SCIENCE CHINA
Information Sciences

• RESEARCH PAPER •

August 2016, Vol. 59 082303:1–082303:11
doi: 10.1007/s11432-015-5473-9

# On the criteria for designing complex orthogonal space-time block codes

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Received June 18, 2015; accepted August 30, 2015; published online April 6, 2016

**Abstract** A complex orthogonal design (COD) used in space-time block codes is a kind of combinatorial design. It has been well studied because it has a fast maximum-likelihood decoding algorithm and achieves full diversity. When designing CODs, there are seven characteristics that should be considered, which include code rate, decoding delay, transceiver signal linearization, peak-to-average power ratio, etc. In this paper, we study the relationships among these design criteria. We prove that the maximum rate of generalized CODs (GCODs) that meet the last five criteria is 1/2. Moreover, we present a new method to construct GCODs based on CODs. Using this method on balanced complex orthogonal designs (BCODs), we obtain a novel class of GCODs that performs almost perfectly with respect to all the seven properties.

**Keywords** space-time block code, complex orthogonal design, decoding delay, transceiver signal linearization, peak-to-average power ratio (PAPR)

**Citation** Kan H B, Liu X D, Han G Y. On the criteria for designing complex orthogonal space-time block codes. Sci China Inf Sci, 2016, 59(8): 082303, doi: 10.1007/s11432-015-5473-9

## 1 Introduction

In recent years, the investigation of MIMO systems for wireless communications has developed in a variety of directions. It is widely acknowledged that, in real-world wireless systems, the complexity of the decoding algorithm is critical for the employment of multiple-antenna communication techniques. Since the pioneering work by Alamouti [1] and Tarokh et al. [2], the complex orthogonal design (COD) has become a powerful tool for designing space-time block codes (STBCs). The COD plays an important role in wireless communications because it has a fast decoupled maximum-likelihood decoding algorithm and achieves full diversity.

**Definition 1.** A COD $\boldsymbol{G}[p,n,k]$ is a $p\times n$ matrix with elements in the form of 0, $\pm z_i$ or $\pm z_i^*$ such that

$$\boldsymbol{G}^{\mathrm{H}}\cdot\boldsymbol{G}=\left(\sum_{i=1}^{k}|z_i|^2\right)\boldsymbol{I}_n,$$

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where $z_1, \ldots, z_n$ are complex variables, $\boldsymbol{G}^{\mathrm{H}}$ denotes the Hermitian transpose of $\boldsymbol{G}$, and $\boldsymbol{I}_n$ represents the $n \times n$ identity matrix. It is said to be combinatorial in the sense that there are no linear combinations of complex variables in the matrix.

Linear combinations can be allowed in applications in the real world without increasing the complexity. In this sense, we can generalize the orthonormality property to orthogonality.

**Definition 2.** A generalized COD (GCOD) $\boldsymbol{G}[p,n,k]$ is a $p \times n$ matrix on $k$ complex variables $z_1, z_2, \ldots, z_k$ such that

$$\boldsymbol{G}^{\mathrm{H}} \cdot \boldsymbol{G} = \sum_{i=1}^{k} |z_i|^2 \boldsymbol{D}_i,$$

where $\boldsymbol{D}_i$, $i = 1, 2, \ldots, k$, is an $n \times n$ diagonal constant matrix with all diagonal elements positive.

Adams et al. [3] summarize several characteristics that should be considered when designing CODs as follows.

C1. The code rate [2] (defined as $k/p$).

C2. The decoding delay [1] (defined as $p$).

C3. The property of transceiver signal linearization (conjugation separation) [4]. This means that all elements in the same row have the same conjugation.

C4. The peak-to-average power ratio (PAPR) of the code [5]. A code containing no zero entries reaches the highest PAPR.

C5. The property of power-balance [5]. If every variable has the same number of instances in each column, then the code is said to be power-balanced.

C6. There are no irrational coefficients [6].

C7. There are no linear processes [7].

However, we cannot optimize all those properties at the same time except with the Alamouti code [1]. Liang [8] determined the tight upper bound for the code rate, which converges to 1/2 while $n$ tends to infinity, and presented a class of maximum-rate CODs that reaches a minimum decoding delay when $n \equiv 1, 2, 3 \mod 4$. Lu et al. [9] constructed the first class of maximum-rate CODs with the minimum decoding delay for $n \equiv 0 \mod 4$. When the rate reaches the upper bound, Adams et al. [10,11] found the greatest lower bound on the decoding delay, which grows in factorial order as $n$ increases. Furthermore, Li and Kan [12] give a global explicit-form construction for all the "first type" CODs, which include all the maximum-rate CODs.

As the number of antennas increases, the decoding delay of a maximum-rate COD grows very quickly. It may be worth decreasing the rate a little bit to decrease the decoding delay significantly. Tarokh et al. [2] use a rate 1 $[\nu(n), n, \nu(n)]$ real orthogonal design (ROD) for constructing a rate 1/2 $[2\nu(n), n, \nu(n)]$ COD with decoding delay $2\nu(n)$, where $\nu(n) = 2^{\delta(n)}$, and $\delta(n)$ is given by the following formula:

$$\delta(n) = \begin{cases} 4l, & \text{if } n = 8l+1; \\ 4l+1, & \text{if } n = 8l+2; \\ 4l+2, & \text{if } n = 8l+3 \text{ or } 8l+4; \\ 4l+3, & \text{if } n = 8l+5, 8l+6, 8l+7 \text{ or } 8l+8. \end{cases}$$

Liang [8] observes that any rate 1 $[\nu(n), n, \nu(n)]$ ROD is itself a rate 1/2 $[\nu(n), n, \nu(n)/2]$ COD. Das and Rajan [13] constructed a class of rate 1/2 $[\nu(n), n, \nu(n)/2]$ scaled CODs that satisfies design criteria C4–C7. Adams et al. [3] constructed a class of balanced complex orthogonal designs (BCODs), which has rate 1/2 and decoding delay $2^{\lceil \frac{n}{2} \rceil}$. They presented a new class of rate 1/2 GCODs based on BCODs that satisfies design criteria C4–C7 [3]. In [14], Liu et al. define the standard form for BCODs, and prove the conjecture that the minimum coding delay for BCODs with $2m$ columns is $2^m$.

