A novel weighting scheme for random k-SAT

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Abstract Consider a random k-conjunctive normal form $F_k(n, rn)$ with n variables and rn clauses. We prove that if the probability that the formula $F_k(n, rn)$ is satisfiable tends to 0 as $n \to \infty$, then $r \geqslant 2.83$, 8.09, 18.91, 40.81, and 84.87, for $k = 3$, 4, 5, 6, and 7, respectively.

1 Introduction

Let V be a set of n Boolean variables and their negations. A k-clause c on V is a disjunction of k literals selected from V. Let $C_k(V) = \{c = \ell_1 \vee \cdots \vee \ell_k : \ell_i \in V, i = 1, \ldots, k\}$ be the set of all $(2n)^k$ k-clauses on V. A random k-conjunctive normal form (k-CNF) $F_k(n, rn)$ on V is a conjunction of rn clauses selected uniformly, independently and with replacement from $C_k(V)$ [1–3].

If k is growing with n, Frieze and Wormald [2] proved that random k-SAT has a sharp threshold around $-n \ln 2 / \ln(1 - 2^{-k}) = n(2^k + O(1)) \ln 2$, provided that $k - \log_2 n \to +\infty$. A few years later, the condition was relaxed from $k - \log_2 n \to +\infty$ to $k \geqslant (1/2 + \epsilon) \log_2 n$ for any fixed $\epsilon > 0$ [3].

For each fixed $k \geqslant 2$, let

$$r_k = \sup \left\{ r : \lim_{n \to \infty} \boldsymbol{P}\big[F_k(n, rn) \text{ is satisfiable}\big] = 1 \right\},$$

$$r_k^* = \inf \left\{ r : \lim_{n \to \infty} \boldsymbol{P}\big[F_k(n, rn) \text{ is satisfiable}\big] = 0 \right\}.$$

The *Satisfiability Threshold Conjecture* asserts that $r_k = r_k^*$ for all $k \geqslant 3$.

Over the past few decades, a lot of attention has been paid to this conjecture, crossing theoretical computer science, artificial intelligence, combinatorics, and statistical physics. Most of the work has focused on proving lower bounds for r_k, or upper bounds for r_k^*, and obtaining tight bounds for r_k, or r_k^*, is a benchmark problem for wider applicability of those analytic and combinatorial techniques [4–9].

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For every truth assignment $\sigma \in \{0,1\}^n$ and every clause $c = \ell_1 \vee \cdots \vee \ell_k$, let $d = d(\sigma, c)$ be the number of satisfied literal occurrences among the k literals $\ell_1, \ldots, \ell_k$ under σ. For fixed $\lambda > 0$, let

$$\omega(\sigma, c) \propto \begin{cases} 0, & d = 0, \\ \lambda^d, & \text{otherwise.} \end{cases} \tag{1}$$

Applying the above weighting scheme (1), Achlioptas and Peres [1] proved the following significant result.

Theorem 1.1 (Achlioptas and Peres [1]). There exists a positive sequence $\delta_k \to 0$ such that $r_k \geqslant 2^k \ln 2 - (k+1)\frac{\ln 2}{2} - 1 - \delta_k, k = 3, 4, \ldots$.

Specifically, by using the weighting scheme (1), for all $k \geqslant 4$, Achlioptas and Peres [1] improved previously known lower bounds for r_k.

Later in 2014, Ding, Sly and Sun [10] proved that there exists an absolute constant k_0 such that for all $k \geqslant k_0$, the satisfiability threshold for random k-SAT exists, and gave explicit value of the threshold. But for small k, it is still very difficult to estimate r_k, or r_k^*. Our main result establishes lower bounds for r_k, $3 \leqslant k \leqslant 7$.

In this paper, we propose a novel weighting scheme, which is a revised version of (1),

$$\omega(\sigma, c) \propto \begin{cases} 0, & d = 0, \\ \lambda(1 + \beta), & d = 1, \\ \lambda^d, & \text{otherwise,} \end{cases} \tag{2}$$

where $\beta > -1$ and $\lambda > 0$ are fixed.

