SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

September 2016, Vol. 59 092202:1–092202:16
doi: 10.1007/s11432-015-5498-0

Modeling of nonlinear dynamical systems based on deterministic learning and structural stability

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Received August 25, 2015; accepted October 28, 2015; published online May 27, 2016

Abstract Recently, a deterministic learning (DL) theory was proposed for accurate identification of system dynamics for nonlinear dynamical systems. In this paper, we further investigate the problem of modeling or identification of the partial derivative of dynamics for dynamical systems. Firstly, based on the locally accurate identification of the unknown system dynamics via deterministic learning, the modeling of its partial derivative of dynamics along the periodic or periodic-like trajectory is obtained by using the mathematical concept of directional derivative. Then, with accurately identified system dynamics and the partial derivative of dynamics, a C^1-norm modeling approach is proposed from the perspective of structural stability, which can be used for quantitatively measuring the topological similarities between different dynamical systems. This provides more incentives for further applications in the classification of dynamical systems and patterns, as well as the prediction of bifurcation and chaos. Simulation studies are included to demonstrate the effectiveness of this modeling approach.

Keywords system modeling, system identification, deterministic learning, nonlinear dynamics, structural stability, topological equivalence

Citation Chen D F, Wang C, Dong X D. Modeling of nonlinear dynamical systems based on deterministic learning and structural stability. Sci China Inf Sci, 2016, 59(9): 092202, doi: 10.1007/s11432-015-5498-0

1 Introduction

Over the past decades, modeling or identification of nonlinear systems becomes increasingly important and a lot of approaches have been investigated [1–4]. Due to the universal approximation ability of the neural networks (NNs), identification algorithms based on neural network (NN) have attracted a lot of attention [1, 5–7]. In particular, the Lyapunov stability theory [8] is commonly introduced to design and analyze the rule for updating the weights of NN. To achieve accurate nonlinear modeling and the convergence of estimating NN weights, however, the difficulty lies in that the network input is normally required to be persistently exciting [9–11]. Initial results on the persistent excitation (PE) condition for identification algorithms based on the radial basic function (RBF) were presented in [12–14]. Besides, as

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shown by [15], the traditionally defined PE condition was further relaxed to a cooperative PE condition by considering the network topology for a group of continuous time systems.

Recently, a deterministic learning (DL) [16, 17] approach was proposed for identification, recognition and control of nonlinear dynamical systems. Through DL, the unknown system dynamics can be accurately modeled in a local region along the recurrent trajectories of nonlinear dynamical systems. Then, a time-varying dynamical pattern that generated from dynamical systems can be effectively represented in a time-invariant way by using constant neural weights [16]. Similarity definition based on the system dynamics and pattern states was given [18]. With these results, a rapid recognition mechanism was proposed according to a kind of internal and dynamical matching on system dynamics for dynamical systems and patterns [18].

For nonlinear dynamical systems, structural stability is a fundamental concept since it provides a qualitative tool for analyzing the equivalent relation between a nonlinear dynamical system and its perturbed system [19]. More specifically, a dynamical system $\dot{x} = f(x)$ is structurally stable if its system trajectories are qualitatively similar (topologically equivalent) to the trajectories of any perturbed system $\dot{x} = f'(x)$ in the ϵ-neighborhood of system $\dot{x} = f(x)$ in the sense of C^1-closed measure

$$\|f - f'\|_{C^1} = \sup_{x\in\Omega} \left\{ \|f - f'\|_{C^0} + \left\| \frac{\partial f}{\partial x} - \frac{\partial f'}{\partial x} \right\|_{C^0} \right\} \leqslant \epsilon, \tag{1}$$

in which, $\|\cdot\|_{C^0}$ denotes a vector norm in R^n. Thus, a structurally stable dynamical system $\dot{x} = f(x)$ and its C^1-closed system $\dot{x} = f'(x)$ possess similar qualitative structures and can be classified as the same class of dynamical systems. While structural stability is related to similar dynamical systems and system trajectories, the concept of structural instability yields bifurcation, implying the topological nonequivalence of system trajectories as a parameter-dependent dynamical system varies its parameters across a critical value [17]. Thus, the bifurcation points can be taken as the boundaries between different subclasses of dynamical systems and patterns.

Currently, most researches about structural stability [20–22], including some applications in practical systems such as the power system [19,23], are mainly limited to qualitative analyses. As for its quantitative study, it is a completely new problem and to the best of our knowledge, has not been investigated in the literature.

According to the definition of structural stability, two parts of system dynamics are contained in the C^1-closed measure, namely, the system dynamics f and its partial derivative of dynamics $\frac{\partial f}{\partial x}$. This means that the calculation of the C^1-norm measure largely depends on the accurate modeling of both parts of system dynamics. As mentioned before, the unknown nonlinear dynamics f can be locally-accurately identified along the periodic or recurrent trajectory through deterministic learning [16]. Thus, the remaining key problem for the quantitative property of structural stability lies in modeling of the partial derivative of dynamics $\frac{\partial f}{\partial x}$.

In this paper, we firstly investigate the problem of modeling or identification of the partial derivative of dynamics $\frac{\partial f}{\partial x}$ based on the identified system dynamics f. Precisely, based on the locally accurate identification of system dynamics f, the modeling of the partial derivative of dynamics $\frac{\partial f}{\partial x}$ is obtained along the system trajectory by introducing the mathematical concept of directional derivative. Then, a C^1-norm ($\|\cdot\|_{C^1}$) modeling approach is proposed based on structural stability. With the identification of both parts of dynamics f and $\frac{\partial f}{\partial x}$, the C^1-norm modeling can be used for quantitatively measuring the topological differences between nonlinear systems. This will provide more incentives for further applications in the classification for nonlinear systems and dynamical patterns [24, 25], as well as the prediction of bifurcation and chaos. Simulation studies based on the Duffing oscillator [26] and the Rossler system [27] are included to demonstrate the effectiveness of the proposed approach.

The rest of the paper is organized as follows. In Section 2, we firstly discuss the problems to be solved in this paper, and then briefly review the deterministic learning theory and the concept of structural stability. The main results are given in Section 3, including the modeling of the partial derivative of dynamics $\frac{\partial f}{\partial x}$, and the C^1-norm ($\|\cdot\|_{C^1}$) modeling of system dynamics. Section 4 demonstrates the simulation results. In Section 5, the conclusion of this paper is given firstly, and then is a concise discussion on further applications of this modeling approach.

2 Problem formulation and preliminaries

2.1 Problem formulation

Consider a general nonlinear dynamical system in the following form:

$$\dot{x} = f(x;p), \quad x(t_0) = x_{\xi 0}, \tag{2}$$

where $x = [x_1, x_2, \dots, x_n]^{\mathrm{T}} \in R^n$ is the state vector of the system, $p \in R^m$ is the parameter vector that different parameter value may produce different dynamical behaviors. $f(x;p) = [f_1(x;p), \dots, f_n(x;p)]^{\mathrm{T}}$ is a continues nonlinear vector field, with $f(x;p)$ and its partial derivative $\frac{\partial f}{\partial x}$ both representing the corresponding system dynamics.

Assumption 1. The dynamical systems are unknown nonlinear but smooth. The state x remains uniformly bounded, that is, $x(t) \in \Omega \subset R^n, \forall t \geqslant t_0$, where Ω is a compact set. Moreover, the system trajectory starting from point $x(t_0) = x_{\xi 0}$ that denoted as $\varphi_\xi(x_0)$ is in either a periodic or periodic-like (recurrent) motion.

The recurrent trajectory represents a large class of trajectories generated from nonlinear dynamical systems, including not only periodic trajectories, but also quasi-periodic, almost-periodic, and even some chaotic trajectories. Particularly, many practical dynamical systems, such as rotating machineries, electronic systems, power systems, communication network, ECG systems, etc., exhibit such kind of property of recurrent trajectories or oscillations.

The objective of the paper is twofold: (1) to model or identify the partial derivative of dynamics $\frac{\partial f}{\partial x}$ along the system trajectory, and (2) to provide a quantitative modeling of system dynamics in the sense of C^1-norm given by

$$\|f\|_{C^1} = \sup_{x\in\Omega} \left\{ \|f\|_{C^0} + \left\| \frac{\partial f}{\partial \varphi_\xi} \right\|_{C^0} \right\}, \tag{3}$$

in which $\|\cdot\|_{C^0}$ denotes a vector norm in R^n defined as

$$\|\cdot\|_{C^0} = \max_{x,p\in\varphi_\xi} |\cdot|, \tag{4}$$

$\frac{\partial f}{\partial \varphi_\xi}$ is the identification result of the partial derivative of dynamics $\frac{\partial f}{\partial x}$, with φ_ξ being the corresponding system trajectory.