In this paper, we study the tradeoff among the design criteria (Section 3). We find the tight upper bound of the code rate for a special class of GCODs, which is the only result in the literature on the maximum rate after Wang and Xia [15]. We also present a new method to construct GCODs based on CODs (Section 4). Using the method with BCODs, we obtain a novel class of almost perfect GCODs

(APGCODs), which performs remarkably well with respect to all the properties C1–C7 (Section 5). Compared with existing constructions, it has rate 1/2, which is optimal asymptotically, and decoding delay $2^{\lceil \frac{n}{2} \rceil}$, which is the lowest in most cases, and it satisfies all the criteria C3–C7. With this novel construction, we reduce the upper bound for the minimum decoding delay of APGCODs by half in most cases.

## 2 Definitions and notations

**Definition 3.** [12] There are several operations that can be performed on any COD:

(1) Transform the order of the rows/columns ("row/column transformation");

(2) Multiply a certain row/column by $-1$ ("row/column negation");

(3) Multiply all the instances of a certain variable by $-1$ ("variable negation");

(4) Conjugate all the instances of a certain variable ("variable conjugation");

(5) Change the index of a certain variable ("variable renaming").

They are called equivalence operations since they do not cancel the orthogonality of CODs. If two CODs are identical under equivalence operations, then they are said to be isomorphic.

For any row or column vector $\boldsymbol{r}$ of a COD $\boldsymbol{G}$, the zero pattern of $\boldsymbol{r}$ is defined as a binary vector $\boldsymbol{s_r}$ such that $\boldsymbol{s_r}(i) = 0$ iff $\boldsymbol{r}(i) = 0$, and the weight of $\boldsymbol{r}$ is defined as the number of nonzero entries of $\boldsymbol{s_r}$, i.e., $w(\boldsymbol{r}) = \sum \boldsymbol{s_r}(i)$. To simplify the notation, we denote the instances of the complex variable $z_i$ by $z[i]$ up to negation and conjugation, i.e., $z[i] \in \{\pm z_i, \pm z_i^*\}$. Note that different instances may share the same notation $z[i]$.

Denote

$$\boldsymbol{A} = \begin{pmatrix} z_i & z_j \\ -z_j^* & z_i^* \end{pmatrix}, \quad \boldsymbol{T} = \begin{pmatrix} z_i & 0 \\ 0 & z_i \end{pmatrix}, \quad \text{and } \boldsymbol{D} = \begin{pmatrix} z_i & 0 \\ 0 & z_i^* \end{pmatrix}.$$

A $2 \times 2$ submatrix that is isomorphic to $\boldsymbol{A}$ is called an Alamouti $2 \times 2$ form, isomorphic to $\boldsymbol{T}$ a trivial $2 \times 2$ form, and isomorphic to $\boldsymbol{D}$ a diagonal $2 \times 2$ form.

**Definition 4.** Following Liang's definition in [8], the $(u, v) - \boldsymbol{B}_j$ form of a COD $\boldsymbol{G}[p, n, k]$ is defined as

$$\boldsymbol{B}_j = \begin{pmatrix} z_j \boldsymbol{I}_u & \boldsymbol{M}_j \\ -\boldsymbol{M}_j^{\mathrm{H}} & z_j^* \boldsymbol{I}_v \end{pmatrix},$$

where $u + v = n$ and $\boldsymbol{M}_j$ is a $u \times v$ matrix. It is called the $\boldsymbol{B}_j$ form for short.

For any COD $\boldsymbol{G}$ with $n$ columns, and for any fixed $z_j$, $1 \leqslant j \leqslant k$, we can transform $\boldsymbol{G}$ through equivalence operations to make the $\boldsymbol{B}_j$ form appear within the top $n$ rows [8]. If $\boldsymbol{B}_j$ can be created without column transformations, then $\boldsymbol{G}$ is said to be in the $\boldsymbol{B}_j$ form. Based on this concept, Liang [8] solved the maximum-rate problem.

**Theorem 1.** [8] Let $\boldsymbol{G}[p, n, k]$ be a COD with $n = 2m - 1$ or $2m$. Then the maximum rate of $\boldsymbol{G}$ is upper bounded by $(m + 1)/2m$. The only case in which this upper bound can be achieved is when no diagonal $2 \times 2$ form exists in $\boldsymbol{G}$, and for all $1 \leqslant i \leqslant k$, the submatrix $\boldsymbol{B}_i$ is in the form $(m - 1, m) - \boldsymbol{B}_i$ or $(m, m - 1) - \boldsymbol{B}_i$ if $n = 2m - 1$, and in the form $(m, m) - \boldsymbol{B}_i$ if $n = 2m$.

The minimum decoding delay problem was solved by Adams et al. [10,11]. Their proof is based on the concept of zero pattern.

**Theorem 2.** Let $\boldsymbol{G}[p, n, k]$ be a COD with $n = 2m - 1$ or $2m$. If the rate of $\boldsymbol{G}$ reaches the upper bound, then the tight lower bound of $p$ is $\binom{2m}{m+1}$ if $n \equiv 0, 1, 3 \mod 4$ [10], and $2\binom{2m}{m+1}$ otherwise [11].

