Proof. This lemma contains a combination of Refs. [14] and [16].

Remark 1. Using the following relation:

$$\sum_{i=1}^{k-1}(1-\alpha)^i \leqslant \sum_{i=1}^{\infty}(1-\alpha)^i < \frac{1}{\alpha}, \tag{52}$$

inequality (51) can be rewritten in the form [14]:

$$\mathrm{E}\left\{\|\zeta_k\|^2\right\} \leqslant \frac{\bar{\upsilon}}{\underline{\upsilon}}\mathrm{E}\left\{\|\zeta_0\|^2\right\}(1-\alpha)^k + \frac{\mu}{\underline{\upsilon}\alpha}. \tag{53}$$

In this subsection, the positive definiteness and boundedness of $(\boldsymbol{P}_k)_{\mathrm{CPF}}$ are established next.

Theorem 1. For the nonlinear stochastic system given by Eq.(3), if there exist positive real numbers $0 < z \leqslant 1$, $\bar{k} > 0$, $\bar{r} \geqslant \underline{r} \geqslant 0$ and $\bar{q} \geqslant \underline{q} > 0$, such that the following bounds are satisfied for every $k \geqslant 0$.

$$\begin{aligned}
&0 \leqslant \boldsymbol{A}_k\boldsymbol{A}_k^{\mathrm{T}} \leqslant (1-z)\,\boldsymbol{I}, \quad \underline{r}\boldsymbol{I} \leqslant \boldsymbol{R} \leqslant \bar{r}\boldsymbol{I}, \quad \underline{q}\boldsymbol{I} \leqslant \boldsymbol{Q}_k \leqslant \bar{q}\boldsymbol{I},\\
&0 \leqslant \frac{1}{4}(\Delta t)^2\left[\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{f}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k} + \left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{\eta}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}\right]\\
&\quad\cdot\left[\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{f}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k} + \left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{\eta}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}\right]^{\mathrm{T}} \leqslant \bar{k}\boldsymbol{I},
\end{aligned} \tag{54}$$

and the initial condition $(\boldsymbol{P}_0)_{\mathrm{CPF}}$ satisfies $\underline{p}\boldsymbol{I} \leqslant (\boldsymbol{P}_0)_{\mathrm{CPF}} \leqslant \bar{p}\boldsymbol{I}$ for real numbers $\bar{p} \geqslant \underline{p} > 0$, then the recursion (46) is bounded as

$$\underline{p}\boldsymbol{I} \leqslant (\boldsymbol{P}_k)_{\mathrm{CPF}} \leqslant \bar{p}\boldsymbol{I}, \tag{55}$$

for every $k \geqslant 0$.

Proof. Let

$$\underline{q} = \underline{p}, \ \ \bar{q} = \bar{p}z + \bar{p}^2\bar{k}. \tag{56}$$

Then the following two inequalities can be derived from conditions Eq. (54), as follows:

$$\begin{aligned}&\underline{p}\boldsymbol{A}_k\boldsymbol{A}_k^{\mathrm{T}}-\left\{\frac{1}{4}(\Delta t)^2\underline{p}^2\left[\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{f}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}+\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{\eta}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}\right]\right.\\&\quad\left.\cdot\left[\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{f}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}+\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{\eta}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}\right]^{\mathrm{T}}\right\}+\boldsymbol{Q}_k\geqslant\boldsymbol{I},\\&\bar{p}\boldsymbol{A}_k\boldsymbol{A}_k^{\mathrm{T}}-\left\{\frac{1}{4}(\Delta t)^2\bar{p}^2\left[\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{f}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}+\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{\eta}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}\right]\right.\\&\quad\left.\cdot\left[\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{f}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}+\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{\eta}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}\right]^{\mathrm{T}}\right\}+\boldsymbol{Q}_k\leqslant\bar{p}\boldsymbol{I}.\end{aligned}\tag{57}$$

If the initial condition satisfies $\underline{p}\boldsymbol{I}\leqslant(\boldsymbol{P}_0)_{\mathrm{CPF}}\leqslant\bar{p}\boldsymbol{I}$, it is readily obtained from the above inequalities that

$$\underline{p}\boldsymbol{I}\leqslant(\boldsymbol{P}_1)_{\mathrm{CPF}}=\boldsymbol{A}_0(\boldsymbol{P}_0)_{\mathrm{CPF}}\boldsymbol{A}_0^{\mathrm{T}}-\boldsymbol{K}_0\boldsymbol{K}_0^{\mathrm{T}}+\boldsymbol{Q}_0\leqslant\bar{p}\boldsymbol{I}\tag{58}$$

yielding $\underline{p}\boldsymbol{I}\leqslant(\boldsymbol{P}_1)_{\mathrm{CPF}}\leqslant\bar{p}\boldsymbol{I}$. Repeating the process, it can readily establish that $\underline{p}\boldsymbol{I}\leqslant(\boldsymbol{P}_k)_{\mathrm{CPF}}\leqslant\bar{p}\boldsymbol{I}$ for $k\geqslant 0$.

After the recursion Eq. (58) for the state estimation error and the corresponding covariance matrix are derived, the stochastic boundedness analysis of the CPF is discussed in the following section.

5 Stochastic stability analysis of CPF

With the discussions above, we are able to state the main results of the error boundedness and steady for the CPF in this paper.

Theorem 2. Consider a nonlinear stochastic system given by Eq. (3) and the CPF given in Section 2. Let the following assumptions hold:

(1) There are positive real numbers $\bar{a},\underline{a},\bar{p},\underline{p},\bar{q},\underline{q},\underline{r},\bar{r},\underline{\lambda},\bar{\lambda},\underline{s},\bar{s},\bar{k}>0$, such that the following bounds on various matrices are fulfilled for every $k\geqslant 0$:

$$\begin{aligned}&\underline{a}\leqslant\|\boldsymbol{A}_k\|\leqslant\bar{a},\quad\left\|\frac{1}{2}\left(\Delta t\right)\left[\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{f}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}+\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{\eta}\left(\boldsymbol{x}\right)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}\right]\right\|\leqslant\sqrt{\bar{k}},\\&\underline{q}\boldsymbol{I}\leqslant\boldsymbol{Q}_k\leqslant\bar{q}\boldsymbol{I},\quad\underline{r}\boldsymbol{I}\leqslant\boldsymbol{R}\leqslant\bar{r}\boldsymbol{I},\quad 0\leqslant\boldsymbol{W}\leqslant\bar{w}\boldsymbol{I},\\&\underline{\Lambda}\leqslant\|\boldsymbol{\Lambda}\left(\Delta t\right)\|\leqslant\bar{\lambda},\quad\underline{s}\leqslant\|(\boldsymbol{S}\left(\hat{\boldsymbol{x}}_k\right))_{\mathrm{CPF}}\|\leqslant\bar{s},\quad\underline{p}\boldsymbol{I}\leqslant(\boldsymbol{P}_k)_{\mathrm{CPF}}\leqslant\bar{p}\boldsymbol{I}.\end{aligned}\tag{59}$$

(2) $\boldsymbol{A}_k$ is invertible for every $k\geqslant 0$.

(3) There exist positive real numbers $\kappa_{\bar{\mu}},\varepsilon'>0$, such that the nonlinear function given by Eq. (39) is bounded,

$$\left\|\bar{\boldsymbol{\mu}}_k\left(\boldsymbol{x}_k,\boldsymbol{d}_k,(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}},\left(\hat{\boldsymbol{d}}_k\right)_{\mathrm{CPF}}\right)\right\|\leqslant\kappa_{\bar{\mu}}\|\boldsymbol{x}_k-(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^2,\quad\|\boldsymbol{x}_k-(\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|\leqslant\varepsilon'.\tag{60}$$

Then the state estimation error given by Eq. (39) is exponentially bounded in mean square and bounded with probability one, provided that the initial estimation error satisfies

$$\|(\tilde{\boldsymbol{x}}_0)_{\mathrm{CPF}}\|\leqslant\varepsilon,\tag{61}$$

for some $\varepsilon>0$, where $\|\cdot\|$ is the Euclidian norm of real vectors or the spectral norm of real matrices.

The proof of this theorem is divided into several lemmas.