Remark 1. As for the C^1-norm modeling approach, the motivation comes from the concept of structural stability. As a fundamental property of nonlinear dynamical system, structural stability was mainly used as a qualitative tool for analyzing the equivalent relation between dynamical systems. According to the definition of the structural stability, the equivalent relation between a nonlinear system and its perturbed system can be described through the C^1-norm measure of their system dynamics. Thus, in this paper a C^1-norm modeling approach is proposed for quantitatively measuring the topological similarities of dynamical systems. In the C^1-norm modeling, two parts of dynamics are contained, i.e., the system dynamics f and its partial derivative of dynamics $\frac{\partial f}{\partial x}$. This means that the quantitative calculation of the C^1-norm depends on the accurate modeling of both parts of system dynamics. Whereas the identification of the system dynamics f has been achieved through deterministic learning, the remaining problem is the modeling of the partial derivative of dynamics.

2.2 Deterministic learning theory

In what follows, we briefly review the deterministic learning algorithm and the concept of structural stability.

The deterministic learning (DL) approach [16,17] was proposed for identification of nonlinear dynamical systems according to the following elements: (1) employment of localized radial basis function (RBF) networks [9, 28]; (2) satisfaction of a partial persistent excitation (PE) condition [29, 30]; (3) exponential stability of the adaptive system along the period or recurrent orbit; (4) locally-accurate NN approximation of the unknown system dynamics [31].

Consider the following dynamical RBF network for identification of the unknown system dynamics $f_i(x;p)$ of the nonlinear system $\dot{x}_i = f_i(x;p)$ under the assumption that the system state x remains uniformly bounded and the corresponding trajectory $\varphi_\xi(x_0)$ starting from x_0 is a recurrent trajectory [32]:

$$\dot{\hat{x}}_i = -a_i(\hat{x}_i - x_i) + \hat{W}_i^{\mathrm{T}} S_i(x), \quad i = 1, \ldots, n, \tag{5}$$

where $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_n]^{\mathrm{T}}$ is the state vector, $x = [x_1, \ldots, x_n]^{\mathrm{T}}$ is the system state, $a_i > 0$ is the design constant, and $\hat{W}_i^{\mathrm{T}} S_i(x)$ is a RBF network used to approximate the unknown nonlinearity $f_i(x;p)$. The weight estimates $\hat{W}_i$ are updated by using the following Lyapunov-based learning law:

$$\dot{\hat{W}}_i = -\mathit{\Gamma}_i S_i(x)\tilde{x}_i, \tag{6}$$

where $\mathit{\Gamma}_i = \mathit{\Gamma}_i^{\mathrm{T}} > 0$ and $\tilde{x}_i = \hat{x}_i - x_i$.

Define $\tilde{W}_i = \hat{W}_i - W_i^*$, where $\hat{W}_i$ is the estimation of W_i^*. According to the properties of localized RBF networks, for each recurrent trajectory $\varphi_\xi(x_0)$ generated from system $\dot{x}_i = f_i(x;p)$, the corresponding identification error system can be derived in the following form:

$$\begin{bmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{W}}_{\xi i} \end{bmatrix} = \begin{bmatrix} -a_i & S_{\xi i}(\varphi_\xi)^{\mathrm{T}} \\ \mathit{\Gamma}_{\xi i} S_{\xi i}(\varphi_\xi) & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_i \\ \tilde{W}_{\xi i} \end{bmatrix} + \begin{bmatrix} \epsilon'_{\xi i} \\ 0 \end{bmatrix}, \tag{7}$$

where $\epsilon'_{\xi i} = \epsilon_{\xi i} - \hat{W}_{\bar{\xi} i}^{\mathrm{T}} S_{\bar{\xi} i}(\varphi_\xi) = 0(\epsilon_{\xi i}) = 0(\epsilon_i)$, and the subscripts $(\cdot)_{\xi i}$ and $(\cdot)_{\bar{\xi} i}$ stand for the regions close to and away from the trajectory, respectively. $S_{\xi i}(\varphi_\xi)$ is a subvector of $S_i(x)$, $\hat{W}_{\xi i}$ is the corresponding weight subvector, and $|\epsilon'_{\xi i}|$ is close to ϵ_i^*, where ϵ_i^* is the ideal approximation error. Based on the properties of RBF networks, almost any periodic or recurrent trajectory $\varphi_\xi(x_0)$ ensures PE of the regressor subvector $S_{\xi i}(\varphi_\xi)$. Then, a locally accurate NN approximation of the unknown dynamics $f_i(x;p)$ is obtained along the trajectory $\varphi_\xi(x_0)$ as

$$f_i(\varphi_\xi;p) = \hat{W}_{\xi i}^{\mathrm{T}} S_{\xi i}(\varphi_\xi) + \epsilon_{\xi i_1} = \hat{W}_{\xi i}^{\mathrm{T}} S_{\xi i}(\varphi_\xi) + \hat{W}_{\bar{\xi} i}^{\mathrm{T}} S_{\bar{\xi} i}(\varphi_\xi) + \epsilon_{\xi i_1} - \hat{W}_{\bar{\xi} i}^{\mathrm{T}} S_{\bar{\xi} i}(\varphi_\xi) = \hat{W}_i^{\mathrm{T}} S_i(\varphi_\xi) + \epsilon_{i_1}, \tag{8}$$

where $\epsilon_{i_1} = \epsilon_{\xi i_1} - \hat{W}_{\bar{\xi} i}^{\mathrm{T}} S_{\bar{\xi} i}(\varphi_\xi) = 0(\epsilon_{\xi i_1}) = 0(\epsilon_i)$. Moreover, based on the convergence result, a constant vector of neural weights is obtained by using the mathematical sense of arithmetic mean [17]: $\bar{W}_i = \mathrm{mean}_{t \in [t_a, t_b]} \hat{W}_i(t)$, in which $[t_a, t_b]$, $t_b > t_a > T$ represents a time segment after the transient process. Hence, accurate NN approximation of the system dynamics in a local region along a periodic or recurrent trajectory is obtained:

$$f_i(\varphi_\xi;p) = \hat{W}_{\xi i}^{\mathrm{T}} S_{\xi i}(\varphi_\xi) + \epsilon_{\xi i_1} = \bar{W}_{\xi i}^{\mathrm{T}} S_{\xi i}(\varphi_\xi) + \epsilon_{\xi i_2} = \bar{W}_i^{\mathrm{T}} S_i(\varphi_\xi) + \epsilon_{i_2}, \tag{9}$$

in which $\epsilon_{i_2} = \epsilon_{\xi i_2} - \bar{W}_{\bar{\xi} i}^{\mathrm{T}} S_{\bar{\xi} i}(\varphi_\xi) = 0(\epsilon_{\xi i_2}) = 0(\epsilon_i)$ is the practical approximation error by using $\bar{W}_i^{\mathrm{T}} S_i$. Thus, accurate identification of the unknown dynamics is achieved by using the entire RBF network within a local region along the recurrent orbit.

2.3 Structural stability

As a fundamental property of nonlinear dynamical systems, structural stability provides a justification for applying the qualitative theory of dynamical systems to analyze physical systems [33]. Unlike Lyapunov stability [34], which studies the influence of perturbations of the initial condition on dynamical behaviors of a dynamical system itself, structural stability reveals that the qualitative dynamical behaviors (i.e. the system trajectories) of a nonlinear dynamical system will be unchanged or unaffected by small parameter

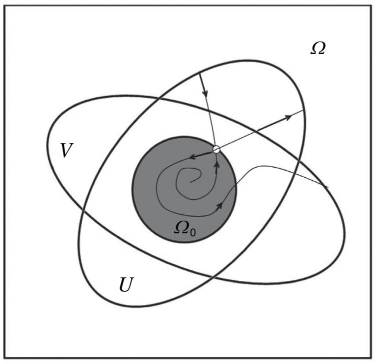

Figure 1 Andronov's structural stability [35].

perturbations [19]. The unchanging property of the topological structures or the qualitative dynamical behaviors is closely related to the concept of topological equivalence.

Consider the following nonlinear dynamical system:

$$\dot{x} = f'(x;p'), \quad x(t_0) = x_{\zeta 0}, \tag{10}$$

where, $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in R^n$ is the system state, $p' \in R^m$ is the parameter vector. $f'(x;p') = [f'_1(x;p'), \ldots, f'_n(x;p')]^{\mathrm{T}}$ is the unknown but nonlinear smooth function vector.