A COD $\boldsymbol{G}[p, n, k]$ is said to be atomic if any submatrix consisting of some but not all rows of $\boldsymbol{G}$ is not a COD. For an atomic COD $\boldsymbol{G}[p, n, k]$, given $1 \leqslant u, v \leqslant k$, there exist $u = m_1, m_2, \ldots, m_t = v$ such that $\boldsymbol{B}_{m_i}$ and $\boldsymbol{B}_{m_{i+1}}$ share some common rows for all $1 \leqslant i \leqslant t - 1$. Or equivalently, given two rows $\boldsymbol{u}$ and $\boldsymbol{v}$, there exist $t$ distinct rows $\boldsymbol{u} = \boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_t = \boldsymbol{v}$ such that $\boldsymbol{r}_i$ and $\boldsymbol{r}_{i+1}$ share a common variable for all $1 \leqslant i \leqslant t - 1$.

These results are obtained via combinatorial methods due to the existence of the $\boldsymbol{B}_j$ form in CODs. However, combinatorial methods are invalid in the study of GCODs. The best-known results were obtained by Wang and Xia [15].

**Theorem 3.** [15] Let $\boldsymbol{G}[p,n,k]$ be a GCOD with $n > 2$. Then the maximum rate of $\boldsymbol{G}$ is upper bounded by 4/5. Furthermore, if $\boldsymbol{G}$ satisfies the condition

$$\boldsymbol{G}^{\mathrm{H}} \cdot \boldsymbol{G} = \left(\sum_{i=1}^{k} |z_i|^2\right)\boldsymbol{I}_n,$$

then the maximum rate of $\boldsymbol{G}$ is upper bounded by 3/4.

## 3 Relationships among design criteria

In this section, we study the relationships among the seven criteria for designing CODs. Since neither an irrational coefficient nor a linear process can be found in any COD, and each variable appears exactly once in every column, we merely study the relationships among criteria C1–C4.

From the $\boldsymbol{B}_j$ form, we note that, when $n \geqslant 3$, there must be at least one diagonal $2 \times 2$ form existing in any maximum-rate COD $\boldsymbol{G}[p,n,k]$. Hence, criteria C1 and C4 can never be satisfied at the same time.

**Corollary 1.** Let $\boldsymbol{G}[p,n,k]$ be a maximum-rate COD with $n \geqslant 3$, then $\boldsymbol{G}$ must contain zero entries. In other words, C1 and C4 are mutually exclusive.

Liang [8] and Su et al. [16] presented a class of conjugation-separated maximum-rate CODs independently, which reaches the minimum decoding delay if the number of columns $n \equiv 1, 2, 3 \mod 4$ [10, 11]. This simply implies the following corollary.

**Corollary 2.** There exists a COD $\boldsymbol{G}[p,n,k]$ that satisfies criteria C1–C3 when $n \equiv 1, 2, 3 \mod 4$.

However, that construction (presented in [8] and [16]) has a decoding delay twice that of the greatest lower bound when $n \equiv 0 \mod 4$. In other words, C2 and C3 are mutually exclusive in this case. This theorem is presented by Adams et al. [17], and we provide a novel proof in this paper. First, we present several lemmas.

**Lemma 1.** [10] Let $\boldsymbol{G}[p, 2m-1, k]$ be a maximum-rate COD. Then every possible binary vector of length $2m-1$ with weight $m$ or $m+1$ must be the zero pattern of a distinct row of $\boldsymbol{G}$.

**Lemma 2.** Let $\boldsymbol{G}[p, 2m-1, k]$ be an atomic maximum-rate COD that is conjugation-separated. Then either all the conjugated rows have weight $m+1$ and all the unconjugated rows have weight $m$, or all the conjugated rows have weight $m$ and all the unconjugated rows have weight $m+1$.

*Proof.* Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two conjugated rows of $\boldsymbol{G}$. If they share a common variable $z_i$, we can make it into a $\boldsymbol{B}_i$ form without using the operation of variable conjugation. Since any two rows with the same conjugation in the $\boldsymbol{B}_i$ form have the same weight, $\boldsymbol{u}$ and $\boldsymbol{v}$ should have the same weight as well.

Now we assume that the variables contained in $\boldsymbol{u}$ and $\boldsymbol{v}$ are different. Since $\boldsymbol{G}$ is an atomic COD, there exist $t$ distinct conjugated rows $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_t$ for some positive integer $t$ such that $\boldsymbol{u}$ and $\boldsymbol{r}_1$ share a common variable, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ share a common variable, ..., $\boldsymbol{r}_t$ and $\boldsymbol{v}$ share a common variable. Thus, we have $w(\boldsymbol{u}) = w(\boldsymbol{r}_1) = \cdots = w(\boldsymbol{r}_t) = w(\boldsymbol{v})$.

Because $\boldsymbol{u}$ and $\boldsymbol{v}$ are chosen arbitrarily, we conclude that all the conjugated rows have the same weight and so do the unconjugated rows. If $\boldsymbol{G}$ contains an $(m-1, m) - \boldsymbol{B}_1$ form, then all the unconjugated rows are weighted $m+1$, and all the conjugated rows are weighted $m$. If $\boldsymbol{G}$ contains an $(m, m-1) - \boldsymbol{B}_1$ form, then all the unconjugated rows are weighted $m$, and all the conjugated rows are weighted $m+1$.

Now we can prove the theorem that was first presented in [17].

**Theorem 4.** [17] When $n \equiv 0 \mod 4$, there does not exist a COD $\boldsymbol{G}[p,n,k]$ that satisfies criteria C1–C3.

*Proof.* Let $n = 4m$, and $\boldsymbol{G}[\binom{4m}{2m+1}, 4m, \binom{4m-1}{2m}]$ be a maximum-rate COD with minimum decoding delay. Assume that $\boldsymbol{G}$ is conjugation-separated. By deleting the last column of $\boldsymbol{G}$, we obtain another COD $\boldsymbol{G}'[\binom{4m}{2m+1}, 4m-1, \binom{4m-1}{2m}]$, which has the maximum rate and reaches the minimum decoding delay.