Lemma 2. Under the conditions of Theorem 2, there is a real number $0<\alpha<1$ such that $\Pi_k=\left(\boldsymbol{P}_k^{-1}\right)_{\mathrm{CPF}}$ satisfies the inequality

$$\left(\boldsymbol{A}_k-\boldsymbol{K}_k\boldsymbol{C}_k\right)^{\mathrm{T}}\Pi_{k+1}\left(\boldsymbol{A}_k-\boldsymbol{K}_k\boldsymbol{C}_k\right)\leqslant(1-\alpha)\,\Pi_k,\tag{62}$$

where

$$\boldsymbol{C}_k = \left((\boldsymbol{P}_k^{-1})_{\rm CPF}\boldsymbol{A}_k^{-1}\boldsymbol{K}_k\right)^{\rm T}, \quad \alpha = 1 - 1\Bigg/\left(1 + \frac{q}{\bar{p}\left(\bar{a}+\bar{p}^2\bar{k}/\underline{ap}\right)^2}\right), \tag{63}$$

for every $k \geqslant 0$.

Proof. From Eqs. (46) and (63) we have

$$(\boldsymbol{P}_{k+1})_{\rm CPF} = \boldsymbol{A}_k(\boldsymbol{P}_k)_{\rm CPF}\boldsymbol{A}_k^{\rm T} + \boldsymbol{Q}_k - \boldsymbol{A}_k(\boldsymbol{P}_k)_{\rm CPF}\boldsymbol{C}_k^{\rm T}\boldsymbol{K}_k^{\rm T}, \tag{64}$$

and rearranging the terms yields

$$(\boldsymbol{P}_{k+1})_{\rm CPF} = (\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)(\boldsymbol{P}_k)_{\rm CPF}(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{\rm T} + \boldsymbol{Q}_k + \boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{P}_k)_{\rm CPF}(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{\rm T}. \tag{65}$$

The next step we consider the term $\boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{P}_k)_{\rm CPF}(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{\rm T}$ on the right-hand side of Eq. (65). With Eq. (63) it can be verified that

$$\boldsymbol{A}_k^{-1}(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)(\boldsymbol{P}_k)_{\rm CPF} = (\boldsymbol{P}_k)_{\rm CPF} - (\boldsymbol{P}_k)_{\rm CPF}\boldsymbol{C}_k^{\rm T}\boldsymbol{C}_k(\boldsymbol{P}_k)_{\rm CPF} \tag{66}$$

is a symmetric matrix. From Eqs. (39), (40) and (46), the linear term and the first order term of the Taylor series expansion of $(\boldsymbol{x}_{k+1})_{\rm CPF}$ are $\boldsymbol{A}_k \cdot (\tilde{\boldsymbol{x}}_k)_{\rm CPF}$, as well as the expectation of the 2nd order term is $\boldsymbol{K}_k$. It is obvious that the zeroth and first order terms of any nonlinear function account for the major than that of the residual terms. In other words, the following inequality exists:

$$|\boldsymbol{K}_k| < |\boldsymbol{A}_k \cdot (\tilde{\boldsymbol{x}}_k)_{\rm CPF}|. \tag{67}$$

So, Eq. (67) can be rewritten in the following form:

$$\frac{|\boldsymbol{K}_k|}{|\boldsymbol{A}_k \cdot (\tilde{\boldsymbol{x}}_k)_{\rm CPF}|}|(\tilde{\boldsymbol{x}}_k)_{\rm CPF}| < |(\tilde{\boldsymbol{x}}_k)_{\rm CPF}|. \tag{68}$$

Using Eq. (68), we establish that

$$\begin{aligned}&\left((\boldsymbol{A}_k^{-1})\boldsymbol{K}_k\right)\left((\boldsymbol{A}_k^{-1})\boldsymbol{K}_k\right)^{\rm T} < (\boldsymbol{P}_k)_{\rm CPF},\\ &(\boldsymbol{P}_k^{-1})_{\rm CPF}\left((\boldsymbol{A}_k^{-1})\boldsymbol{K}_k\right)\left((\boldsymbol{A}_k^{-1})\boldsymbol{K}_k\right)^{\rm T}(\boldsymbol{P}_k^{-1})_{\rm CPF} < (\boldsymbol{P}_k^{-1})_{\rm CPF},\end{aligned} \tag{69}$$

because equality $(\boldsymbol{P}_k^{-1})_{\rm CPF} = \left((\boldsymbol{P}_k^{-1})_{\rm CPF}\right)^{\rm T}$ exists, and inserting into Eq. (63) leads to

$$\boldsymbol{C}_k\boldsymbol{C}_k^{\rm T} < (\boldsymbol{P}_k^{-1})_{\rm CPF}. \tag{70}$$

Substituting Eq. (70) into Eq. (66), and we have form Eq. (66) using $(\boldsymbol{P}_k^{-1})_{\rm CPF} > 0$ and $(\boldsymbol{P}_k)_{\rm CPF} > 0$,

$$\boldsymbol{A}_k^{-1}(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)(\boldsymbol{P}_k)_{\rm CPF}{=}(\boldsymbol{P}_k)_{\rm CPF} - (\boldsymbol{P}_k)_{\rm CPF}\boldsymbol{C}_k^{\rm T}\boldsymbol{C}_k(\boldsymbol{P}_k)_{\rm CPF} > 0. \tag{71}$$

Moreover, we have the other form that

$$(\boldsymbol{A}_k^{-1})\boldsymbol{K}_k\boldsymbol{C}_k{=}(\boldsymbol{A}_k^{-1}\boldsymbol{K}_k)(\boldsymbol{A}_k^{-1}\boldsymbol{K}_k)^{\rm T}\left((\boldsymbol{P}_k^{-1})_{\rm CPF}\right)^{\rm T} \geqslant 0. \tag{72}$$

Combining Eqs. (71) and (72), we establish that

$$\boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{P}_k)_{\rm CPF}(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{\rm T} = \boldsymbol{A}_k\left[\boldsymbol{A}_k^{-1}\boldsymbol{K}_k\boldsymbol{C}_k\right]\left[\boldsymbol{A}_k^{-1}(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)(\boldsymbol{P}_k)_{\rm CPF}\right]^{\rm T}\boldsymbol{A}_k^{\rm T} \geqslant 0 \tag{73}$$

holds, and inserting into Eq. (65) leads to

$$(\boldsymbol{P}_{k+1})_{\rm CPF} \geqslant (\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)(\boldsymbol{P}_k)_{\rm CPF}(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{\rm T} + \boldsymbol{Q}_k. \tag{74}$$

Inequality Eq. (71) implies that $(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{-1}$ exists and therefore we may write

$$\begin{aligned}(\boldsymbol{P}_{k+1})_{\rm CPF} \geqslant (\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)\Big[&(\boldsymbol{P}_k)_{\rm CPF} + (\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{-1}\\ &\cdot\boldsymbol{Q}_k \cdot (\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{-\rm T}\Big](\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{\rm T}.\end{aligned} \tag{75}$$

From Eq. (59) we have

$$\|\boldsymbol{K}_k\| \leqslant \bar{p}\sqrt{\bar{k}}, \|\boldsymbol{C}_k\| \leqslant \bar{p}\sqrt{\bar{k}}\Big/\underline{ap}, \tag{76}$$

and Eq. (75) can be rewritten in the form:

$$(\boldsymbol{P}_{k+1})_{\mathrm{CPF}} \geqslant (\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)\left[(\boldsymbol{P}_k)_{\mathrm{CPF}} + \frac{q}{\left(\bar{a} + \bar{p}^2\bar{k}/\underline{ap}\right)^2}\right](\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{\mathrm{T}}. \tag{77}$$

Since $(\boldsymbol{P}_k)_{\mathrm{CPF}} > \underline{p}\boldsymbol{I}$ and $\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k$ are nonsingular. Applying the matrix inversion lemma [17] and taking the inverse of both sides, multiplying from left and right with $(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{\mathrm{T}}$ and $(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)$, and using Eq. (59) we obtain finally with $\Pi_k = \left(\boldsymbol{P}_k^{-1}\right)_{\mathrm{CPF}}$,

$$(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)^{\mathrm{T}}\Pi_{k+1}(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k) \leqslant \left[1 + \frac{q}{\bar{p}\left(\bar{a} + \bar{p}^2\bar{k}/\underline{ap}\right)^2}\right]^{-1}\Pi_k, \tag{78}$$

i.e., inequality Eq. (62) with

$$1 - \alpha = 1\bigg/\left(1 + \frac{q}{\bar{p}\left(\bar{a} + \bar{p}^2\bar{k}/\underline{ap}\right)^2}\right). \tag{79}$$

Lemma 3. Let the conditions of Theorem 2 be fulfilled, let $\Pi_k = \left(\boldsymbol{P}_k^{-1}\right)_{\mathrm{CPF}}$ and $\boldsymbol{K}_k, \bar{\boldsymbol{\mu}}_k$ be given by Eqs. (46) and (40). Then there are positive real numbers $\boldsymbol{\kappa}_1, \varepsilon' > 0$ such that

$$\begin{aligned}&(\boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}))^{\mathrm{T}}\Pi_k\left[2(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) + \boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) + \bar{\boldsymbol{\mu}}_k\right]\\ &\quad \leqslant \boldsymbol{\kappa}_1\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^2\end{aligned} \tag{80}$$

holds for $\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| \leqslant \varepsilon'$.