Two dynamical systems $\dot{x} = f(x;p)$ and $\dot{x} = f'(x;p')$ given in (2) and (10) are considered as topologically equivalent if their phase portraits are qualitatively similar. That is, for two topologically equivalent systems (2) and (10), the portrait of one of the system can be obtained from the other by a continuous transformation or a homeomorphism mapping, an invertible map that both the map and its inverse are continuous [32, 35]. Based on the concept of topological equivalence, structural stability is defined as follows.

Definition 1 (Andronov's structural stability [35]). A system (2) defined in a region $\Omega \subset R^n$ is called structurally stable in a region $\Omega_0 \subset \Omega$ if for any sufficiently C^1-close in Ω system (10), there are regions $U, V \subset \Omega$, $\Omega_0 \subset U$ such that system (2) is topologically equivalent in U to system (10) in V (the relationship between these regions is demonstrated in Figure 1 and the C^1-close measure is shown in Eq. (11) as follows):

$$\|f - f'\|_{C^1} = \sup_{x \in \Omega} \left\{ \|f - f'\|_{C^0} + \left\| \frac{\partial f}{\partial x} - \frac{\partial f'}{\partial x} \right\|_{C^0} \right\}, \tag{11}$$

in which $\|\cdot\|_{C^0}$ denotes a vector norm in R^n.

3 Main results

In this section, we firstly investigate the modeling or identification of the partial derivative of dynamics $\frac{\partial f}{\partial x}$ along system trajectory, and then propose the C^1-norm ($\|\cdot\|_{C^1}$) modeling of system dynamics to further obtain the quantitative property of structure stability.

3.1 Modeling of the partial derivative of dynamics $\frac{\partial f}{\partial x}$

As presented by [35], the appearance of the first partial derivative of system function $\frac{\partial f}{\partial x}$ shown in (11) is natural if one wants to ensure that neighboring systems have the same topological type. It implies the corresponding dynamical information for its neighboring systems. To accurately model the partial derivative of dynamics $\frac{\partial f}{\partial x}$, we introduce the mathematical concept of directional derivative [36].

The directional derivative of a scalar function $f(X) = f(x_1, \ldots, x_n)$ along a vector $V = (v_1, \ldots, v_n)$ is the function defined by the following limit:

$$\nabla_V f(X) = \lim_{h \to 0} \frac{f(X + hV) - f(X)}{h}, \tag{12}$$

in which, if the function f is differentiable at X, then the directional derivative exists along any vector V, and one has

$$\nabla_v f(X) = \nabla f(X) \cdot V, \tag{13}$$

where the sign ∇ on the right of Eq. (13) denotes the gradient. It reflects the rate of change of the system function f along the direction given by vector V at a point x [36].

As described by the Assumption 1, the nonlinear systems considered in this paper are unknown nonlinear but smooth. Thus, the directional derivative along any direction within the domain of state space exists. Then, by considering the directions along the system trajectory, the concept of directional derivative along system trajectory (DDST) denoted as $\frac{\partial f}{\partial \varphi_\xi(x_0)}$ is obtained. For each subfunction $f_i(x;p)$ $(i=1,\ldots,n)$ of the nonlinear function vector $f(x;p)$ given by (2), there is

$$\frac{\partial f_i}{\partial \varphi_\xi(x_0)} = \frac{\partial f_i}{\partial x_1}\cos\alpha_1 + \cdots + \frac{\partial f_i}{\partial x_n}\cos\alpha_n = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\cos\alpha_j, \tag{14}$$

in which $\cos\alpha_i (i=1,\ldots,n)$ is the directional cosine, and $\varphi_\xi(x_0)$ denotes the system trajectory of the dynamical system (2) starting from the point $x(t_0) = x_{\xi 0}$.

Considering the whole nonlinear function vector $f(x;p) = [f_1(x;p),\ldots,f_n(x;p)]^{\mathrm{T}}$, we have

$$\frac{\partial f}{\partial \varphi_\xi(x_0)} = \left[\frac{\partial f_1}{\partial \varphi_\xi(x_0)},\ldots,\frac{\partial f_n}{\partial \varphi_\xi(x_0)}\right]^{\mathrm{T}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \cos\alpha_1 \\ \vdots \\ \cos\alpha_n \end{bmatrix}. \tag{15}$$

Thus, the modeling of the partial derivative of dynamics $\frac{\partial f}{\partial x}$ is then obtained along the system trajectory by $\frac{\partial f}{\partial \varphi_\xi(x_0)}$.

In the following, we take the dynamical patterns φ_ξ^i $(i=1,2,3)$ generated from the Duffing oscillator [26] as examples to intuitively demonstrate the effects of the above modeling approach. A dynamical pattern is defined as a recurrent system trajectory generated from certain dynamical system, including periodic, quasi-periodic, almost periodic and even chaotic trajectories. As a typical nonlinear vibration system model, the Duffing oscillator can produce different dynamical motions with the variation of system parameters and has been widely used for practical engineering, which is given as

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -p_2 x_1 - p_3 x_1^3 - p_1 x_2 + q\cos(wt), \tag{16}$$

where $x = [x_1, x_2]^{\mathrm{T}}$ is the state vector, p_1, p_2, p_3, w and q are constant parameters, $q\cos(wt)$ is a known periodic term. Initial condition is given as $x(0) = [x_1(0), x_2(0)]^{\mathrm{T}} = [0.0, -1.8]^{\mathrm{T}}$.

In Figure 2, three different kinds of dynamical patterns are shown, corresponding to three different types of topological structures of their phase portraits. They are a periodic-1 limit cycle with $p_1 = 0.4, q = 0.620$ of pattern φ_ξ^1; a periodic-2 limit cycle with $p_1 = 0.65, q = 1.498$ of pattern φ_ξ^2; and a chaotic orbit with $p_1 = 0.35, q = 1.498$ of pattern φ_ξ^3. The modeling results of the partial derivative of dynamics $\frac{\partial f_2}{\partial \varphi_\xi}$ in the phase space for these three dynamical patterns are demonstrated in Figure 3, presenting three different kinds of dynamical motions.

Remark 2. It is noticed that the portraits of the partial derivative of dynamics $\frac{\partial f_2}{\partial \varphi_\xi}$ demonstrated in Figure 3 possess similar structure properties with their phase portrait as shown in Figure 2 for different dynamical patterns. This implies that the modeling of the partial derivative of dynamics is important as well as meaningful, since it reflects the corresponding dynamical information of certain dynamical systems from another point of view.

As mentioned before, the DL algorithm [16] is capable of achieving locally-accurate NN approximation of the unknown system dynamics along the recurrent system trajectory. That is, the unknown system dynamics f_i can be accurately identified by using the constant neural networks $\bar{W}_i^{\mathrm{T}} S_i(x)$ along the system trajectory:

$$f_i(x) = \hat{W}_i^{\mathrm{T}} S_i(x) + \epsilon_{i1} = \bar{W}_i^{\mathrm{T}} S_i(x) + \epsilon_{i2}, \tag{17}$$

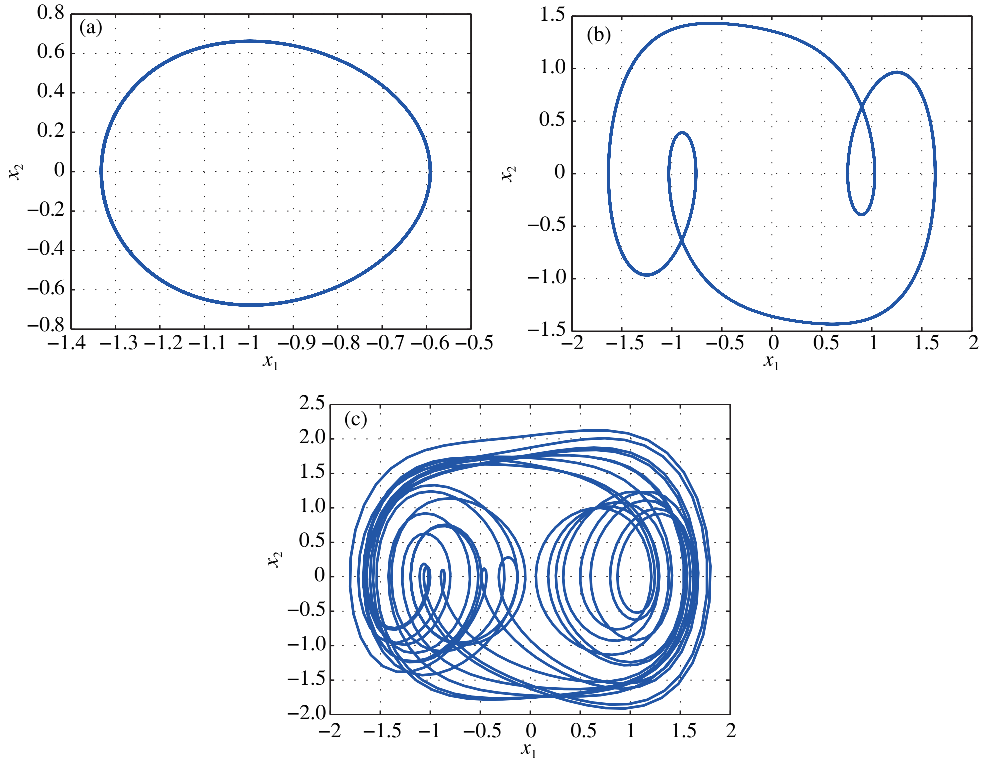

Figure 2 (Color online) Phase portrait of patterns (a) φ_ξ^1, (b) φ_ξ^2 and (c) φ_ξ^3 generated from the Duffing oscillator.