Denote by $\boldsymbol{r}$ and $\boldsymbol{r}'$ that row before and after the deletion. Without loss of generality, we may assume that $\boldsymbol{G}'$ is in $(2m-1,2m)-\boldsymbol{B}_1'$ form. By Lemma 2, we know that all the unconjugated rows of $\boldsymbol{G}'$ have weight $2m+1$, and all the conjugated rows of $\boldsymbol{G}'$ have weight $2m$. Now we consider the conjugation of the (unique) row in $\boldsymbol{G}$ that contains a $z[1]$ in the last column. Denote it by $\boldsymbol{u}$.

Because $\boldsymbol{G}$ is in $(2m,2m)-\boldsymbol{B}_1$ form and $\boldsymbol{G}'$ is in $(2m-1,2m)-\boldsymbol{B}_1$ form, $\boldsymbol{u}(4m)$ must be $\pm z_1$. Hence, $\boldsymbol{u}$ and $\boldsymbol{u}'$ should be two unconjugated rows. On the other hand, since $w(\boldsymbol{u})=2m+1$, we have $w(\boldsymbol{u}')=2m$. From Lemma 2, we know that, $\boldsymbol{u}'$ must be a conjugated row. This leads to a contradiction. Thus, $\boldsymbol{G}[\binom{4m}{2m+1},4m,\binom{4m-1}{2m}]$ cannot be a conjugation-separated COD.

**Corollary 3.** When $n=2m$, the minimum decoding delay of conjugation-separated, maximum-rate CODs with $n$ columns is $2\binom{2m}{m+1}$.

*Proof.* The minimum decoding delay of maximum-rate CODs is $2\binom{2m}{m+1}$ if $m$ is odd. For the even case, we know from Theorem 4 that, if $\boldsymbol{G}[p,n,k]$ is a conjugation-separated, maximum-rate COD, then $p>\binom{2m}{m+1}$. Theorem 3.2 in [10] tells us that, if $p>\binom{2m}{m+1}$, then $p\geqslant 2\binom{2m}{m+1}$. In both cases, this lower bound can be reached according to Liang's constructions [8].

However, there are many fewer results in designing GCODs in the literature. Even determining the maximum rate for general $n$ is still an open problem. In this paper, we find an upper bound for the rate of a special class of GCODs that performed well with respect to criteria C3–C7.

**Theorem 5.** If a GCOD $\boldsymbol{G}[p,n,k]$ satisfies criteria C3–C7, then the rate of $\boldsymbol{G}$ must be the reciprocal of some integer $t$, i.e., $k/p=1/t$.

*Proof.* Since $\boldsymbol{G}$ is power-balanced, we may assume that every variable appears exactly $t$ times in each column. On the one hand, the number of nonzero entries in each column is $kt$ since $\boldsymbol{G}$ contains no linear processes. On the other hand, since $\boldsymbol{G}$ reaches the lowest PAPR, all its entries should be nonzero, and hence, each column contains exactly $p$ nonzero entries. Thus, we must have $p=kt$. Or equivalently, $k/p=1/t$.

Since no rate 1 GCOD exists for $n\geqslant 3$, the maximum rate of GCODs satisfying criteria C3–C7 should be 1/2. This upper bound can be reached due to the work by Tarokh et al. [2]. Thus, it is an important problem to determine the minimum decoding delay for rate 1/2 GCODs.

**Theorem 6.** [2, 13] For $n\leqslant 9$, the minimum decoding delay for rate 1/2 GCODs with $n$ columns is $\nu(n)$.

**Conjecture 1.** [13] The minimum decoding delay for rate 1/2 GCODs with $n$ columns is $\nu(n)$.

Das and Rajan [13] provide a class of rate 1/2 GCODs with parameters $[\nu(n),n,\nu(n)/2]$. This construction allows us to gain some confidence in Conjecture 1, but of course it does not affect the conjectural nature of the statement. However, this design does not admit transceiver signal linearization for general $n$. Tarokh et al. [2] construct a class of rate 1/2 transceiver linearized GCODs based on rate 1 RODs. However, the decoding delay of their construction is twice that of the preceding one. It would be interesting to find transceiver linearized GCODs with a lower decoding delay.

**Definition 5.** A GCOD is called almost perfect if it satisfies all the criteria C3–C7 and has rate 1/2, which reaches the upper bound.

An APGCOD satisfies almost all the design criteria except for the decoding delay. To become entirely perfect, we have to determine the minimum decoding delay for the APGCOD. The only known APGCOD, which is presented by Tarokh et al. [2], has decoding delay $2\nu(n)$, and hence, $2\nu(n)$ is an upper bound for the minimum decoding delay. In the following, we present a new class of APGCODs with decoding delay $2^{\lceil \frac{n}{2}\rceil}$, which equals $\nu(n)$ in most cases. In this sense, we reduce the upper bound by half in most cases.

## 4 New construction method

In this section, we present a new method for constructing GCODs based on CODs.

**Theorem 7.** Let $\boldsymbol{G}[p,2m,k]=(\boldsymbol{c}_1,\boldsymbol{c}_2,\ldots,\boldsymbol{c}_{2m})$ be a COD with an even number of columns. For any $1\leqslant i\leqslant m$, define $\boldsymbol{x}_i=\boldsymbol{c}_i+\boldsymbol{c}_{m+i}$ and $\boldsymbol{y}_i=\boldsymbol{c}_i-\boldsymbol{c}_{m+i}$. Then $\boldsymbol{O}=(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m,\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m)$ is a GCOD that has the same rate and decoding delay as $\boldsymbol{G}$.