Proof. From Eq. (76) we have

$$\|\boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}})\| \leqslant \frac{\bar{p}^2\bar{k}}{\underline{ap}}\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| = k_1\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|, \tag{81}$$

where $k_1 = \bar{p}^2\bar{k}/\underline{ap}$.

From Eqs. (60) and (76), the conditions of $\Pi_k = \left(\boldsymbol{P}_k^{-1}\right)_{\mathrm{CPF}}$ and $\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| \leqslant \varepsilon'$ are satisfied, and we obtain

$$\begin{aligned}&(\boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}))^{\mathrm{T}}\Pi_k\left[2(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) + \boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) + \bar{\boldsymbol{\mu}}_k\right]\\ &\quad \leqslant k_1\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|\frac{1}{\underline{p}}\left((2\bar{a} + k_1)\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| + \kappa_{\bar{\mu}}\varepsilon'\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|\right)\\ &\quad \leqslant \left(\frac{k_1}{\underline{p}}\left((2\bar{a} + k_1) + \kappa_{\bar{\mu}}\varepsilon'\right)\right)\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^2,\end{aligned} \tag{82}$$

i.e., Eq. (80) with

$$\boldsymbol{\kappa}_1 = \left(\frac{k_1}{\underline{p}}\left((2\bar{a} + k_1) + \kappa_{\bar{\mu}}\varepsilon'\right)\right). \tag{83}$$

Lemma 4. Under the conditions of Theorem 2, let $\Pi_k = \left(\boldsymbol{P}_k^{-1}\right)_{\mathrm{CPF}}$ and $\boldsymbol{K}_k, \bar{\boldsymbol{\mu}}_k$ be given by Eqs. (46) and (40). Then there are positive real numbers $\boldsymbol{\kappa}_2, \varepsilon' > 0$ such that

$$\bar{\boldsymbol{\mu}}_k^{\mathrm{T}}\Pi_k\left[2(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) + \boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) + \bar{\boldsymbol{\mu}}_k\right] \leqslant \boldsymbol{\kappa}_2\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^3 \tag{84}$$

holds $\|\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| \leqslant \varepsilon'$.

Proof. From Eqs. (60) and (82) we have

$$\bar{\boldsymbol{\mu}}_k^{\mathrm{T}}\Pi_k\left[2(\boldsymbol{A}_k - \boldsymbol{K}_k\boldsymbol{C}_k)(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) + \boldsymbol{K}_k\boldsymbol{C}_k(\boldsymbol{x}_k - (\hat{\boldsymbol{x}}_k)_{\mathrm{CPF}}) + \bar{\boldsymbol{\mu}}_k\right]$$

$$\leqslant \kappa_{\bar{\mu}}\|\boldsymbol{x}_k-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\|^2\frac{1}{\underline{p}}\left((2\bar{a}+k_1)\|\boldsymbol{x}_k-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\|+\kappa_{\bar{\mu}}\varepsilon'\|\boldsymbol{x}_k-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\|\right)$$

$$\leqslant \left(\frac{\kappa_{\bar{\mu}}}{\underline{p}}\left((2\bar{a}+k_1)+\kappa_{\bar{\mu}}\varepsilon'\right)\right)\|\boldsymbol{x}_k-(\hat{\boldsymbol{x}}_k)_{\rm CPF}\|^3, \tag{85}$$

i.e., Eq. (84) with

$$\boldsymbol{\kappa}_2=\left(\frac{\kappa_{\bar{\mu}}}{\underline{p}}\left((2\bar{a}+k_1)+\kappa_{\bar{\mu}}\varepsilon'\right)\right). \tag{86}$$

Lemma 5. Let the conditions of Theorem 2 hold, let $\Pi_k=\left(\boldsymbol{P}_k^{-1}\right)_{\rm CPF}$ and $\boldsymbol{K}_k,\bar{\boldsymbol{\mu}}_k$ be given by Eqs. (46) and (40). Then there is a positive real number $\boldsymbol{\kappa}_3>0$ independent of δ, such that

$$\mathrm{E}\left\{\boldsymbol{s}_k^{\rm T}\Pi_k\boldsymbol{s}_k\right\}\leqslant\boldsymbol{\kappa}_3\delta \tag{87}$$

holds.

Proof. From Eq. (40) we focus on the following terms:

$$\boldsymbol{s}_k^{\rm T}\Pi_k\boldsymbol{s}_k=(\boldsymbol{B}_k\nu_k)^{\rm T}\Pi_k(\boldsymbol{B}_k\nu_k)\leqslant\frac{1}{\underline{p}}\nu_k^{\rm T}\boldsymbol{B}_k^{\rm T}\boldsymbol{B}_k\nu_k=\frac{1}{\underline{p}}\mathrm{tr}\left(\nu_k^{\rm T}\boldsymbol{B}_k^{\rm T}\boldsymbol{B}_k\nu_k\right). \tag{88}$$

Using the well-known matrix identity

$$\mathrm{tr}(\Gamma\Delta)=\mathrm{tr}(\Delta\Gamma), \tag{89}$$

where Γ, Δ are such matrices that the above matrix multiplication. From the reference [18], Eq. (88) can be rewritten in the form, such that

$$\boldsymbol{s}_k^{\rm T}\Pi_k\boldsymbol{s}_k\leqslant\frac{1}{\underline{p}}\mathrm{tr}\left(\boldsymbol{B}_k\nu_k\nu_k^{\rm T}\boldsymbol{B}_k^{\rm T}\right), \tag{90}$$

and taking the mean value yields

$$\mathrm{E}\left\{\boldsymbol{s}_k^{\rm T}\Pi_k\boldsymbol{s}_k\right\}\leqslant\frac{1}{p}\mathrm{tr}\left(\boldsymbol{B}_k\mathrm{E}\left(\nu_k\nu_k^{\rm T}\right)\boldsymbol{B}_k^{\rm T}\right). \tag{91}$$

Because ν_k is standard vector-valued white noise processes, the conditions

$$\mathrm{E}\left(\nu_k\nu_k^{\rm T}\right)=\boldsymbol{R}. \tag{92}$$

From Eqs. (91), (92) and (59), we get

$$\mathrm{E}\left\{\boldsymbol{s}_k^{\rm T}\Pi_k\boldsymbol{s}_k\right\}\leqslant\frac{\bar{r}}{\underline{p}}\mathrm{tr}\left(\boldsymbol{B}_k\boldsymbol{B}_k^{\rm T}\right). \tag{93}$$

Using Eqs. (7) and (59), we obtain

$$\mathrm{tr}\left(\boldsymbol{B}_k\boldsymbol{B}_k^{\rm T}\right)\leqslant\frac{\bar{r}^2\bar{\lambda}^2\bar{s}^2}{\underline{r}^2\underline{\lambda}^4\underline{s}^4}\mathrm{tr}(\boldsymbol{I})\leqslant\frac{\bar{r}^2\bar{\lambda}^2\bar{s}^2}{\underline{r}^2\underline{\lambda}^4\underline{s}^4}n, \tag{94}$$

where n is the number of the row for matrix $\boldsymbol{B}_k$. Therefore, it yields the desired inequality Eq. (87),

$$\mathrm{E}\left\{\boldsymbol{s}_k^{\rm T}\Pi_k\boldsymbol{s}_k\right\}\leqslant\frac{\bar{r}}{\underline{p}}\frac{\bar{r}^2\bar{\lambda}^2\bar{s}^2}{\underline{r}^2\underline{\lambda}^4\underline{s}^4}n=\boldsymbol{\kappa}_3\delta, \tag{95}$$

where $\boldsymbol{\kappa}_3=\frac{\bar{r}^3\bar{\lambda}^2\bar{s}^2}{\underline{p}\underline{r}^2\underline{\lambda}^4\underline{s}^4}n,\ \bar{r}=\delta$.