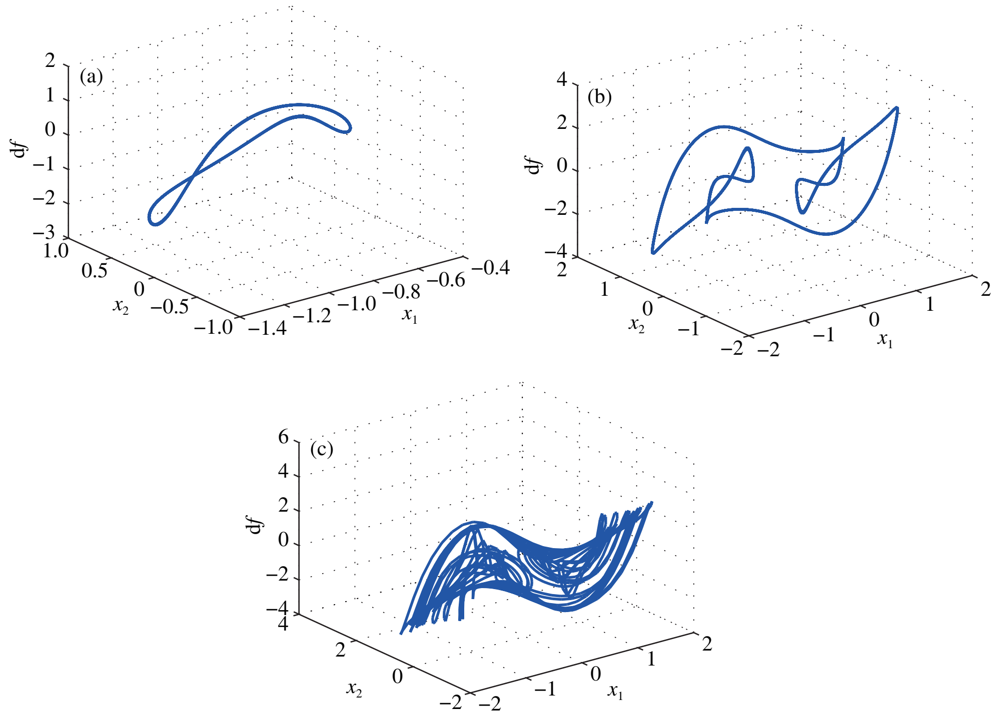

Figure 3 (Color online) Trajectory of the partial derivative of dynamics $\frac{\partial f_2}{\partial \varphi_\xi}$ for pattern (a) φ_ξ^1, (b) φ_ξ^2 and (c) φ_ξ^3.

in which $\hat{W}_i$ is the estimate of the ideal weight W_i^*. $\bar{W}_i$ is the mean value of the neural weights $\hat{W}_i$ (9). $\epsilon_{i1} = f_i(x) - \hat{W}_i^{\mathrm{T}} S_i(x)$ is the estimate error and ϵ_{i2} is the practical approximation error by using $\bar{W}_i^{\mathrm{T}} S_i(x)$.

Based on this result, we will further show that the unknown partial derivative of dynamics $\frac{\partial f}{\partial x}$ modeled by DDST $\frac{\partial f}{\partial \varphi_\xi(x_0)}$ also can be accurately identified. That is,

$$\begin{aligned}\frac{\partial f_i(x)}{\partial \varphi_\xi(x_0)} &= \sum_{j=1}^{n} \frac{\partial f_i(x)}{\partial x_j}\cos\alpha_j = \sum_{j=1}^{n} \frac{\partial(\hat{W}_i^{\mathrm{T}} S_i(x) + \epsilon_{i1})}{\partial x_j}\cos\alpha_j \\ &= \sum_{j=1}^{n} \frac{\partial(\bar{W}_i^{\mathrm{T}} S_i(x) + \epsilon_{i2})}{\partial x_j}\cos\alpha_j = \sum_{j=1}^{n} \frac{\partial \bar{W}_i^{\mathrm{T}} S_i(x)}{\partial x_j}\cos\alpha_j,\end{aligned} \tag{18}$$

then, according to (14), we have

$$\frac{\partial f_i}{\partial \varphi_\xi(x_0)} = \frac{\partial \bar{W}_i^{\mathrm{T}} S_i}{\partial \varphi_\xi(x_0)}. \tag{19}$$

Similarly, for the function vector $f(x;p)$, we get

$$\frac{\partial f}{\partial \varphi_\xi(x_0)} = \left[\frac{\partial f_1}{\partial \varphi_\xi(x_0)}, \ldots, \frac{\partial f_n}{\partial \varphi_\xi(x_0)}\right]^{\mathrm{T}} = \left[\frac{\partial \bar{W}_1^{\mathrm{T}} S_1}{\partial \varphi_\xi(x_0)}, \ldots, \frac{\partial \bar{W}_n^{\mathrm{T}} S_n}{\partial \varphi_\xi(x_0)}\right]^{\mathrm{T}}. \tag{20}$$

Remark 3. In [37, 38], the performance of deterministic learning theory has been investigated. It is shown that the learning accuracy (identification error) increases with the level of persistency of excitation. Particularly, when the level of excitation is large enough, locally-accurate learning can be achieved to the desired accuracy, whereas low level of PE may result in the deterioration of the learning performance. That is, by appropriately designing the network parameters for certain dynamical system in the learning process, high level of PE can be guaranteed, resulting in desirable identification accuracy of the unknown system dynamics f. Thus, the identification error ϵ_{i2} as given in Eq. (17) will be as small as possible. This further guarantees the accuracy of the identification of its partial derivative of dynamics $\frac{\partial f}{\partial x}$.

Based on (15) and (20), the locally accurate modeling or identification of the unknown partial derivative of dynamics $\frac{\partial f}{\partial x}$ has been successfully obtained. Since different dimensions of the state space are considered in the partial derivative of dynamics, revealing more detailed dynamical information of system dynamics, it will help us learn more about the dynamical system and its nonlinear behaviors through the modeling result, especially for those complex and high-dimensional systems.

3.2 C^1-norm modeling of system dynamics

By using the directional derivative along the system trajectory, the calculation of the partial derivative of dynamics is achieved. This makes it more valid and more feasible for quantitative measuring of structural stability, i.e., based on the modeling of the partial derivative of dynamics, the topological similarity between different dynamical systems can be calculated in the sense of the C^1-norm measure, that is

$$\|f\|_{C^1} = \sup_{x\in\Omega}\left\{\|f\|_{C^0} + \left\|\frac{\partial f}{\partial \varphi_\xi(x_0)}\right\|_{C^0}\right\}, \tag{21}$$

where $\|\cdot\|_{C^0}$ is given in Eq. (4). f and $\frac{\partial f}{\partial \varphi_\xi(x_0)}$ are nonlinear dynamics of system (2), with $\frac{\partial f}{\partial \varphi_\xi(x_0)}$ being the modeling result of the partial derivative of dynamics $\frac{\partial f}{\partial x}$ obtained in (15). Ω is a recurrent trajectory set, including not only periodic trajectories, but also quasi-periodic, almost-periodic and even some chaotic trajectories.

Remark 4. According to [39], a C^0-close measure is too strong and may destroy any singularity or the structure of orbits (phase portraits) if consider the C^0-topology of a certain dynamical system; as for the $C^r(r>1)$-topology, the C^1-closed measure remains valid since all C^1-small perturbations are also C^r-small. That is, both parts of dynamics contained in the C^1-norm modeling (21) are meaningful as well as important for revealing the topological property of a dynamical system and its system trajectory.