*Proof.* Since $\boldsymbol{G}$ is a COD, we know that for all $1\leqslant i,j\leqslant 2m$,

$$\boldsymbol{c}_i\cdot\boldsymbol{c}_j=\delta_{ij}\sum_{t=1}^{k}|z_t|^2,$$

while $\delta_{ij}$ equals 1 if $i=j$ and 0 otherwise. In the following we will show that

$$\boldsymbol{O}^{\mathrm{H}}\cdot\boldsymbol{O}=2\left(\sum_{t=1}^{k}|z_t|^2\right)\boldsymbol{I}_{2m}.$$

(1) For any $1\leqslant i\leqslant m$,

$$\begin{aligned}\boldsymbol{x}_i^{\mathrm{H}}\cdot\boldsymbol{x}_i &= (\boldsymbol{c}_i+\boldsymbol{c}_{m+i})^{\mathrm{H}}\cdot(\boldsymbol{c}_i+\boldsymbol{c}_{m+i})=\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_i+\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_{m+i}+2\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_{m+i}=2\sum_{t=1}^{k}|z_t|^2,\\ \boldsymbol{y}_i^{\mathrm{H}}\cdot\boldsymbol{y}_i &= (\boldsymbol{c}_i-\boldsymbol{c}_{m+i})^{\mathrm{H}}\cdot(\boldsymbol{c}_i-\boldsymbol{c}_{m+i})=\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_i+\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_{m+i}-2\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_{m+i}=2\sum_{t=1}^{k}|z_t|^2.\end{aligned}$$

(2) When $1\leqslant i<j\leqslant m$,

$$\begin{aligned}\boldsymbol{x}_i^{\mathrm{H}}\cdot\boldsymbol{x}_j &= (\boldsymbol{c}_i+\boldsymbol{c}_{m+i})^{\mathrm{H}}\cdot(\boldsymbol{c}_j+\boldsymbol{c}_{m+j})=\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_j+\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_j+\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_{m+j}+\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_{m+j}=0,\\ \boldsymbol{y}_i^{\mathrm{H}}\cdot\boldsymbol{y}_j &= (\boldsymbol{c}_i-\boldsymbol{c}_{m+i})^{\mathrm{H}}\cdot(\boldsymbol{c}_j-\boldsymbol{c}_{m+j})=\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_j-\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_j-\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_{m+j}+\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_{m+j}=0.\end{aligned}$$

(3) When $1\leqslant i,j\leqslant m$,

$$\begin{aligned}\boldsymbol{x}_i^{\mathrm{H}}\cdot\boldsymbol{y}_j &= (\boldsymbol{c}_i+\boldsymbol{c}_{m+i})^{\mathrm{H}}\cdot(\boldsymbol{c}_j-\boldsymbol{c}_{m+j})\\ &= \boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_j+\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_j-\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_{m+j}-\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_{m+j}\\ &= \boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_j-\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_{m+j}.\end{aligned}\tag{1}$$

If $i=j$, then expression (1) is equal to $\boldsymbol{c}_i^{\mathrm{H}}\cdot\boldsymbol{c}_i-\boldsymbol{c}_{m+i}^{\mathrm{H}}\cdot\boldsymbol{c}_{m+i}=0$. If $i\neq j$, then both of the two items are equal to 0. Anyway, expression (1) is equal to 0 all the time.

In conclusion,

$$(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m,\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m)^{\mathrm{H}}\cdot(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_m,\boldsymbol{y}_1,\ldots,\boldsymbol{y}_m)=2\left(\sum_{t=1}^{k}|z_t|^2\right)\boldsymbol{I}_{2m}.$$

Thus, $\boldsymbol{O}[p,2m,k]$ is a GCOD with the same rate and decoding delay as $\boldsymbol{G}$.

**Example 1.** Let $\boldsymbol{G}_1[4,4,3]$ be a maximum-rate COD with four columns:

$$\boldsymbol{G}_1=\begin{pmatrix} z_1 & 0 & z_2 & z_3\\ 0 & z_1 & z_3^* & -z_2^*\\ -z_2^* & -z_3 & z_1^* & 0\\ -z_3^* & z_2 & 0 & z_1^*\end{pmatrix}.$$

Using the new method, we obtain a rate 3/4 GCOD $\boldsymbol{O}_1[4,4,3]$ with the same parameters as $\boldsymbol{G}_1$:

$$\boldsymbol{O}_1=\begin{pmatrix} z_1+z_2 & z_3 & z_1-z_2 & -z_3\\ z_3^* & z_1-z_2^* & -z_3^* & z_1+z_2^*\\ -z_2^*+z_1^* & -z_3 & -z_2^*-z_1^* & -z_3\\ -z_3^* & z_2+z_1^* & -z_3^* & z_2-z_1^*\end{pmatrix}.$$

**Theorem 8.** Let $\boldsymbol{G}[p, 2m+1, k] = (\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_{2m+1})$ be a COD with an odd number of columns. Define $\boldsymbol{x}_i = \boldsymbol{c}_i + \boldsymbol{c}_{m+i}$ and $\boldsymbol{y}_i = \boldsymbol{c}_i - \boldsymbol{c}_{m+i}$ for all $1 \leqslant i \leqslant m$. Then $\boldsymbol{O} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_m, \boldsymbol{c}_{2m+1})$ is a GCOD that has the same rate and decoding delay as $\boldsymbol{G}$.

The proof is omitted here since it is almost the same as the one of Theorem 7. Note that in this case,

$$\boldsymbol{O}^{\mathrm{H}} \cdot \boldsymbol{O} = \left( \sum_{t=1}^{k} |z_t|^2 \right) \begin{pmatrix} 2\boldsymbol{I}_{2m} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}.$$

**Example 2.** Let $\boldsymbol{G}_2[4,3,3]$ be a maximum-rate COD with three columns:

$$\boldsymbol{G}_2 = \begin{pmatrix} z_1 & z_2 & z_3 \\ -z_2^* & z_1^* & 0 \\ -z_3^* & 0 & z_1^* \\ 0 & -z_3^* & z_2^* \end{pmatrix}.$$

We obtain another rate 3/4 GCOD $\boldsymbol{O}_2[4,3,3]$ by our new method, which has the same parameters as $\boldsymbol{G}_2$:

$$\boldsymbol{O}_2 = \begin{pmatrix} z_1 + z_2 & z_1 - z_2 & z_3 \\ -z_2^* + z_1^* & -z_2^* - z_1^* & 0 \\ -z_3^* & -z_3^* & z_1^* \\ -z_3^* & z_3^* & z_2^* \end{pmatrix}.$$

Note that in this case, the new method does not change the transceiver signal linearization property of $\boldsymbol{G}_2$.