By choosing β and λ properly, we will prove that $r_3 \geqslant 2.83$ (the best result remains the algorithmic lower bound $r_3 \geqslant 3.52$ of Kaporis, Kirousis and Lalas [11]), $r_4 \geqslant 8.09$ (this result is consistent with the second moment method lower bound of Vorobyev [12]), $r_5 \geqslant 18.91$, $r_6 \geqslant 40.81$ and $r_7 \geqslant 84.87$, sharpening the lower bounds $r_3 \geqslant 2.68$, $r_4 \geqslant 7.91$, $r_7 \geqslant 84.82$ obtained in [1], and $r_5 \geqslant 18.79$, $r_6 \geqslant 40.74$ obtained by using the same method in [1].

2 The second moment method

For any non-negative random variable X, making use of the second moment $\boldsymbol{E}[X^2]$ is called the second moment method. In this paper, we use the second moment method in the following form.

Lemma 2.1. For any random variable $X \geqslant 0$, we have

$$\boldsymbol{P}[X > 0] \geqslant \boldsymbol{E}[X]^2 / \boldsymbol{E}[X^2].$$

Note that for any non-negative random variable Y, if $Y > 0$ implies that $X > 0$, then

$$\boldsymbol{P}[X > 0] \geqslant \boldsymbol{P}[Y > 0] \geqslant \boldsymbol{E}[Y]^2 / \boldsymbol{E}[Y^2].$$

Friedgut [8] established the existence of a non-uniform threshold for random k-SAT.

Theorem 2.2 (Friedgut [8]). For any fixed $k \geqslant 2$, there exists a sequence $r_k(n)$ such that for any fixed $\epsilon > 0$,

$$\lim_{n \to \infty} \boldsymbol{P}\left[F_k(n, rn) \text{ is satisfiable}\right] = \begin{cases} 1, & r = (1 - \epsilon)r_k(n), \\ 0, & r = (1 + \epsilon)r_k(n). \end{cases}$$

Given a random k-CNF formula F on V, let $S(F) = \{\sigma : \sigma \text{ satisfies } F\} \subseteq \{0,1\}^n$ be the set of all satisfying truth assignments of F and let $X = X(F) \geqslant 0$ be a random variable such that $X > 0$ implies that $S(F) \neq \emptyset$. The following class of random variables $X = \sum_\sigma \omega(\sigma, F)$ clearly has this property if $\omega(\sigma, F) \geqslant 0$ and $\omega(\sigma, F) > 0$ implies that $\sigma \in S(F)$.

An immediate corollary of Theorem 2.2 is as follows.

Corollary 2.3. For any fixed $k \geqslant 2$, if $\liminf_{n \to \infty} \boldsymbol{P}\big[F_k(n, rn) \text{ is satisfiable}\big] > 0$, then $r_k \geqslant r$.

Thus, for any $r > 0$, if $\boldsymbol{E}[X^2] = O(\boldsymbol{E}[X]^2)$, then $r_k \geqslant r$.

Since F is formed by some independent clauses, it is natural to require that $\omega(\sigma, F)$ has product structure over these clauses, i.e., $\omega(\sigma, F) = \prod_c \omega(\sigma, c)$, and because these clauses are chosen independently, we have $\boldsymbol{E}[\omega(\sigma, F)] = \prod_c \boldsymbol{E}[\omega(\sigma, c)]$. With this in mind, let us consider the following class of random variables:

$$X = \sum_\sigma \prod_c \omega(\sigma, c), \tag{3}$$

where $\omega(\sigma, c) \geqslant 0$ and $\omega(\sigma, c) = 0$ if σ falsifies c.

For every truth assignment σ and every clause $c = \ell_1 \vee \cdots \vee \ell_k$, we require that $\omega(\sigma, c) = \omega(\boldsymbol{v})$, where $\boldsymbol{v} = (v_1, \ldots, v_k)$, $v_i = +1$ if ℓ_i is satisfied under σ and -1 otherwise. Since every ℓ_i in c has equal chance, it is natural to require that $\omega(\boldsymbol{v}) = \omega(|\boldsymbol{v}|)$, where $|\boldsymbol{v}|$ denotes the number of $+1$s in $\boldsymbol{v}$.