Proof of Theorem 2. We choose

$$\boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\rm CPF}\right)=\left(\tilde{\boldsymbol{x}}_k^{\rm T}\right)_{\rm CPF}\Pi_k(\tilde{\boldsymbol{x}}_k)_{\rm CPF} \tag{96}$$

with $\Pi_k = \left(\boldsymbol{P}_k^{-1}\right)_{\mathrm{CPF}}$, which exists since $(\boldsymbol{P}_k)_{\mathrm{CPF}}$ is positive definite. From Eq. (59) we have

$$\frac{1}{\bar{p}}\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^2 \leqslant \boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right) \leqslant \frac{1}{\underline{p}}\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^2. \tag{97}$$

To satisfy the requirements for an application of Lemma 1, we need an upper bound on $\mathrm{E}\{\boldsymbol{V}_{k+1}((\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}})|\,(\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\}$. From Eq. (39) we have

$$\begin{aligned}\boldsymbol{V}_{k+1}\left((\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right) &= \left[\left(\tilde{\boldsymbol{x}}_k^{\mathrm{T}}\right)_{\mathrm{CPF}}\cdot \boldsymbol{A}_k^{\mathrm{T}} + \bar{\boldsymbol{\mu}}_k^{\mathrm{T}} + \boldsymbol{s}_k^{\mathrm{T}}\right]\Pi_{k+1}\left[\boldsymbol{A}_k\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \bar{\boldsymbol{\mu}}_k + \boldsymbol{s}_k\right]\\ &= \left[\left(\tilde{\boldsymbol{x}}_k^{\mathrm{T}}\right)_{\mathrm{CPF}}\cdot(\boldsymbol{A}_k-\boldsymbol{K}_k\boldsymbol{C}_k)^{\mathrm{T}} + \left(\tilde{\boldsymbol{x}}_k^{\mathrm{T}}\right)_{\mathrm{CPF}}\cdot(\boldsymbol{K}_k\boldsymbol{C}_k)^{\mathrm{T}} + \bar{\boldsymbol{\mu}}_k^{\mathrm{T}} + \boldsymbol{s}_k^{\mathrm{T}}\right]\Pi_{k+1}\\ &\quad\cdot\left[(\boldsymbol{A}_k-\boldsymbol{K}_k\boldsymbol{C}_k)\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \boldsymbol{K}_k\boldsymbol{C}_k\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \bar{\boldsymbol{\mu}}_k + \boldsymbol{s}_k\right],\end{aligned} \tag{98}$$

and applying Lemma 2 we obtain with (96)

$$\begin{aligned}\boldsymbol{V}_{k+1}\left((\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right) \leqslant{}& (1-\alpha)\,\boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right) + \left(\boldsymbol{K}_k\boldsymbol{C}_k\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right)^{\mathrm{T}}\Pi_k\cdot\left[2\left(\boldsymbol{A}_k-\boldsymbol{K}_k\boldsymbol{C}_k\right)\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \boldsymbol{K}_k\boldsymbol{C}_k\right.\\ &\left.\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \bar{\boldsymbol{\mu}}_k\right] + \bar{\boldsymbol{\mu}}_k^{\mathrm{T}}\Pi_k[2\left(\boldsymbol{A}_k-\boldsymbol{K}_k\boldsymbol{C}_k\right)\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \boldsymbol{K}_k\boldsymbol{C}_k\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}} + \bar{\boldsymbol{\mu}}_k]\\ &+ \boldsymbol{s}_k^{\mathrm{T}}\Pi_k\boldsymbol{s}_k + \boldsymbol{s}_k^{\mathrm{T}}\Pi_k\left[2\left(\boldsymbol{A}_k-\boldsymbol{K}_k\boldsymbol{C}_k\right)\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}} + 2\boldsymbol{K}_k\boldsymbol{C}_k\cdot(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}} + 2\bar{\boldsymbol{\mu}}_k\right].\end{aligned} \tag{99}$$

Taking the conditional expectation $\mathrm{E}\left\{\boldsymbol{V}_{k+1}\left((\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right)|\,(\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right\}$ and considering the white noise property, it can be seen that the last term in Eq. (99) vanishes since it depend on ν_k. The remaining terms are estimated via Lemmas 2.2–2.4 yielding

$$\begin{aligned}&\mathrm{E}\left\{\boldsymbol{V}_{k+1}\left((\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right)|\,(\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right\} - \boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right)\\ &\quad\leqslant -\alpha\boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right) + \boldsymbol{\kappa}_1\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^2 + \boldsymbol{\kappa}_2\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^3 + \boldsymbol{\kappa}_3\bar{r},\end{aligned} \tag{100}$$

for $\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| \leqslant \varepsilon'$. Defining

$$\varepsilon = \min\left(\varepsilon', \frac{\alpha}{2\bar{p}\boldsymbol{\kappa}_2}\right). \tag{101}$$

We obtain with Eqs. (96) and (97) for $\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| \leqslant \varepsilon$,

$$\boldsymbol{\kappa}_2\left\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right\|\left\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right\|^2 \leqslant \frac{\alpha}{2\bar{p}}\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^2 \leqslant \frac{\alpha}{2}\boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right). \tag{102}$$

Inserting into Eq. (100) yields

$$\begin{aligned}&\mathrm{E}\left\{\boldsymbol{V}_{k+1}\left((\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right)|\,(\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right\} - \boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right) \leqslant -\tfrac{\alpha}{2}\boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right) + \tilde{\boldsymbol{\kappa}}_3\delta,\\ &\tilde{\boldsymbol{\kappa}}_3 = \boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_1\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^2\Big/\delta,\end{aligned} \tag{103}$$

for $\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| \leqslant \varepsilon$. Therefore we are able to apply Lemma 1 with $\|(\tilde{\boldsymbol{x}}_0)_{\mathrm{CPF}}\| \leqslant \varepsilon$, $\underline{v} = 1/\bar{p}, \bar{v} = 1/\underline{p}$ and $\tilde{\boldsymbol{\mu}} = \tilde{\boldsymbol{\kappa}}_3\delta$. However, we have to take care that for $\tilde{\varepsilon} \leqslant \|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| \leqslant \varepsilon$ with some $\tilde{\varepsilon} < \varepsilon$ the supermartingale inequality

$$\mathrm{E}\left\{\boldsymbol{V}_{k+1}\left((\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right)|\,(\tilde{\boldsymbol{x}}_{k+1})_{\mathrm{CPF}}\right\} - \boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right) \leqslant -\frac{\alpha}{2}\boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right) + \tilde{\boldsymbol{\kappa}}_3\delta \leqslant 0 \tag{104}$$

is fulfilled to guarantee the boundedness of the estimation error [19]. Choosing

$$\delta = \frac{\alpha\tilde{\varepsilon}^2}{2\bar{p}\tilde{\boldsymbol{\kappa}}_3} \tag{105}$$

with some $\tilde{\varepsilon} < \varepsilon$, we have $\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\| \geqslant \tilde{\varepsilon}$,

$$\tilde{\boldsymbol{\kappa}}_3\delta = \frac{\alpha\tilde{\varepsilon}^2}{2\bar{p}} \leqslant \frac{\alpha}{2\bar{p}}\|(\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\|^2 \leqslant \frac{\alpha}{2}\boldsymbol{V}_k\left((\tilde{\boldsymbol{x}}_k)_{\mathrm{CPF}}\right), \tag{106}$$

i.e., Eq. (104) holds. Therefore, we conclude that the estimation error remains bounded if the initial error and the noise terms are bounded.