Generally speaking, the new method does not change the code rate and decoding delay, and keeps the property of conjugation separation if the primary COD is conjugation-separated. However, it may also bring in linear processes in the obtained GCOD, and even power-unbalance odd columns. Thus, we want to find CODs that can help us to avoid these disadvantages. Fortunately, this kind of COD does exist. In the next section, we use this new method to construct APGCODs based on BCODs.

## 5 Novel construction of APGCODs

Note that this new method does not perform well in general cases. Fortunately, there does exist a special class of CODs that performs perfectly with respect to all the seven design criteria. To begin the construction, we first introduce the BCOD, which was presented for the first time by Adams et al. [3], and simplified by Liu et al. [14].

**Definition 6.** [3, 14] A BCOD $\boldsymbol{G}[2k, 2m, k]$ is a COD that satisfies the following conditions:

(1) $\boldsymbol{G}$ is conjugation-separated.

(2) Each row of $\boldsymbol{G}$ contains exactly $m$ nonzero entries (and of course $m$ zeros).

(3) For any $1 \leqslant j \leqslant k$, the summatrix $\boldsymbol{M}_j$ in the $\boldsymbol{B}_j$ form is skew-symmetric, i.e., $\boldsymbol{M}_j^{\mathrm{T}} = -\boldsymbol{M}_j$.

By deleting one column from $\boldsymbol{G}$, a BCOD with parameters $[2k, 2m-1, k]$ is obtained.

**Definition 7.** [14] We say a BCOD $\boldsymbol{G}[2k, 2m, k]$ is in standard form iff it is already in $\boldsymbol{B}_j$ form for some $1 \leqslant j \leqslant k$.

**Theorem 9.** [14] Let $\boldsymbol{G}[2k, 2m, k]$ be a standard BCOD. Then

(1) For any $1 \leqslant i \leqslant m$, the zero patterns of the $i$th column and $(m+i)$th column of $G$ are mutually complementary.

(2) For any $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$, the instances of $z[j]$ in the $i$th column and $(m+i)$th column have different conjugations.

(3) For any $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$, the instances of $z[j]$ in the $i$th column and $(m+i)$th column are in a diagonal $2 \times 2$ form.

Note that the columns of a BCOD have correspondingly complementary zero patterns. This is the key property that can help us to obtain an elegant design via our new method. In fact, we have the following theorem.

**Theorem 10.** Let $\boldsymbol{G}[2k, 2m, k]$ be a standard BCOD. Then by using the new method on $\boldsymbol{G}$, an APG-COD $\boldsymbol{O}[2k, 2m, k]$ that has rate $1/2$ and satisfies design criteria C3–C7 is obtained.

*Proof.* Obviously, $\boldsymbol{O}$ is conjugation-separated and contains no irrational coefficients. Since $\boldsymbol{c}_i$ and $\boldsymbol{c}_{m+i}$ have complementary zero patterns for all $1 \leqslant i \leqslant m$, the vectors $\boldsymbol{c}_i \pm \boldsymbol{c}_{m+i}$ contain neither zero entries nor linear processes. Additionally, each variable appears exactly twice in both $\boldsymbol{c}_i + \boldsymbol{c}_{m+i}$ and $\boldsymbol{c}_i - \boldsymbol{c}_{m+i}$. Thus, $\boldsymbol{O}$ is power-balanced. Finally, the rate of $\boldsymbol{O}$ is $1/2$, just as the same as $\boldsymbol{G}$.

**Example 3.** Consider the standard BCOD $\boldsymbol{G}_3[16, 8, 8]$ when $m = 4$:

$$\boldsymbol{G}_3 = \begin{pmatrix} 0 & 0 & 0 & z_1 & -z_2 & -z_3 & -z_4 & 0 \\ 0 & 0 & z_1 & 0 & -z_5 & -z_6 & 0 & z_4 \\ 0 & z_1 & 0 & 0 & -z_7 & 0 & z_6 & z_3 \\ z_1 & 0 & 0 & 0 & 0 & z_7 & z_5 & z_2 \\ 0 & -z_4 & z_3 & -z_6 & z_8 & 0 & 0 & 0 \\ z_4 & 0 & -z_2 & z_5 & 0 & z_8 & 0 & 0 \\ z_3 & -z_2 & 0 & z_7 & 0 & 0 & -z_8 & 0 \\ z_6 & -z_5 & z_7 & 0 & 0 & 0 & 0 & z_8 \\ 0 & 0 & 0 & z_8^* & z_6^* & -z_5^* & z_7^* & 0 \\ 0 & 0 & -z_8^* & 0 & z_3^* & -z_2^* & 0 & z_7^* \\ 0 & z_8^* & 0 & 0 & z_4^* & 0 & -z_2^* & z_5^* \\ z_8^* & 0 & 0 & 0 & 0 & -z_4^* & z_3^* & -z_6^* \\ 0 & z_7^* & z_5^* & z_2^* & z_1^* & 0 & 0 & 0 \\ -z_7^* & 0 & z_6^* & z_3^* & 0 & z_1^* & 0 & 0 \\ -z_5^* & -z_6^* & 0 & z_4^* & 0 & 0 & z_1^* & 0 \\ -z_2^* & -z_3^* & -z_4^* & 0 & 0 & 0 & 0 & z_1^* \end{pmatrix}.$$