In view of the concept of structural stability, the equivalent relation between different dynamical systems can be qualitatively analyzed via the C^1-closed measure of system dynamics (11) in the sense of C^1-topology. Inspired by this analysis, a C^1-norm based measure can be induced from the C^1-norm ($\|\cdot\|_{C^1}$) modeling (21) for different dynamical systems (2) and (10), that is

$$\|f-f'\|_{C^1}=\sup_{x\in\Omega}\left\{\|f-f'\|_{C^0}+\left\|\frac{\partial f}{\partial\varphi_\xi(x_0)}-\frac{\partial f'}{\partial\varphi_\zeta(x_0)}\right\|_{C^0}\right\},\tag{22}$$

in which $\frac{\partial f}{\partial\varphi_\xi(x_0)}$ and $\frac{\partial f'}{\partial\varphi_\zeta(x_0)}$ are the modeling results of the partial derivative of dynamics $\frac{\partial f}{\partial x}$ and $\frac{\partial f'}{\partial x}$ for systems (2) and (10), respectively. $\varphi_\zeta(x_0)$ is the trajectory of system (10) starting from point x_0.

With the identification results obtained in (17) and (20), the nonlinear dynamics f together with its partial derivative of dynamics $\frac{\partial f}{\partial x}$ can be replaced by $\bar{W}^{\mathrm{T}}S(x)$ and $\frac{\partial\bar{W}^{\mathrm{T}}S}{\partial\varphi_\xi(x_0)}$ by using the constant neural networks $\bar{W}^{\mathrm{T}}$S, respectively. In this way, the quantification of the C^1-closed measure (22) is achieved. This can be further applied for quantitatively analyzing the equivalent relation and topological similarity between different dynamical systems and nonlinear behaviors from the perspective of structural stability.

4 Simulations

In what follows, we demonstrate the effectiveness of the identification (approximation) of the partial derivative of dynamics $\frac{\partial f}{\partial\varphi_\xi}$ for different dynamical systems. A class of dynamical patterns generated from the Duffing oscillator [26] and the Rossler system [27] with different system parameters is considered.

4.1 Duffing oscillator

Firstly, consider the Duffing oscillator [26] again:

$$\dot{x}_1=x_2,\quad \dot{x}_2=-p_2x_1-p_3x_1^3-p_1x_2+q\cos(wt),\tag{23}$$

where $x=[x_1,x_2]^{\mathrm{T}}$ is the state, p_1,p_2,p_3,w and q are constant parameters. The system dynamics $f_2(x,p)=-p_2x_1-p_3x_1^3-p_1x_2$ is an unknown, smooth nonlinear function and $q\cos(wt)$ is a known periodic term. Initial condition is given as $x(0)=[x_1(0),x_2(0)]^{\mathrm{T}}=[0.0,-1.8]^{\mathrm{T}}$.

The three different kinds of dynamical patterns $\varphi_\xi^{1,2,3}$, namely, a periodic-1 limit cycle with $p_1=0.4, q=0.620$ of pattern φ_ξ^1, a periodic-2 limit cycle with $p_1=0.65, q=1.498$ of pattern φ_ξ^2 and a chaotic orbit with $p_1=0.35, q=1.498$ of φ_ξ^3, can be seen from Figure 2.

According to the DL algorithm, the unknown system dynamics $f_2(x,p)$ can be accurately identified by using the dynamical RBF network $\dot{\hat{x}}_2=-a_2(\hat{x}_2-x_2)+\hat{W}_2^{\mathrm{T}}S(x)+q\cos(wt)$, in which the weights of the RBF networks are updated by $\dot{\hat{W}}_i=\dot{\tilde{W}}_i=-\Gamma_iS_i(x)\tilde{x}_i-\sigma_i\Gamma_i\hat{W}_i$ (please check [18] for more information). The RBF network $\hat{W}_2^{\mathrm{T}}S_2(x)$ is constructed in a regular lattice, with nodes $N=441$, the network centers μ_i evenly spaced on $[-3.0,3.0]\times[-3.0,3.0]$, and the widths $\eta_i=0.3$. The mentioned parameters are designed as $a_2=2$, $\Gamma_2=3$, and $\sigma_2=0.001$. $\hat{W}_2(0)=0.0$ is the initial weights.

Through simulation studies, we can obtain the NN approximation of dynamics f_i and its partial derivative of dynamics $\frac{\partial f_i}{\partial x}$ both in the phase space and in the time domain. For conciseness of presentation, only the approximation of the dynamics f_2 and $\frac{\partial f_2}{\partial x}$ is demonstrated in this paper. From Figures 4 and 5, we show that no matter in the phase space or in the time domain, the unknown dynamics f_2 for the three different dynamical patterns can all be well approximated along their trajectories. Similarly, the corresponding approximation effects of the partial derivative of dynamics $\frac{\partial f_2}{\partial\varphi_\xi}$ in the phase space and in the time domain are shown in Figures 6 and 7, respectively.

4.2 Rossler system

Then, consider the following Rossler system [27], a three dimensional dynamical system, to further verify the effectiveness of the modeling approach:

$$\dot{x}_1=-x_1-x_3,\quad \dot{x}_2=x_1+p_1x_2,\quad \dot{x}_3=p_2+x_3(x_1-p_3),\tag{24}$$

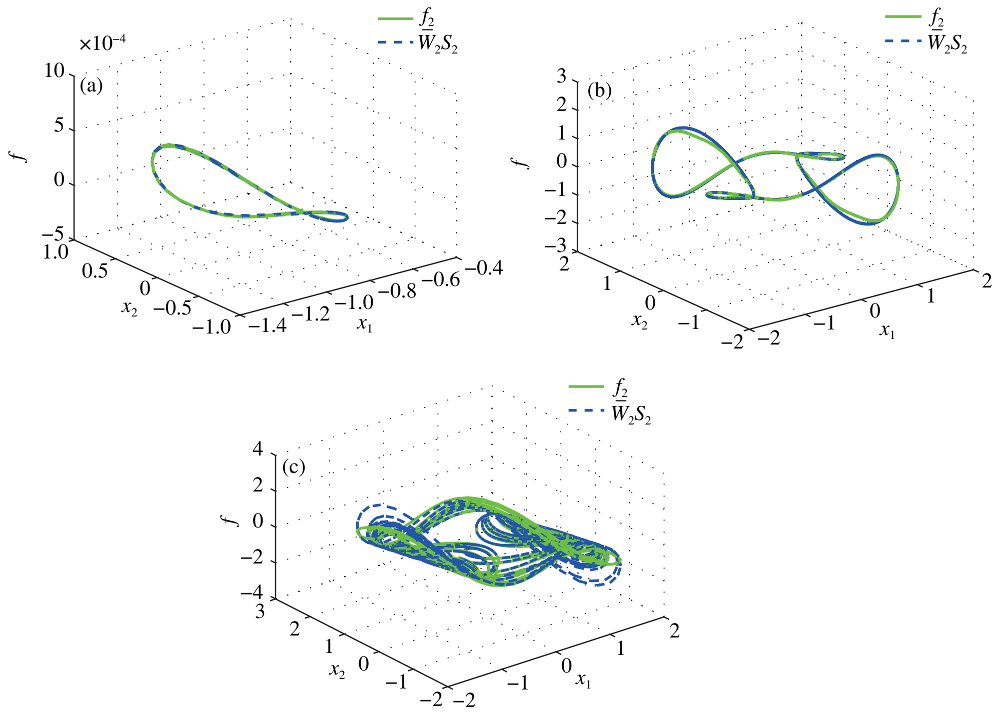

Figure 4 (Color online) Approximation of system dynamics $f_2(x;p)$ in the phase space for patterns (a) φ_ξ^1, (b) φ_ξ^2 and (c) φ_ξ^3.