Table 1 Different working conditions for the numerical simulations

	Initial error	Noise	Model error	Weighting matrix	Figure
Normal case	$\Theta=5°$ $\boldsymbol{\omega}^{\mathrm{T}}=[0,0,0]\,(°/\mathrm{s})$	$20''$	$\Delta\boldsymbol{N}_c=0.001\times[4,-5,-3]^{\mathrm{T}}$ (Nm) $\Delta\boldsymbol{N}_e=0.001\cos(10\omega_\circ t)\times[-3,2,3]^{\mathrm{T}}$ (Nm)	10^5	1
Serious case	$\Theta=5°$ $\boldsymbol{\omega}^{\mathrm{T}}=[0,0,0]\,(°/\mathrm{s})$	$100''$	$\Delta\boldsymbol{N}_c=0.005\times[4,-5,-3]^{\mathrm{T}}$ (Nm) $\Delta\boldsymbol{N}_e=0.005\cos(10\omega_\circ t)\times[-3,2,3]^{\mathrm{T}}$ (Nm)	10^5	2
Large initial error	$\Theta=85°$ $\boldsymbol{\omega}^{\mathrm{T}}=[1.7,1.7,1.7]\,(°/\mathrm{s})$	$20''$	$\Delta\boldsymbol{N}_c=0.001\times[4,-5,-3]^{\mathrm{T}}$ (Nm) $\Delta\boldsymbol{N}_e=0.001\cos(10\omega_\circ t)\times[-3,2,3]^{\mathrm{T}}$ (Nm)	10^5	3
Large measurement noise	$\Theta=5°$ $\boldsymbol{\omega}^{\mathrm{T}}=[0,0,0]\,(°/\mathrm{s})$	$1800''$	$\Delta\boldsymbol{N}_c=0.001\times[4,-5,-3]^{\mathrm{T}}$ (Nm) $\Delta\boldsymbol{N}_e=0.001\cos(10\omega_\circ t)\times[-3,2,3]^{\mathrm{T}}$ (Nm)	10^5	4
Large model error	$\Theta=5°$ $\boldsymbol{\omega}^{\mathrm{T}}=[0,0,0]\,(°/\mathrm{s})$	$20''$	$\Delta\boldsymbol{N}_c=0.023\times[4,-5,-3]^{\mathrm{T}}$ (Nm) $\Delta\boldsymbol{N}_e=0.023\cos(10\omega_\circ t)\times[-3,2,3]^{\mathrm{T}}$ (Nm)	10^5	5

Based on the preceding results, we can make further analysis. On the analogy of Eq. (59),

$$\|\boldsymbol{A}_k\|, \quad (\boldsymbol{P}_k)_{\mathrm{CPF}} \quad \text{and} \quad \left\|\frac{1}{2}(\Delta t)\left[\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{f}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}+\left(\nabla^{\mathrm{T}}\nabla\right)\boldsymbol{\eta}(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}\right]\right\|$$

remain bounded; therefore, the elements of these formulas also have bounded. In response to Eq. (46),

$$\|\boldsymbol{G}_\eta\|, \quad \left\|\frac{\left(\nabla^{\mathrm{T}}(\boldsymbol{P}_k)_{\mathrm{CPF}}\nabla\right)\eta(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}}{2!}\right\|$$

remain bounded apparently. Considering the assumption $\boldsymbol{g}(\boldsymbol{x})=\boldsymbol{I}$ exists in Eq. (3), we can obtain $-\boldsymbol{d}(\hat{\boldsymbol{x}}_k)=\boldsymbol{\eta}(\hat{\boldsymbol{x}}_k)$; therefore, the formulas $\boldsymbol{G}_\eta=-\boldsymbol{G}_d$ and

$$\frac{\left(\nabla^{\mathrm{T}}(\boldsymbol{P}_k)_{\mathrm{CPF}}\nabla\right)\eta(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}}{2!}=-\frac{\left(\nabla^{\mathrm{T}}(\boldsymbol{P}_k)_{\mathrm{CPF}}\nabla\right)d(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}}{2!}$$

both exist. We can obtain the expression as follows:

$$\bar{\boldsymbol{d}}_k=\mathrm{E}[\boldsymbol{d}_k]=\hat{\boldsymbol{d}}(\boldsymbol{x}_k)-\boldsymbol{G}_d\tilde{\boldsymbol{x}}_k-\frac{\left(\nabla^{\mathrm{T}}(\boldsymbol{P}_k)_{\mathrm{CPF}}\nabla\right)d(\boldsymbol{x})|_{\boldsymbol{x}=\hat{\boldsymbol{x}}_k}}{2!}-\tilde{\varepsilon}(\hat{\boldsymbol{x}}_k), \tag{107}$$

where, $\tilde{\varepsilon}(\hat{\boldsymbol{x}}_k)$ is the high-order discretization error. In the process of proving convergence for the covariance $(\boldsymbol{P}_k)_{\mathrm{CPF}}$, the high-order error $\tilde{\varepsilon}(\hat{\boldsymbol{x}}_k)$ is included into the term of $\boldsymbol{Q}_k$ in Eq. (46). Because the covariance $(\boldsymbol{P}_k)_{\mathrm{CPF}}$ is verified convergent and bounded; therefore, $\boldsymbol{Q}_k$ also remains bounded. It is obvious that the high-order error $\tilde{\varepsilon}(\hat{\boldsymbol{x}}_k)$ must be bounded. $\tilde{\boldsymbol{d}}(\boldsymbol{x}_k)$ is the estimation of the model error in the CPF. If the CPF is convergent, the estimation must remain bounded. We can obtain the conclusion that if the CPF is convergent, the expectation of the actual model error must be bounded, so that the actual model error also requires bounded.

In this section, we have proved that the estimation error of the CPF remains bounded as long as the initial error, the disturbing noise terms and the model error terms have boundedness. This is an important result, since the stability and convergence of the CPF have been analyzed.

6 Numerical simulations

The results discussed above show that the CPF algorithm based on the cubature-points remains bounded if certain conditions are satisfied. These conditions include the requirements of a small enough initial error and small enough noise as well as small enough model error. To demonstrate the performance of the CPF and the significance of these conditions, in this section we apply the CPF to an example and verify the estimate performance and the error behavior by numerical simulations. We consider a nonlinear