Under our new construction method, we obtain an APGCOD $\boldsymbol{O}_3[16, 8, 8]$ as follows:

$$\boldsymbol{O}_3 = \begin{pmatrix} -z_2 & -z_3 & -z_4 & z_1 & z_2 & z_3 & z_4 & z_1 \\ -z_5 & -z_6 & z_1 & z_4 & z_5 & z_6 & z_1 & -z_4 \\ -z_7 & z_1 & z_6 & z_3 & z_7 & z_1 & -z_6 & -z_3 \\ z_1 & z_7 & z_5 & z_2 & z_1 & -z_7 & -z_5 & -z_2 \\ z_8 & -z_4 & z_3 & -z_6 & -z_8 & -z_4 & z_3 & -z_6 \\ z_4 & z_8 & -z_2 & z_5 & z_4 & -z_8 & -z_2 & z_5 \\ z_3 & -z_2 & -z_8 & z_7 & z_3 & -z_2 & z_8 & z_7 \\ z_6 & -z_5 & z_7 & z_8 & z_6 & -z_5 & z_7 & -z_8 \\ z_6^* & -z_5^* & z_7^* & z_8^* & -z_6^* & z_5^* & -z_7^* & z_8^* \\ z_3^* & -z_2^* & -z_8^* & z_7^* & -z_3^* & z_2^* & -z_8^* & -z_7^* \\ z_4^* & z_8^* & -z_2^* & z_5^* & -z_4^* & z_8^* & z_2^* & -z_5^* \\ z_8^* & -z_4^* & z_3^* & -z_6^* & z_8^* & z_4^* & -z_3^* & z_6^* \\ z_1^* & z_7^* & z_5^* & z_2^* & -z_1^* & z_7^* & z_5^* & z_2^* \\ -z_7^* & z_1^* & z_6^* & z_3^* & -z_7^* & -z_1^* & z_6^* & z_3^* \\ -z_5^* & -z_6^* & z_1^* & z_4^* & -z_5^* & -z_6^* & -z_1^* & z_4^* \\ -z_2^* & -z_3^* & -z_4^* & z_1^* & -z_2^* & -z_3^* & -z_4^* & -z_1^* \end{pmatrix}.$$

**Table 1** Some existing (general) complex orthogonal designs

| | |
|---|---|
| TJC99 | The rate 1/2 COD presented in [2] |
| SX03 | The square COD presented in [18] |
| L03 | The maximum-rate COD presented in [8] |
| LFX05 | The maximum-rate COD presented in [9] |
| SBP06 | The square COD presented in [4] |
| LKYM09 | The GCOD presented in [19] |
| ADK11A | The BCOD presented in [3] |
| ADK11B | The GCOD presented in [3] |
| DR12A | The scaled COD presented in [13] |
| DR12B | The GCOD presented in [13] |

The obtained APGCOD $\boldsymbol{O}_3[16,8,8]$ has rate 1/2, decoding delay 16 and satisfies all the criteria C3–C7. Simply by deleting any column of $\boldsymbol{O}_3$, we obtain another APGCOD $\boldsymbol{O}_3'[16,7,8]$, which has the same decoding delay as $\boldsymbol{O}_3$.

Now we turn to the decoding delay of the obtained APGCODs. Liu et al. [14] determined the minimum decoding delay for BCODs, and hence, for this class of APGCODs.

**Theorem 11.** [14] The minimum decoding delay for BCODs with $n$ columns is $2^{\lceil \frac{n}{2} \rceil}$.

Thus, the decoding delay for the new APGCOD with $2m$ columns is $2^m$. For a BCOD $\boldsymbol{G}'[2^m, 2m-1, 2^{m-1}]$ with an odd number of columns, we do not use our method directly, since it can neither remove all the zero entries nor balance the power. As $\boldsymbol{G}'$ is obtained by deleting one column from a BCOD $\boldsymbol{G}[2^m, 2m, 2^{m-1}]$ with an even number of columns, we first use our method on $\boldsymbol{G}$, and then delete one column of the obtained APGCOD. Under this procedure, we obtain another APGCOD $\boldsymbol{O}'[2^m, 2m-1, 2^{m-1}]$ with $2m-1$ columns, which has decoding delay $2^m$. As a consequence, we have already proven the following theorem.

**Theorem 12.** For any positive integer $n \geqslant 3$, an APGCOD $\boldsymbol{O}[2^{\lceil \frac{n}{2} \rceil}, n, 2^{\lceil \frac{n}{2} \rceil - 1}]$ that has rate 1/2, decoding delay $2^{\lceil \frac{n}{2} \rceil}$ and satisfies all the design criteria C3–C7 can be constructed.

The only known APGCOD presented by Tarokh et al. [2] has decoding delay $2\nu(n)$. Hence, $2\nu(n)$ is an upper bound for the minimum decoding delay of APGCODs with $n$ columns. Due to Theorem 12, this upper bound can be reduced in most cases.

**Corollary 4.** The minimum decoding delay for APGCODs with $n$ columns is upper bounded by $2^{\lceil \frac{n}{2} \rceil}$.

Since $2^{\lceil \frac{n}{2} \rceil} = \nu(n)$ if $n \equiv 2,3,4,5,6 \mod 8$, the upper bound of $2\nu(n)$ is reduced by half in these cases. Hence, we provide a significant improvement for the upper bound in most cases. However, there is still no result about the lower bound yet.

Table 1 lists some existing codes. Compared with these existing designs, our novel APGCOD performs remarkably well with respect to all the design criteria. Previously, TJC99 was the only known APGCOD, but it has larger decoding delay than our new design in most cases. DR12B has smaller decoding delay than our new design when $n \equiv 0,1,7 \mod 8$, and satisfies C4–C7, but it is not conjugation-separated. More comparisons are listed in Tables 2–4.

It seems that transceiver signal linearization is not an extreme condition in designing CODs. In most cases, linearizing the transceiver signal of a code does not increase the decoding delay. For example, the transceiver signal can be linearized for all maximum-rate CODs except when $n \equiv 0 \mod 4$. Similarly, compared with DR12B, our new APGCOD has the same decoding delay for $n \equiv 2,3,4,5,6$, and twice otherwise.

