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# Stabilization of NCSs by random allocation of transmission power to sensors

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**Abstract** This study investigates networked control systems (NCSs), whose sensors communicate with remote controllers via a wireless fading channel. The sensor can choose different power levels at which it can transmit its measurement to the controller. The transmission power is selected according to a given probability distribution. The level of transmission power determines the probability of packet loss. The objective of this study is to find an appropriate transmission power probability distribution and a system controller jointly such that NCSs can be exponentially stabilized within a given energy budget. By the average dwell time technique, sufficient conditions for almost sure stability and an optimal sensor power probability distribution maximizing the stability margin are obtained. The effectiveness of the results is demonstrated by numerical simulations.

**Keywords** networked control systems, power allocation, packet dropout, almost sure stability, transmission energy constraint

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## 1 Introduction

Networked control systems (NCSs) have been researched extensively in the last few years. Growing applications have been found in various areas, for example, transportation systems, power systems, remote monitoring and data acquisition systems, chemical processes, and many manufacturing plants [1,2]. NCSs provide many advantages when compared with traditional point-to-point systems, including low cost, easy maintenance, and high efficiency. However, the use of wireless communication in NCSs brings new challenges in system design. One major drawback is channel fading and interference, which frequently cause random packet errors. In digital communication with error-control coding, if bit errors occur, the packets will be dropped. Consecutive packet dropouts may cause performance degradation or even stability loss. Therefore, it is significant to ensure system performance subject to channel fading.

An intuitive approach to alleviating the effect of channel fading is to adjust the power levels. If high transmission powers are used, then dropouts are less frequent. However, the use of high transmission power is rarely an option, as in most applications, battery-powered sensors are expected to be operational for several years without replacement of batteries and consecutive transmission with high power implies

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that the batteries would have shorter lifetimes [3]. Thus, a device cannot always transmit data with high power, and proper allocation of transmission power is the key to achieving a balance between the performance and the power consumption.

There have been various formulations and approaches in the literature to the power allocation (also known as power control) problem. Ren et al. [4] studied the problem of scheduling one sensor to transmit data to the remote estimator. When the sensor is allowed to transmit data, it will cost a certain amount of energy. Otherwise, the sensor is not used to save energy. Shi et al. [5] investigated an optimal sensor-scheduling problem in which a sensor operates at two transmission power levels: high and low. They provided optimal power schedules that minimize the estimation error covariance. Then, Shi et al. [6] extended the scheduling method to a general high-order Gauss-Markov system, where a sensor has to decide whether to use a high or low transmission power to send a measurement to the remote estimator over a packet-dropping network. Han et al. [7] considered a similar scheduling problem where a sensor needs to decide when to switch between two transmission energy levels. An online sensor power schedule was proposed to minimize the average expected estimation error covariance subject to the available energy budget. Unlike [7], a state-based power-scheduling policy is discussed in [8], where the transmission power depends upon the incremental innovation in the sensor local state estimation compared to the previous time. The results show that the state-based power-scheduling policy can give a better performance than other offline power schedules.

Most of the abovementioned studies investigate a very simple case in which the sensor is assumed to operate in one or two power levels. In this case, the sensor only needs to decide when to work or when to use a high transmission power level to transmit data to the remote estimator (we call it $1W$-scheduling). In real applications, many commercially available sensor nodes have more than two transmission power levels, which can switch between working and sleeping. Then, one needs to determine when to activate the sensor to transmit measurement by which power level (we call it $2W$-scheduling). Such a problem is more complicated than $1W$-scheduling, as the sensor-scheduling problem (switching between the work mode and sleep mode) and power-scheduling problem (switching among different power levels) are considered in the same framework. Besides, most of the existing studies are restricted to a static-period-scheduling policy, which is easily realizable but not adaptable to situations with disturbances or uncertainties.

Here we are interested in an entirely new type of scheduling policy, namely a random scheduling policy. Such a policy allows the sensor to transmit data with a randomly chosen power level. Random scheduling policies have recently arisen as an important tool for resolving network access constraints, especially in wireless communications [9]. However, to the best of the authors' knowledge, they have never been utilized for power-scheduling purposes in the literature. To show how a random scheduling policy will affect a control system, let us consider a linearly networked system whose sensor has $N$ transmission power levels. When the sensor is activated, it sends data to the remote controller with a transmission power level chosen from the $N$ choices. Otherwise, the sensor falls into sleep mode and zero power is used. The chosen transmission power (including zero power in the sleep mode) depends on a random process whose probability distribution is given or known. To be more specific, at each sampling time, the sensor randomly chooses a transmission power level from the set of $N+1$ choices (zero power in sleep mode and the other $N$ power choices in work mode) and then sends data to the remote controller via a wireless fading channel. Our aim is to set up a unified methodology by which the plant controller and probability distribution of transmission power levels can be designed jointly in a unified framework with an average communication energy constraint.