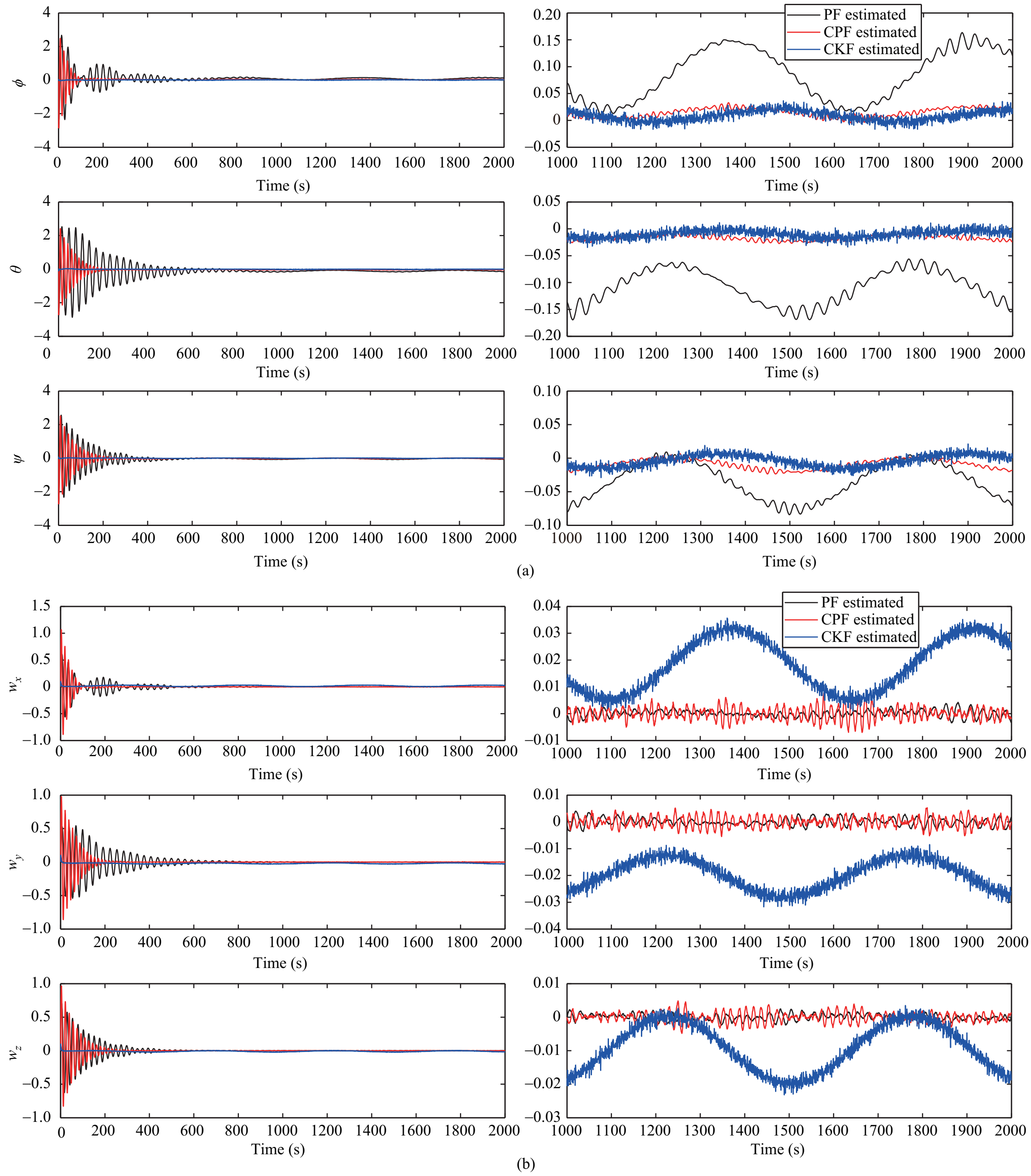

Figure 1 (Color online) Numerical simulations for the nominal case: (a) the estimation error of the attitude; (b) the estimation error of the angular velocity.

system of the spacecraft attitude determination given by

$$\begin{aligned} \dot{\boldsymbol{X}} &= \begin{bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega})\boldsymbol{q} \\ \boldsymbol{J}^{-1}\left[\boldsymbol{\omega}\times(\boldsymbol{J}\boldsymbol{\omega})+\boldsymbol{N}\right]+\boldsymbol{J}^{-1}\boldsymbol{d} \end{bmatrix}, \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{C}_i^b(\boldsymbol{q})\,\nu_1 \\ \boldsymbol{C}_i^b(\boldsymbol{q})\,\nu_2 \end{bmatrix} + \begin{bmatrix} \Delta\nu_1 \\ \Delta\nu_2 \end{bmatrix}, \end{aligned} \tag{108}$$

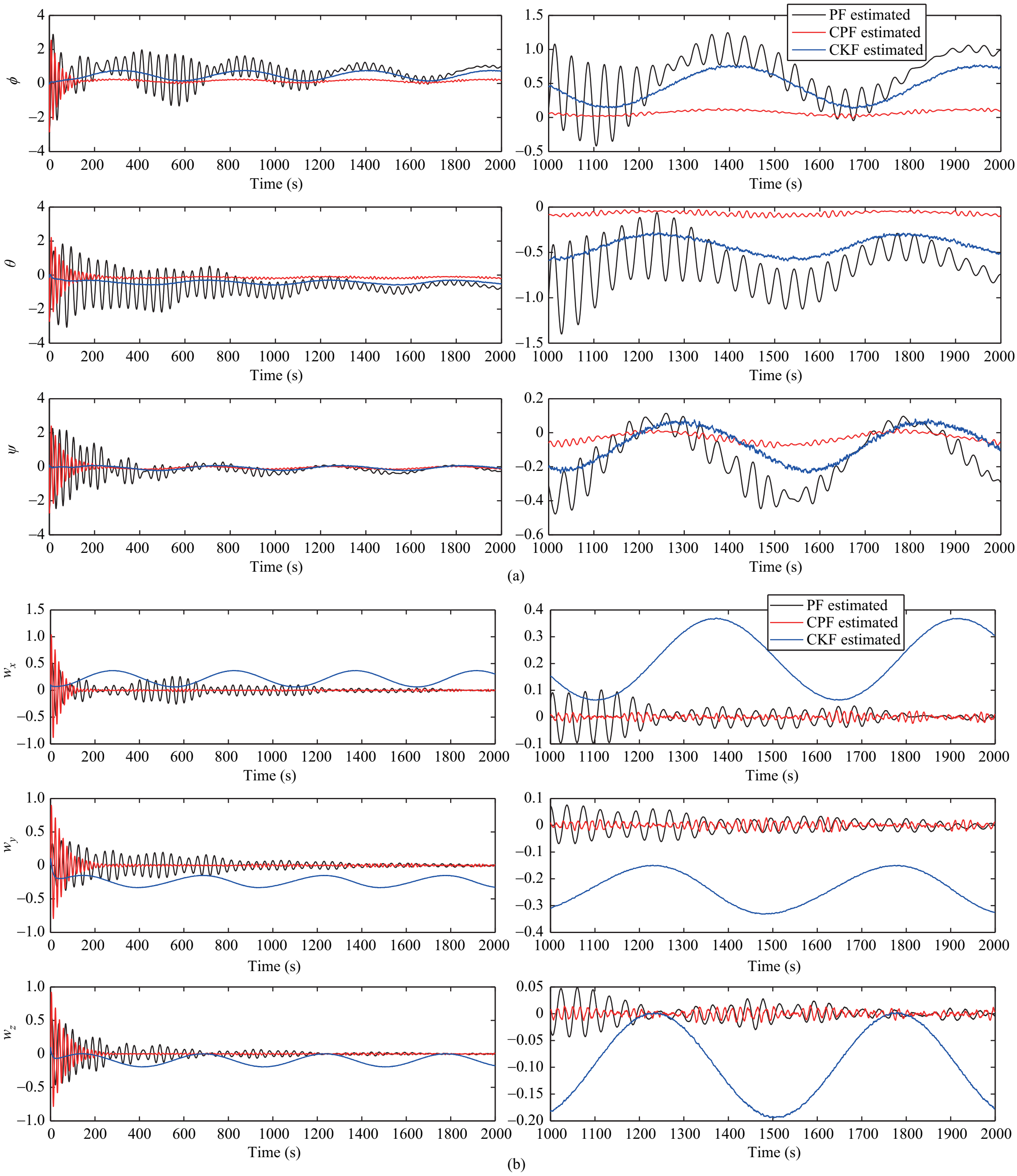

Figure 2 (Color online) Numerical simulations for the serious case: (a) the estimation error of the attitude; (b) the estimation error of the angular velocity.

where,

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\boldsymbol{\omega}^{\mathrm{T}} \\ \boldsymbol{\omega} & -[\boldsymbol{\omega}\times] \end{bmatrix}, \quad [\boldsymbol{\omega}\times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \tag{109}$$

where, $\boldsymbol{q}$ and $\boldsymbol{\omega}$ are the attitude quaternion and the angular velocity of the body-fixed reference relative to the inertial frame; $\boldsymbol{N}$ denotes the total torque vector; $\boldsymbol{J}$ denotes the moment of inertia tensor of the spacecraft; $\boldsymbol{d}$ represents the model error; $\boldsymbol{C}_i^b$ is the transition matrix from the inertial frame to the body-