However, we do not know whether the upper bound of $2^{\lceil \frac{n}{2} \rceil}$ can be reduced without affecting other properties. Since it reaches the upper bound of $\nu(n)$ for rate 1/2 GCODs in most cases, we tend to believe that $2^{\lceil \frac{n}{2} \rceil}$ is also the lower bound for the decoding delay of APGCODs.

**Conjecture 2.** The minimum decoding delay for APGCODs with $n$ columns is $2^{\lceil \frac{n}{2} \rceil}$.

We present the first class of APGCODs with decoding delay $2^{\lceil \frac{n}{2} \rceil}$. While encouraging, this design is clearly not sufficient to prove Conjecture 2.

**Table 2** Comparison of the rate of existing codes and our new APGCOD

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| TJC99 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| SX03 | 0.2 | 0.333 | 0.143 | 0.5 | 0.111 | 0.2 | 0.091 | 0.25 | 0.077 | 0.143 | 0.067 | 0.313 |
| L03 | 0.667 | 0.667 | 0.625 | 0.625 | 0.6 | 0.6 | 0.583 | 0.583 | 0.571 | 0.571 | 0.563 | 0.563 |
| LFX05 | 0.667 | 0.667 | 0.625 | 0.625 | 0.6 | 0.6 | 0.583 | 0.583 | 0.571 | 0.571 | 0.563 | 0.563 |
| SBP06 | 0.2 | 0.333 | 0.143 | 0.25 | 0.111 | 0.2 | 0.091 | 0.167 | 0.077 | 0.143 | 0.067 | 0.125 |
| LKYM09 | 0.667 | 0.667 | 0.625 | 0.625 | 0.6 | 0.6 | 0.583 | 0.583 | 0.571 | 0.571 | 0.563 | 0.563 |
| ADK11A | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| ADK11B | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| DR12A | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| DR12B | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| New | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |

**Table 3** Comparison of decoding delay of existing codes and our new APGCOD

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| TJC99 | 16 | 16 | 16 | 16 | 32 | 64 | 128 | 128 | 256 | 256 | 256 | 256 |
| SX03 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| L03 | 15 | 30 | 56 | 112 | 210 | 420 | 792 | 1584 | 3003 | 6006 | 11440 | 22880 |
| LFX05 | 15 | 30 | 56 | 56 | 210 | 420 | 792 | 792 | 3003 | 6006 | 11440 | 11440 |
| SBP06 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| LKYM09 | 15 | 30 | 56 | 56 | 210 | 420 | 792 | 792 | 3003 | 6006 | 11440 | 11440 |
| ADK11A | 8 | 8 | 16 | 16 | 32 | 32 | 64 | 64 | 128 | 128 | 256 | 256 |
| ADK11B | 8 | 8 | 16 | 16 | 32 | 32 | 64 | 64 | 128 | 128 | 256 | 256 |
| DR12A | 8 | 8 | 8 | 8 | 16 | 32 | 64 | 64 | 128 | 128 | 128 | 128 |
| DR12B | 8 | 8 | 8 | 8 | 16 | 32 | 64 | 64 | 128 | 128 | 128 | 128 |
| New | 8 | 8 | 16 | 16 | 32 | 32 | 64 | 64 | 128 | 128 | 256 | 256 |

**Table 4** Comparison of C3–C7 of existing codes and our new APGCOD

| STBC | Type | C3 | C4 | C5 | C6 | C7 |
|---|---|---|---|---|---|---|
| TJC99 | GCOD | Yes | Yes | Yes | Yes | Yes |
| SX03 | COD | No | No | Yes | Yes | Yes |
| L03 | COD | Yes | No | Yes | Yes | Yes |
| LFX05 | COD | Yes | No | Yes | Yes | Yes |
| SBP06 | COD | Yes | No | Yes | Yes | Yes |
| LKYM09 | GCOD | Yes | No | Yes | No | No |
| ADK11A | COD | Yes | No | Yes | Yes | Yes |
| ADK11B | GCOD | No | Yes | Yes | Yes | Yes |
| DR12A | GCOD | No | No | Yes | Yes | Yes |
| DR12B | GCOD | No | Yes | Yes | Yes | Yes |
| New | GCOD | Yes | Yes | Yes | Yes | Yes |

## 6 Conclusion

In this paper, we study the tradeoff among the seven important criteria for designing CODs. We find that the tight upper bound of the code rate for a special class of GCODs that satisfies criteria C3–C7 is 1/2. It is the only result in the literature on the maximum rate of GCODs after Wang and Xia [15]. Thus, it would be an interesting problem to construct rate 1/2 GCODs that satisfy all criteria C3–C7, which are defined as APGCODs in this paper.

Tarokh et al. [2] presented the first class of APGCODs $\boldsymbol{G}[p, n, k]$ with decoding delay $p = 2\nu(n)$. In this paper, we construct the second class of APGCODs with $n$ columns based on BCODs for any positive integer $n \geqslant 3$. The decoding delay of our construction is $2^{\lceil \frac{n}{2} \rceil}$, which equals $\nu(n)$ if $n \equiv 2, 3, 4, 5, 6$

mod 8 or $2\nu(n)$ if $n \equiv 0, 1, 7 \mod 8$. Compared with the existing APGCODs in [2], the decoding delay is reduced by one-half in our construction for most cases.

A remaining problem is to determine the minimum decoding delay of APGCODs. Due to our construction, the minimum decoding delay is upper bounded by $2^{\lceil \frac{n}{2} \rceil}$. We conjecture that this upper bound is also the lower bound as well.

**Acknowledgements** This work was supported by National Natural Science Foundation of China (Grant No. 61170208), Shanghai Key Program of Basic Research (Grant No. 12JC1401400), National Defense Basic Research Project (Grant No. JCYJ-1408), and Shanghai Excellent Academic Leader Funds (Grant No. 16XD1400200).

**Conflict of interest** The authors declare that they have no conflict of interest.

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