Let $A = \{-1, +1\}^k$ and let $\alpha = z/n$. From [1], we have

$$\boldsymbol{E}[X] = 2^n \left(\sum_{\boldsymbol{v} \in A} \omega(\boldsymbol{v}) 2^{-k} \right)^{rn}, \quad \boldsymbol{E}[X^2] = 2^n \sum_{z=0}^{n} \binom{n}{z} f_\omega(\alpha)^{rn},$$

where

$$f_\omega(\alpha) = \sum_{\boldsymbol{u}, \boldsymbol{v} \in A} \omega(\boldsymbol{u}) \omega(\boldsymbol{v}) 2^{-k} \prod_{i=1}^{k} \left(\alpha^{\mathbf{1}_{u_i = v_i}} (1 - \alpha)^{\mathbf{1}_{u_i \neq v_i}} \right).$$

By using the Laplace method of asymptotic analysis [13], we deduce the following lemma.

Lemma 2.4. Let ϕ be any positive function on $[0, 1]$ and let $S_n = \sum_{z=0}^{n} \binom{n}{z} \phi(\alpha)^n$. Letting $0^0 = 1$, define g on $[0, 1]$ as $g(\alpha) = \phi(\alpha)/(\alpha^\alpha (1-\alpha)^{1-\alpha})$. If there exists $\alpha_{\max} \in (0, 1)$ such that $g(\alpha_{\max}) \equiv g_{\max} > g(\alpha)$ for all $\alpha \neq \alpha_{\max}$ and g, i.e., ϕ is twice differentiable at $\alpha_{\max}$ and $g''(\alpha_{\max}) = -g_{\max} \rho^{-2}/(\alpha_{\max}(1 - \alpha_{\max}))$, where $\rho > 0$, then $\lim_{n \to \infty} S_n/g_{\max}^n = \rho$.

With Lemma 2.4 in mind, let us define $\Lambda_\omega(\alpha) = 2 f_\omega(\alpha)^r/(\alpha^\alpha (1 - \alpha)^{1-\alpha})$. Observe that

$$\Lambda_\omega(1/2)^n = (4 f_\omega(1/2)^r)^n = \boldsymbol{E}[X]^2.$$

Then, for any given $r > 0$, by Lemma 2.4, if Λ_ω has a unique global maximum at $1/2$ on $[0, 1]$ and $\Lambda_\omega''(1/2) < 0$, then $\boldsymbol{E}[X^2] = O\left(\boldsymbol{E}[X]^2\right)$ and we get $r_k \geqslant r$. $\Lambda_\omega'(1/2) = 0$, i.e., $f_\omega'(1/2) = 0$, is equivalent to [1]

$$\sum_{\boldsymbol{v} \in A} \omega(\boldsymbol{v})(2|\boldsymbol{v}| - k) = 0. \tag{4}$$

The specific calculations of weighting scheme (2). For our weighting scheme, as defined in (2), we can rewrite (4) as $\left[\sum_{j=1}^{k} \binom{k}{j} \lambda^j (2j - k) \right] + k(2 - k)\lambda\beta = 0$, i.e.,

$$(1 + \lambda)^{k-1}(1 - \lambda) + (k - 2)\lambda\beta = 1. \tag{5}$$

For every truth assignment σ and every clause $c = \ell_1 \vee \cdots \vee \ell_k$, let $S_1(c) = \{\sigma : d(\sigma, c) = 1\}$ and let $H(\sigma, c) = d(\sigma, c) - (k - d(\sigma, c)) = 2d(\sigma, c) - k$. For any fixed $\gamma > 0$, let

$$X = \sum_\sigma \prod_c \gamma^{H(\sigma, c)} \left(\mathbf{1}_{\sigma \in S(c)} + \beta \times \mathbf{1}_{\sigma \in S_1(c)} \right). \tag{6}$$

(Note that $\gamma^{H(\sigma, c)} = \gamma^{2d(\sigma, c) - k}$, so this is consistent with (2) for $\gamma^2 = \lambda$.)