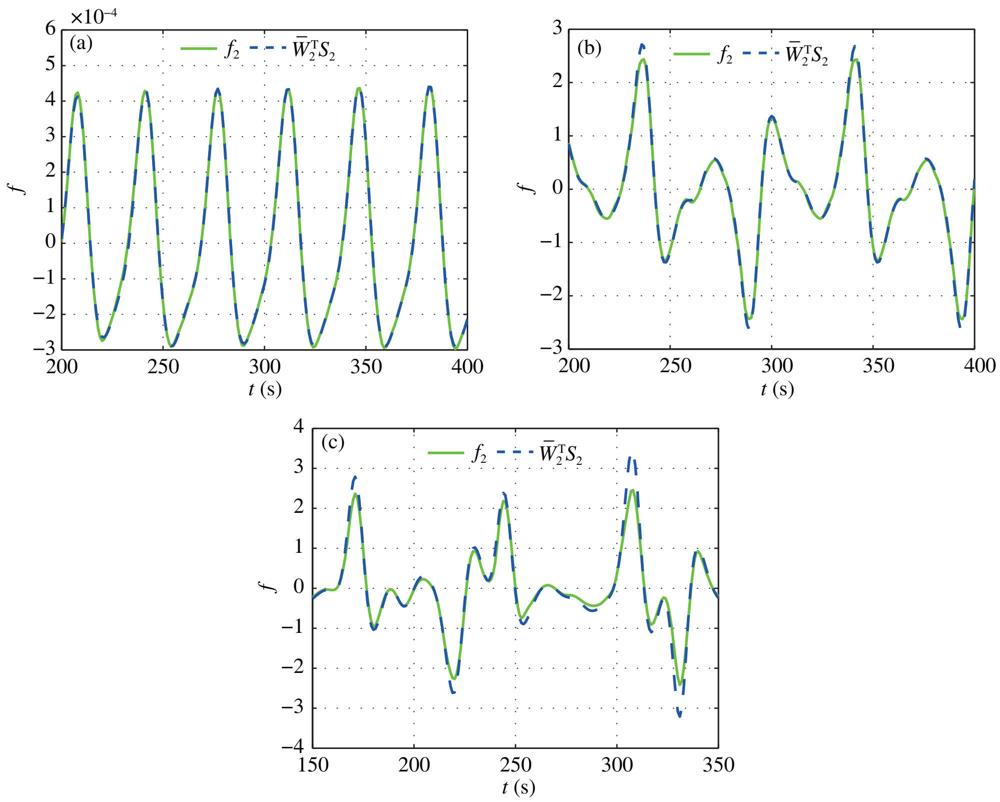

Figure 5 (Color online) Approximation of system dynamics $f_2(x;p)$ in the time domain for patterns (a) φ_ξ^1, (b) φ_ξ^2 and (c) φ_ξ^3.

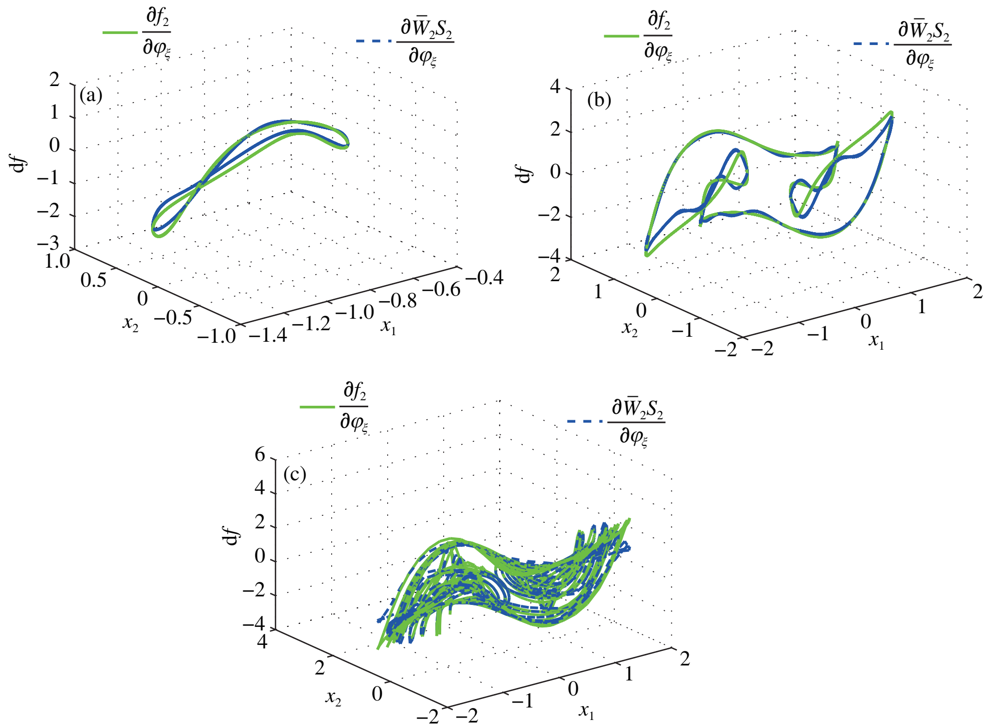

Figure 6 (Color online) Approximation of the partial derivative of dynamics $\frac{\partial f_2}{\partial \varphi_\xi}$ in the phase space for patterns (a) φ_ξ^1, (b) φ_ξ^2 and (c) φ_ξ^3.

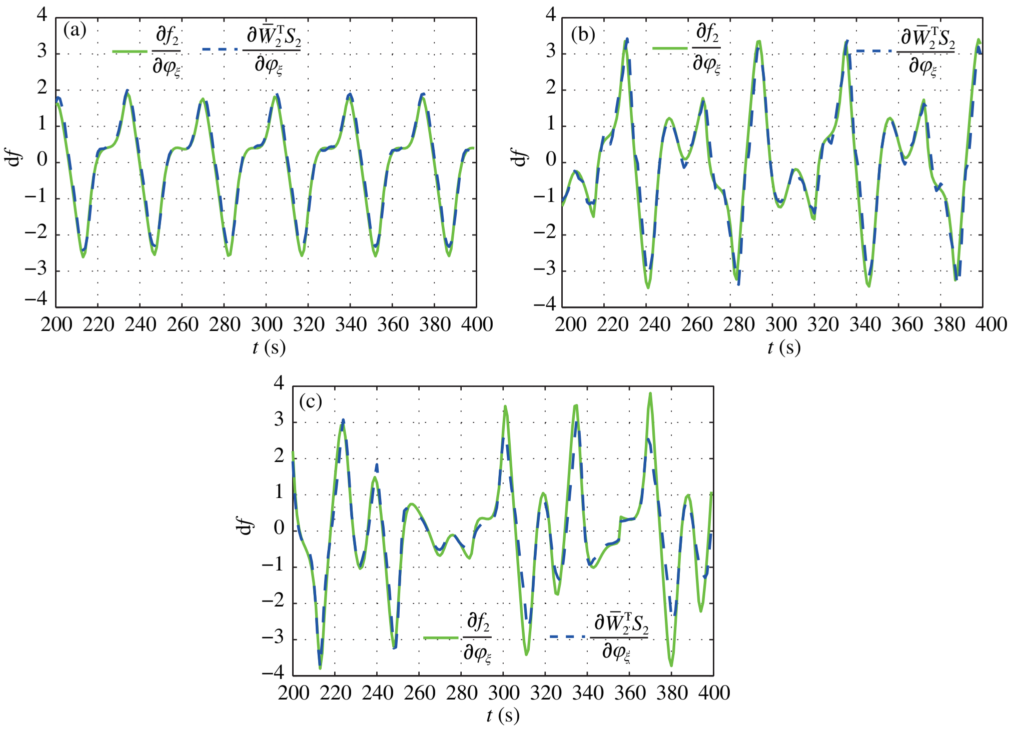

Figure 7 (Color online) Approximation of the partial derivative of dynamics $\frac{\partial f_2}{\partial \varphi_\xi}$ in the time domain for patterns (a) φ_ξ^1, (b) φ_ξ^2 and (c) φ_ξ^3.

where $x = [x_1, x_2, x_3]^{\mathrm{T}} \in R^3$ is the state vector, $p = [p_1, p_2, p_3]^{\mathrm{T}}$ is a constant vector of system parameters. Initial condition is given as $[x_1(0), x_2(0), x_3(0)]^{\mathrm{T}} = [0.5, 0.2, 0.3]^{\mathrm{T}}$. According to [27], by fixing $p_1 = p_2 =$