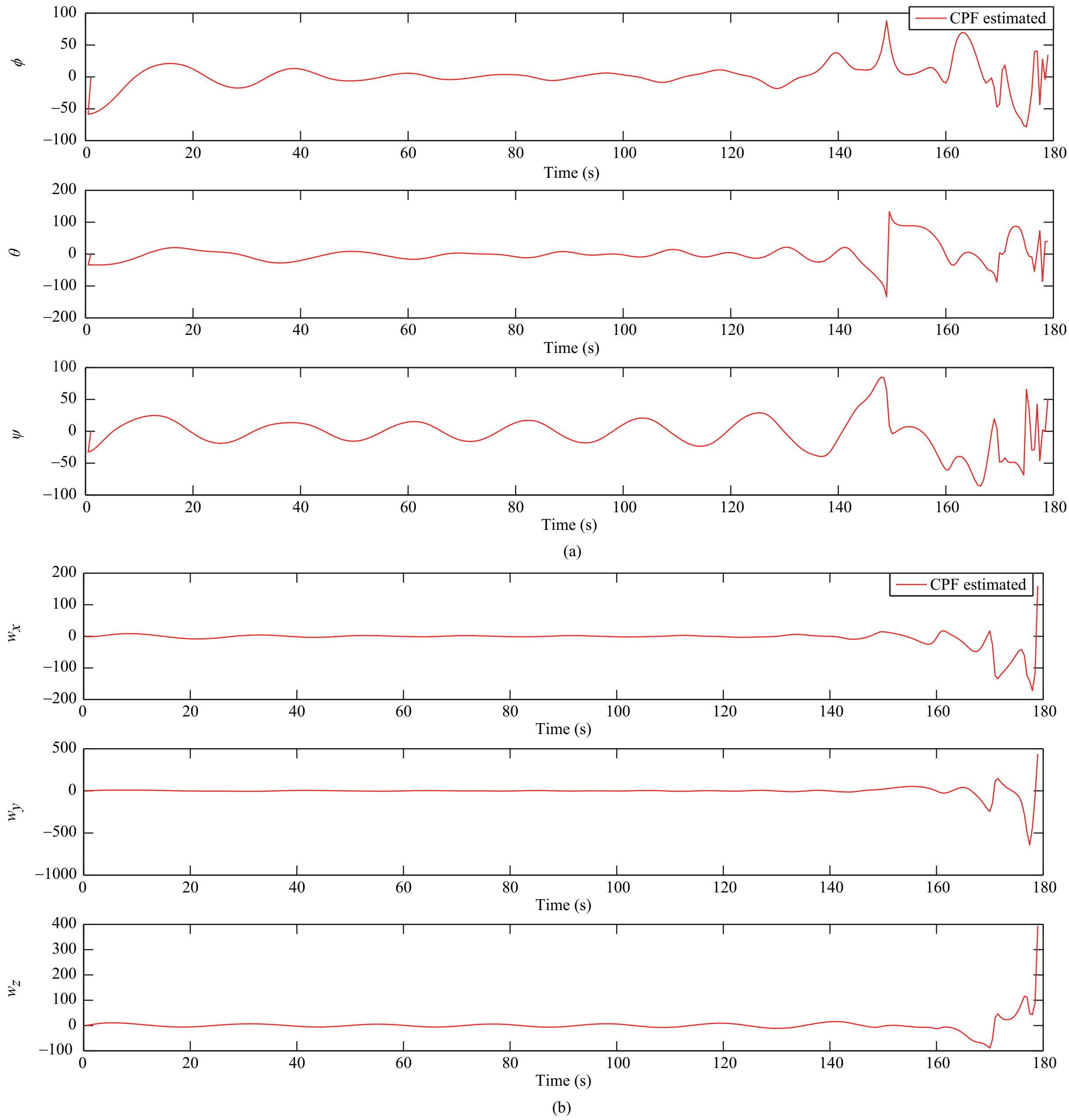

Figure 3 (Color online) Numerical simulations for large initial error: (a) the estimation error of the attitude; (b) the estimation error of the angular velocity.

fixed reference; ν_1, ν_2 are the star sensor vectors in the inertial frame; $\Delta\nu_1, \Delta\nu_2$ are the measurement noises of the star sensor.

The numerical simulation parameters of the nominal case are the following.

The initial value of the attitude quaternion is $\boldsymbol{q} = [\,1\ 0\ 0\ 0\,]^{\mathrm{T}}$; the initial value of the angular velocity is $\boldsymbol{\omega}^{\mathrm{T}} = [\,0.1\ 0.1\ 0.1\,]\,(^\circ/\mathrm{s})$; the initial estimated value of the attitude quaternion is

$$\hat{\boldsymbol{q}} = \left[\cos\frac{\Theta}{2}\ \ \frac{\sqrt{3}}{3}\sin\frac{\Theta}{2}\ \ \frac{\sqrt{3}}{3}\sin\frac{\Theta}{2}\ \ \frac{\sqrt{3}}{3}\sin\frac{\Theta}{2}\right]^{\mathrm{T}}$$

and $\Theta = 5^\circ$; the initial estimated value of the angular velocity is $\boldsymbol{\omega}^{\mathrm{T}} = [\,0\ 0\ 0\,]\,(^\circ/\mathrm{s})$; the mean square of the star sensor measurement noise is $20''$; the actual model error of this system is consist of $\boldsymbol{d} = \Delta\boldsymbol{N}_c + \Delta\boldsymbol{N}_e$, the constant component is $\Delta\boldsymbol{N}_c = 0.001 \times [4, -5, -3]^{\mathrm{T}}$ (Nm), and the periodic component

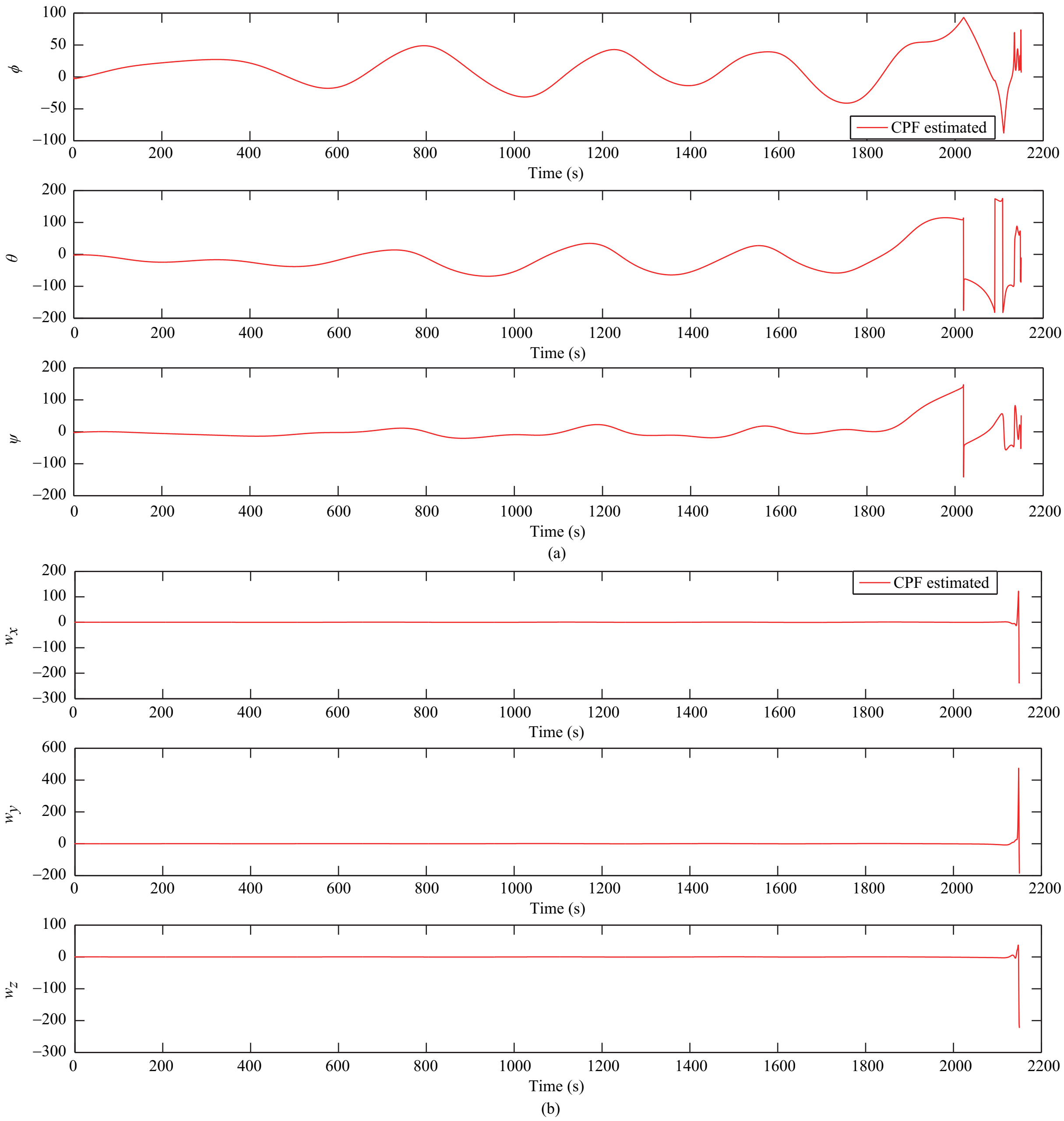

Figure 4 (Color online) Numerical simulations for large noise: (a) the estimation error of the attitude; (b) the estimation error of the angular velocity.

is $\Delta \boldsymbol{N}_e = 0.001\cos(10\omega_\circ t) \times [-3, 2, 3]^{\mathrm{T}}$ (Nm); and $\omega_\circ = 0.066°/\mathrm{s}$; the weighting matrix of the CPF and the PF is $\boldsymbol{W} = 10^5 \boldsymbol{I}_{3\times3}$; the parameters of the symmetrically-distributed set of points are $n = 6, \kappa = 0.8$; the moment of inertia matrix of the spacecraft is $\boldsymbol{J} = \mathrm{diag}\left\{[49.96; 55.40; 63]\left(\mathrm{kg}\cdot\mathrm{m}^2\right)\right\}$.