Thus, we can rewrite (5) as

$$(1 + \gamma^2)^{k-1}(1 - \gamma^2) + (k - 2)\gamma^2\beta = 1. \tag{7}$$

Let σ, τ be any pair of truth assignments that coincide in $z = \alpha n$ elements. Then

$$E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\right] = \left(\alpha\left(\frac{\gamma^2+\gamma^{-2}}{2}\right)+1-\alpha\right)^k,$$

$$E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\mathbf{1}_{\sigma\notin S(c)}\right] = E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\mathbf{1}_{\tau\notin S(c)}\right] = \left(\frac{\alpha\gamma^{-2}+1-\alpha}{2}\right)^k,$$

$$E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\mathbf{1}_{\sigma,\tau\notin S(c)}\right] = \left(\frac{\alpha\gamma^{-2}}{2}\right)^k,$$

$$E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\mathbf{1}_{\sigma\in S_1(c)}\right] = E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\mathbf{1}_{\tau\in S_1(c)}\right] = \binom{k}{1}\left(\frac{\alpha\gamma^2+1-\alpha}{2}\right)\left(\frac{\alpha\gamma^{-2}+1-\alpha}{2}\right)^{k-1},$$

$$E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\mathbf{1}_{\sigma\notin S(c),\tau\in S_1(c)}\right] = E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\mathbf{1}_{\sigma\in S_1(c),\tau\notin S(c)}\right] = \binom{k}{1}\left(\frac{1-\alpha}{2}\right)\left(\frac{\alpha\gamma^{-2}}{2}\right)^{k-1},$$

$$E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\mathbf{1}_{\sigma,\tau\in S_1(c)}\right] = \binom{k}{1}\left(\frac{\alpha\gamma^2}{2}\right)\left(\frac{\alpha\gamma^{-2}}{2}\right)^{k-1} + 2!\binom{k}{2}\left(\frac{1-\alpha}{2}\right)^2\left(\frac{\alpha\gamma^{-2}}{2}\right)^{k-2}.$$

Observe that

$$\left(\mathbf{1}_{\sigma\in S(c)} + \beta\times\mathbf{1}_{\sigma\in S_1(c)}\right)\left(\mathbf{1}_{\tau\in S(c)} + \beta\times\mathbf{1}_{\tau\in S_1(c)}\right)$$
$$= \left(\mathbf{1} - \mathbf{1}_{\sigma\notin S(c)} + \beta\times\mathbf{1}_{\sigma\in S_1(c)}\right)\left(\mathbf{1} - \mathbf{1}_{\tau\notin S(c)} + \beta\times\mathbf{1}_{\tau\in S_1(c)}\right)$$
$$= \mathbf{1} - \mathbf{1}_{\sigma\notin S(c)} - \mathbf{1}_{\tau\notin S(c)} + \mathbf{1}_{\sigma,\tau\notin S(c)} + \beta\times\left(\mathbf{1}_{\sigma\in S_1(c)} + \mathbf{1}_{\tau\in S_1(c)}\right.$$
$$\left. - \mathbf{1}_{\sigma\in S_1(c),\tau\notin S(c)} - \mathbf{1}_{\sigma\notin S(c),\tau\in S_1(c)}\right) + \beta^2\times\mathbf{1}_{\sigma,\tau\in S_1(c)}$$
$$\equiv \Gamma(\sigma,\tau,\beta,c).$$

Therefore,

$$E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\Gamma(\sigma,\tau,\beta,c)\right] = A(\alpha,\gamma) + 2k\times B(\alpha,\gamma)\times\beta + k\times C(\alpha,\gamma)\times\beta^2 \equiv F_{\beta,\gamma}(\alpha),$$

where

$$A(\alpha,\gamma) = \left(\alpha\left(\frac{\gamma^2+\gamma^{-2}}{2}\right)+1-\alpha\right)^k - 2\left(\frac{\alpha\gamma^{-2}+1-\alpha}{2}\right)^k + \left(\frac{\alpha\gamma^{-2}}{2}\right)^k,$$