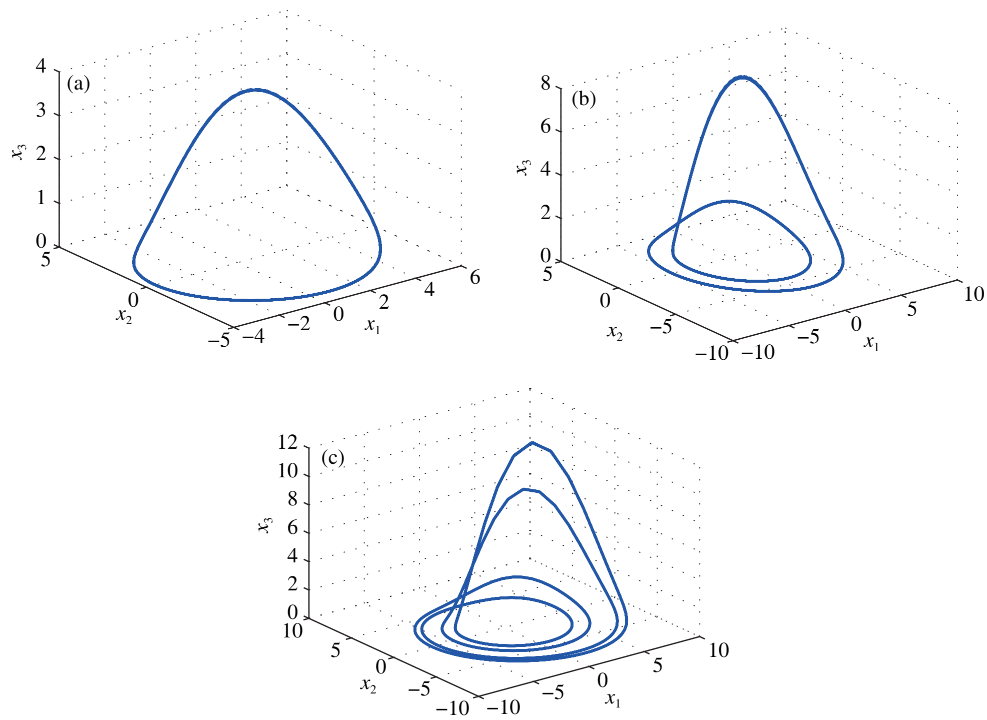

Figure 8 (Color online) Phase portrait of patterns (a) φ_ξ^4, (b) φ_ξ^5 and (c) φ_ξ^6 generated from the Rossler system.

0.2 and varying parameter p_3, the Rossler system presented by (24) can generate different kinds of dynamical behaviors from period-1 limit cycle to chaotic orbit.

As shown in Figure 8, three different dynamical patterns are obtained by setting $p_1 = p_2 = 0.2$ and varying parameter p_3. That is, the pattern φ_ξ^4 exhibits a period-1 orbit with $p_3 = 2.5$; pattern φ_ξ^5 is a period-2 orbit with $p_3 = 3.3$; and pattern φ_ξ^6 is a period-4 orbit when $p_3 = 4.1$. The phase portrait of these three dynamics possess different topological structures and dynamical motions.

In the same way, the DL algorithm is introduced to identify the unknown system dynamics $f_3(x,p) = p_2 + x_3(x_1 - p_3)$. The initial condition is $\hat{W}_3(0) = 0$. We construct RBF networks with the centers μ_3 evenly placed on $[-12, 12] \times [-12, 12]$ and the widths $\eta_3 = 1$.

In Figure 9, the locally accurate approximation of system dynamics f_3 for dynamical patterns φ_ξ^4, φ_ξ^5 and φ_ξ^6 in the phase space is shown. Meanwhile, the approximation effects in the time domain are given in Figure 10. In Figures 11 and 12, we demonstrate the approximation results of the partial derivative of dynamics $\frac{\partial f_3}{\partial \varphi_\xi}$ for these patterns in the phase space and time domain separately.

All these simulation results have demonstrated that under the modeling approach proposed in this paper, the partial derivative of dynamics $\frac{\partial f}{\partial \varphi_\xi}$ can be well modeled and identified along the system trajectory for different dynamical patterns generated from different dynamical systems, no matter they are simple (i.e. period-1 orbit and period-2 orbit) or complex (i.e. the chaotic pattern as well as patterns generated from high dimensional systems). In addition, for a certain dynamical pattern, the phase portrait as well as the trajectories of dynamics f and its partial derivative of dynamics $\frac{\partial f}{\partial x}$ in the phase space all present similar topological features. This means that both dynamics f and $\frac{\partial f}{\partial \varphi_\xi}$ imply the underlying system dynamics and can help us learn more about the system behind diverse nonlinear phenomena.

Remark 5. For intuitive demonstration, the Duffing oscillator and the Rossler system are considered for simulation, whose dynamical trajectories and the modeling results can be clearly shown in the 3D state space. As for high-dimensional case, the DL algorithm can also be implemented, e.g. for the axial flow compressors [40], in which the axial flow compressor is modeled by a 18-dimensional nonlinear system, and accurate modeling of the system dynamics of this high-dimensional compressor model can be achieved via deterministic learning. Based on this analysis, the modeling of the partial derivative of dynamics can be further extended to high-dimensional systems. Simulation studies on high-dimensional

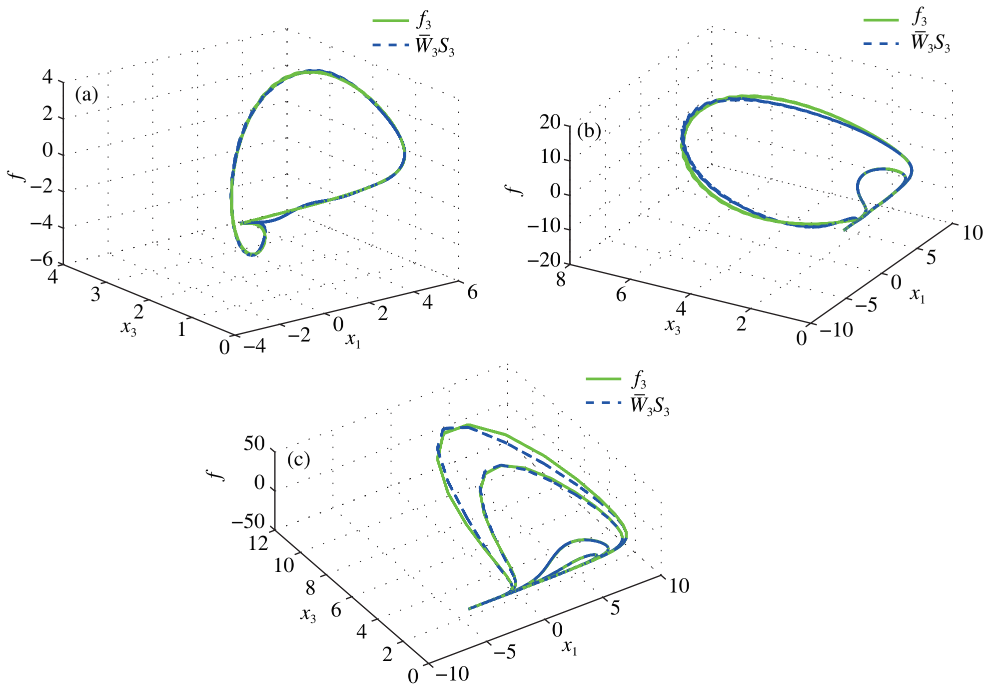

Figure 9 (Color online) Approximation of the dynamics $f_3(x,p)$ in the phase space for patterns (a) φ_ξ^4, (b) φ_ξ^5 and (c) φ_ξ^6.

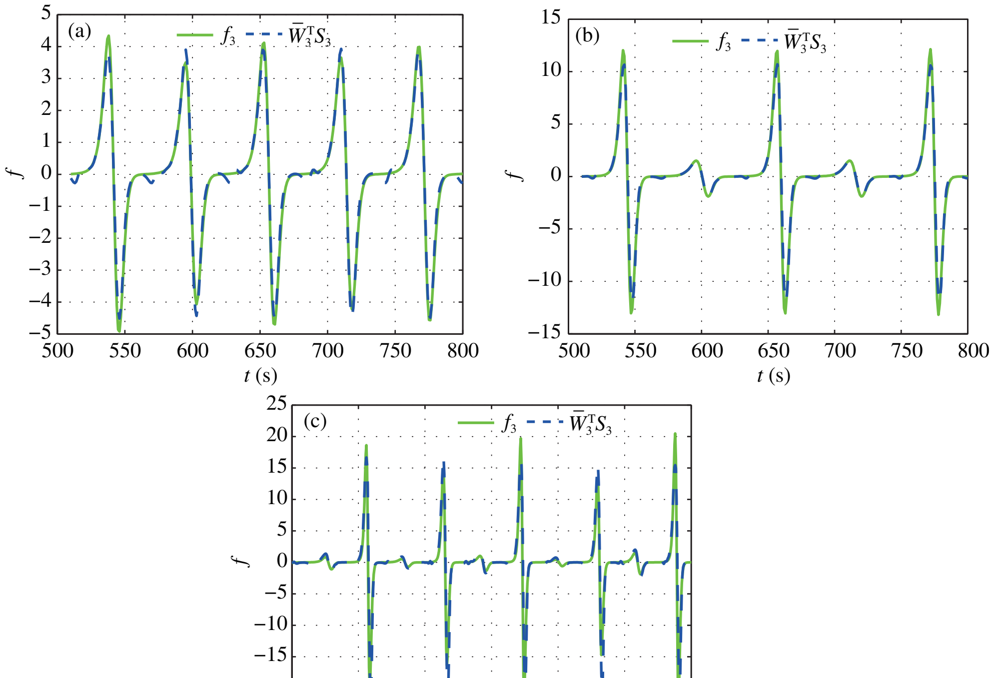

Figure 10 (Color online) Approximation of the dynamics $f_3(x,p)$ in the time domain for patterns (a) φ_ξ^4, (b) φ_ξ^5 and (c) φ_ξ^6.

systems are not included in this paper due to the limitation of space.