To demonstrate the theoretical results in Sections 3 and 5, the comparison of the estimates produced by the CPF and the PF are shown firstly. In addition, the UKF (unscented Kalman filter) is employed to compare with CPF to shown the effectiveness of the proposed CPF. Then the bounded analyses of the estimate error for the CPF are given to verify the theoretical results in Section 5. Due to the fact of the theoretical results discussed above, we consider the following cases: (1) the nominal case; (2) the serious case; (3) large initial error; (4) large measurement noise; (5) large model error. To illustrate the significance of these conditions, in this subsection we adopt the CPF to assess the error behavior by numerical simulations. For all the previously mentioned cases, the simulation figures of the attitude and

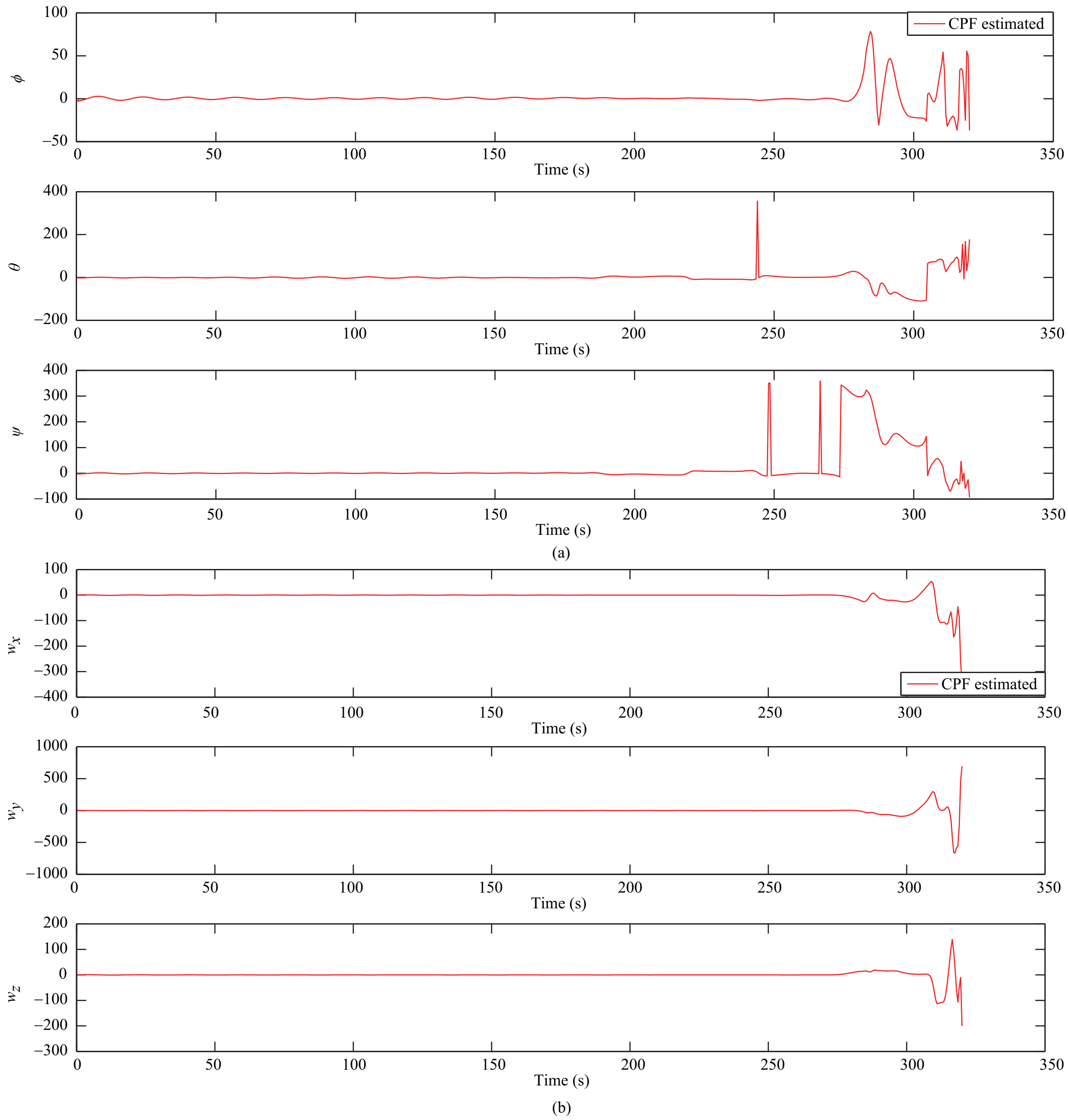

Figure 5 (Color online) Numerical simulations for large model error: (a) the estimation error of the attitude; (b) the estimation error of the angular velocity.

the angular velocity are all the graphs of the estimation error. The relevant parameters are given in the following Table 1.

The simulation results of the nominal case are depicted in Figure 1, which shows the comparison of the CPF, UKF and the PF in the aspects of the attitude estimate error, the angular velocity estimate error as well as the model error estimate error. As can be seen from Figure 1(a) and (b), the convergence speed of the UKF and the CPF is faster than that of the PF and the fluctuations in the convergent process are also smaller and shorter. Particularly, the convergence precision is much higher than that of the PF. It is obvious that the CPF and UKF have nearly the same estimate accuracy of the attitude. However, due to the existence of the model error, the CPF and PF have the better performance than UKF in the angular velocity in Figure 1(b). As evident in Figure 1(a) and (b), the CPF algorithm shows the excellent property in the convergence speed than PF and has better estimate accuracy than UKF.

With the large measurement noise and model errors, the filtering performances of UKF and PF have degraded seriously in Figure 2. However, the CPF still maintains good performance, which illustrates the superiority of the proposed filter. For small initial error and small measurement noise as well as small model error, the estimation error of the CPF remains bounded, as can be verified in Figure 1. However, in the case of large initial error, large measurement noise or large model error, the estimation error is no longer bounded and diverges, as shown in Figures 3–5. This is because of the high nonlinearities of the example system. Furthermore, it is worth pointing out that the parameters in the cases of large initial error, large measurement noise or large model error are much bigger than that of the nominal case in Table 1. The main reason for this phenomenon is that the CPF has a good ability to estimate and deal with the model errors and the large initial error and large noise also can be treated as the special model error to compensate. Obviously, all of the simulation results in this subsection are coincide with the theoretical results in Section 5.

7 Conclusion

In this paper, in a stochastic framework, we have analyzed the error behavior of the CPF when it is applied to general estimation problems for nonlinear system. In Sections 3 and 4, the error analyses of the model error and system state are discussed and the stochastic boundedness has been proved. In Section 5, the proofs show that the estimation error of the CPF is bounded in mean square and bounded with probability one under certain conditions. These conditions include the requirements that the initial estimation error, the measurement noise as well as the model error are small enough. The numerical simulation results in Section 6 verify that the CPF has better estimate performance and the estimate error has proved bounded as long as the initial estimation error, the measurement noise as well as the model error are small enough, which meets the theoretical analysis of Section 5. Moreover, the simulation results indicate that the estimation error is divergent if the initial error, the noise or the model error is large enough. These results presented in this paper will support the theory development and application of the CPF.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant No. 61503414) and State Key Laboratory of Astronautic Dynamics Foundation (ADL) (Grant No. 2016ADL-DW0202).

Conflict of interest The authors declare that they have no conflict of interest.

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