$$B(\alpha,\gamma) = \left(\frac{\alpha\gamma^2+1-\alpha}{2}\right)\left(\frac{\alpha\gamma^{-2}+1-\alpha}{2}\right)^{k-1} - \left(\frac{1-\alpha}{2}\right)\left(\frac{\alpha\gamma^{-2}}{2}\right)^{k-1},$$

$$C(\alpha,\gamma) = \left(\frac{\alpha\gamma^2}{2}\right)\left(\frac{\alpha\gamma^{-2}}{2}\right)^{k-1} + (k-1)\left(\frac{1-\alpha}{2}\right)^2\left(\frac{\alpha\gamma^{-2}}{2}\right)^{k-2}.$$

Then

$$E[X^2] = E\left[\sum_{\sigma,\tau}\prod_c\gamma^{H(\sigma,c)+H(\tau,c)}\Gamma(\sigma,\tau,\beta,c)\right]$$
$$= \sum_{\sigma,\tau}\prod_c E\left[\gamma^{H(\sigma,c)+H(\tau,c)}\Gamma(\sigma,\tau,\beta,c)\right]$$
$$= 2^n\sum_{z=0}^n\binom{n}{z}F_{\beta,\gamma}(\alpha)^{rn}. \tag{8}$$

With Lemma 2.4 in mind, for fixed r, β and γ, let us define $G_{r,\beta,\gamma}(\alpha) = 2F_{\beta,\gamma}(\alpha)^r/(\alpha^\alpha(1-\alpha)^{1-\alpha})$. If $G_{r,\beta,\gamma}$ has a unique global maximum at $\alpha = 1/2$ on $[0,1]$ and $G_{r,\beta,\gamma}''(1/2) < 0$, then $r_k \geqslant r$.

An enhanced method: truncation and weighting. For any given random k-CNF formula F on V, note that $S = S(F) \subseteq \{0,1\}^n$ is the set of all satisfying truth assignments of F. Let $S^+ = \{\sigma \in S : H(\sigma,F) \geqslant 0\}$ and let $X_+ = \sum_{\sigma\in S^+}\prod_c\omega(\sigma,c)$. For any weighting scheme (3), from [12], we have the following Lemma.

Table 1 Lower bounds for k=3, 4, 5, 6, and 7

k	3	4	5	6	7
β	$(0.56, 0.74)$	$(0.13, 0.15)$	$(0.04, 0.06)$	0.02	0.01
r	2.83	8.09	18.91	40.81	84.87

Lemma 2.5. If $f'_\omega(1/2) = 0$, then $\boldsymbol{E}[X_+]/\boldsymbol{E}[X] \to 1/2$ as $n \to \infty$.

For any fixed $\gamma > 0$, let $X_+ = \sum_{\sigma \in S^+} \prod_c \gamma^{H(\sigma,c)} \left(\mathbf{1}_{\sigma \in S(c)} + \beta \times \mathbf{1}_{\sigma \in S_1(c)} \right)$. (This is consistent with those random variables X defined in (6), i.e., consistent with our weighting scheme (2).)

A simple calculation gives

$$
\begin{aligned}
X_+^2 &= \left(\sum_{\sigma \in S^+} \prod_c \omega(\sigma, c) \right) \left(\sum_{\tau \in S^+} \prod_c \omega(\tau, c) \right) \\
&= \sum_{\sigma, \tau} \mathbf{1}_{\sigma, \tau \in S^+} \prod_c \omega(\sigma, c) \omega(\tau, c) \\
&= \sum_{\sigma, \tau} \mathbf{1}_{\sigma, \tau \in S^+} \prod_c \gamma^{H(\sigma,c)+H(\tau,c)} \Gamma(\sigma, \tau, \beta, c).
\end{aligned}
\tag{9}
$$