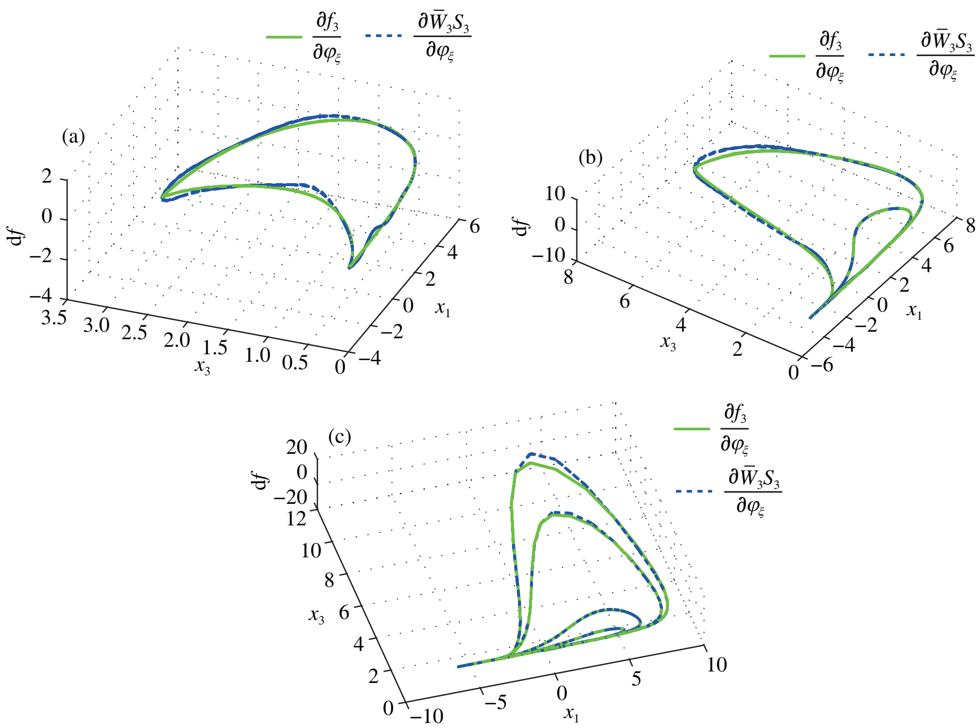

Figure 11 (Color online) Approximation of the partial derivative of dynamics $\frac{\partial f_3}{\partial \varphi_\xi}$ in the phase space for patterns (a) φ_ξ^4, (b) φ_ξ^5 and (c) φ_ξ^6.

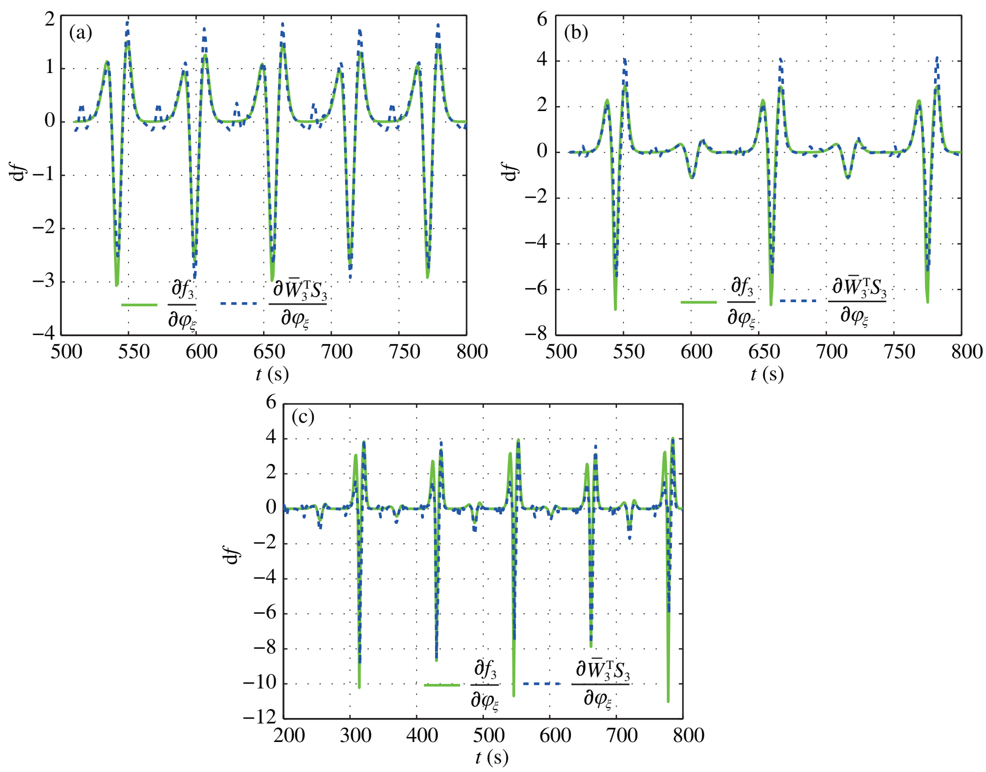

Figure 12 (Color online) Approximation of the partial derivative of dynamics $\frac{\partial f_3}{\partial \varphi_\xi}$ in the time domain for patterns (a) φ_ξ^4, (b) φ_ξ^5 and (c) φ_ξ^6.

5 Discussion and conclusion

In this paper, with the identification of system dynamics f via deterministic learning, the accurate modeling of the partial derivative of dynamics $\frac{\partial f}{\partial x}$ has been achieved along the system trajectory by using the mathematical concept of directional derivative. Based on this result, a C^1-norm modeling (22) has been proposed such that the quantitative calculation ability of structural stability is obtained, which can be further used for some practical applications. Simulations have demonstrated that the partial derivative of dynamics $\frac{\partial f}{\partial x}$ can be well modeled and identified along the system trajectory for different dynamical systems and patterns.

In most existing researches, structural stability is mainly used for analyzing the qualitative behavior of a system under small perturbations. The C^1-norm modeling approach proposed in this paper provides a quantitative tool for actually measuring the dynamical differences between nonlinear systems. By using the directional derivative along the system trajectory, the calculation of the partial derivative of dynamics is achieved. This makes it more valid and more feasible for quantitative measuring of structural stability, i.e., based on the modeling of the partial derivative of dynamics, the topological similarity between different dynamical systems can be calculated in the sense of the C^1-norm measure. This offers more incentives for further applications, such as the classification of nonlinear systems and dynamical patterns [24, 25], as well as the prediction of bifurcation and chaos [27, 41].

As for the classification of dynamical pattern, the error between the test and the template patterns of according to the C^1-norm based measure (22) can be taken as the classification criteria, namely, patterns with the smallest error are regarded as possessing similar qualitative structures as well as nonlinear dynamics, and can be classified into the same class of dynamical patterns. This mechanism is also appropriate for the classification of nonlinear dynamical systems.

In addition, the sudden change of topological structure (i.e. the equivalent relation) with the variation of system parameter, which is the so called bifurcation phenomenon, can be analyzed via the C^1-norm based measure of system dynamics (22). It means that this measure possesses the ability of detecting and predicting of the appearance of bifurcation points. More precisely, as system parameters vary continuously within a certain range (i.e. ϵ-neighborhood range or C^1-closed distance), the measure error between their dynamics will change according but smoothly and slowly; when the parameter reaches a critical value (bifurcation point), the measure result may jump suddenly from one level to another. In this way, the bifurcation is predicted according to the measure of system dynamics. Additionally, since bifurcation is one of the main routes to chaos [27], the prediction of bifurcation [41] will provide valuable information for the analysis of chaotic phenomena.

Acknowledgements This work was supported by National Science Fund for Distinguished Young Scholars (Grant No. 61225014), National Major Scientific Instruments Development Project (Grant No. 61527811), Guangdong Natural Science Foundation (Grant No. 2014A030312005), Guangdong Key Laboratory of Biomedical Engineering, and Space Intelligent Control Key Laboratory of Science and Technology for National Defense.

Conflict of interest The authors declare that they have no conflict of interest.

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