Given a tuple $(\beta_0, \gamma_0) \in (-1, +\infty) \times (0, +\infty)$ which satisfies (7), in particular, if $X_+ = \sum_{\sigma \in S^+} \prod_c \gamma_0^{H(\sigma,c)} \left(\mathbf{1}_{\sigma \in S(c)} + \beta_0 \times \mathbf{1}_{\sigma \in S_1(c)} \right)$, then for any $\gamma \geqslant \gamma_0$, following the derivation of (8) and (9), we deduce that

$$
\begin{aligned}
\boldsymbol{E}[X_+^2] &= \sum_{\sigma, \tau} \boldsymbol{E} \left[\mathbf{1}_{\sigma, \tau \in S^+} \prod_c \gamma_0^{H(\sigma,c)+H(\tau,c)} \Gamma(\sigma, \tau, \beta_0, c) \right] \\
&\leqslant \sum_{\sigma, \tau} \boldsymbol{E} \left[\prod_c \gamma^{H(\sigma,c)+H(\tau,c)} \Gamma(\sigma, \tau, \beta_0, c) \right] \\
&= \sum_{\sigma, \tau} \prod_c \boldsymbol{E} \left[\gamma^{H(\sigma,c)+H(\tau,c)} \Gamma(\sigma, \tau, \beta_0, c) \right] \\
&= 2^n \sum_{z=0}^n \binom{n}{z} F_{\beta_0, \gamma}(\alpha)^{rn}.
\end{aligned}
$$

Therefore $\boldsymbol{E}[X_+^2] \leqslant 2^n \sum_{z=0}^n \binom{n}{z} \inf_{\gamma \geqslant \gamma_0} F_{\beta_0, \gamma}(\alpha)^{rn} \equiv 2^n \sum_{z=0}^n \binom{n}{z} f_{\beta_0, \gamma_0}(\alpha)^{rn}$.

With Lemma 2.4 in mind we take

$$
g_{r, \beta_0, \gamma_0}(\alpha) = \frac{2 f_{\beta_0, \gamma_0}(\alpha)^r}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \leqslant \frac{2 F_{\beta_0, \gamma_0}(\alpha)^r}{\alpha^\alpha (1-\alpha)^{1-\alpha}} = G_{r, \beta_0, \gamma_0}(\alpha).
$$

Suppose that $g_{r, \beta_0, \gamma_0}(1/2) > g_{r, \beta_0, \gamma_0}(\alpha)$ holds for all $\alpha \neq 1/2$ and $G''_{r, \beta_0, \gamma_0}(1/2) < 0$. Note that there exists a constant ϵ such that $G_{r, \beta_0, \gamma_0}(1/2) > G_{r, \beta_0, \gamma_0}(\alpha)$ holds for all $\alpha \in (1/2 - \epsilon, 1/2) \cup (1/2, 1/2 + \epsilon)$. With Lemma 2.4 in mind, we consider the function defined as follows:

$$
\phi_{\beta_0, \gamma_0}(\alpha) = \begin{cases} F_{\beta_0, \gamma_0}(\alpha), & \text{if } \alpha \in (1/2 - \epsilon, 1/2 + \epsilon), \\ f_{\beta_0, \gamma_0}(\alpha), & \text{otherwise.} \end{cases}
$$

It is clear that $\boldsymbol{E}[X_+^2] \leqslant \sum_{z=0}^n \binom{n}{z} \phi_{\beta_0, \gamma_0}(\alpha)^{rn}$, and then by Lemmas 2.4 and 2.5, we get $r_k \geqslant r$.

3 Use of the method

We apply the method to the cases $k = 3$, 4, 5, 6 and 7, respectively. To do this, we demonstrate values β, r and let γ satisfies (7), such that $g_{r, \beta, \gamma}(1/2) > g_{r, \beta, \gamma}(\alpha)$ holds for all $\alpha \neq 1/2$ and $G''_{r, \beta, \gamma}(1/2) < 0$. We obtain Table 1.

4 Conclusion

In this paper, we exploit the power of the weighted second moment method in improving lower bounds for r_k, $3 \leqslant k \leqslant 7$, and some useful techniques on how to choose weights of the second moment are introduced. These techniques can also be of use for other random models when applying the second moment method.

For further work, one interested problem is whether (1) is the optimal weight asymptotically for random k-SAT when $k \to \infty$.

Conflict of interest The authors declare that they have no conflict of interest.

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