The main contributions of this study and comparison with previous studies are summarized as follows:

(1) In this study, we consider a combined issue of control design, sensor scheduling, and power allocation under a random event-driven assignment mechanism and solve it in a unified co-design framework. To the best of our knowledge, this problem's formulation is novel and challenging.

(2) Sufficient conditions for almost sure stability of NCSs with transmission energy constraints are given, which for the first time combine the transmission power and its probability distribution. Based on the results, an optimal probability distribution scheme of sensor power scheduling is derived by maximizing the system stability margin.

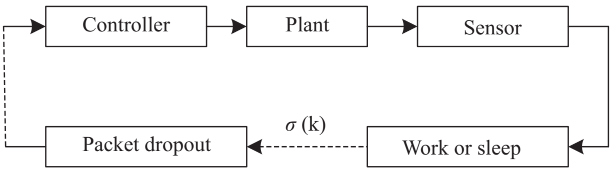


**Figure 1** NCS over a wireless fading channel.

The remainder of this study is organized as follows. In Section 2, we introduce the problem formulation and system models. In Section 3, we provide sufficient conditions for the exponentially almost sure stability of the system. An optimal algorithm is proposed to compute the optimal probability distribution of the sensor transmission power level. A control and scheduling of transmission power co-design procedure is developed for the stabilization of NCSs. Section 4 contains a simulation study to demonstrate the effectiveness of the proposed results. Finally, the conclusion and future work are included in Section 5.

Notation. Throughout this technical note, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ represent the $n$-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. The superscript "T" represents the transpose. For Hermitian matrices $X = X^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and $Y = Y^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $X > Y$ indicates that matrix $X - Y$ is positive-definite. $\mathrm{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure. $||\cdot||$ represents the Euclidean norm for a vector. $\cap$ denotes the intersection set. $\mathrm{prob}\{\cdot\}$ refers to probability and $\mathrm{prob}\{\cdot|\cdot\}$ to conditional probability.

## 2 Problem formulation

Consider an NCS, as shown in Figure 1. At each sampling time, the sensor measures the output of the plant and then sends data to the remote controller via a wireless communication channel. As mentioned above, the communication channel is subject to a fading effect, which may yield packet dropouts. The rate or probability of packet dropping depends on how much power the sensor uses for transmission. A detailed description of this problem is given below.

### 2.1 Packet dropout vs transmission energy

The network channel by which the sensor communicates with the remote controller is assumed to be an additive white Gaussian noise (AWGN) channel with error coding. We suppose that the AWGN channel uses the well-established quantized quadrature amplitude modulation (QAM) mechanism [10]. For an AWGN channel with quantized QAM, if the sensor measurement is quantized into $R$ bits and mapped to one of the $2^R$ available QAM symbols, the symbolic error rate is given by [11]

$$\mathrm{SER} = Q\left(\sqrt{\frac{2\upsilon\sigma(k)}{H_0W}}\right), \tag{1}$$

where $\sigma(k)$ is the transmission power of the sensor at time $k$, $H_0$ is the power spectral density of AWGN noise, $W$ is the channel bandwidth, $\upsilon$ is a constant that depends on $R$, and

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} \exp(-s^2/2)\mathrm{d}s.$$

For a sufficiently large signal-to-noise ratio, we have [12]

$$\mathrm{SER} \approx \exp\left(-\frac{\upsilon\sigma(k)}{H_0W}\right). \tag{2}$$

Throughout this study, we shall assume that the communication channel is time-invariant, i.e., $\upsilon$, $H_0$, and $W$ are constants. Then, the symbolic error rate varies inversely with the transmission power. Assume that the sensor has $N$ transmission power levels to be selected, i.e., $\sigma_1, \ldots, \sigma_N$, where $\sigma_1 < \cdots < \sigma_N$. Then, according to (2), we know that

$$\mathrm{SER}_1 > \cdots > \mathrm{SER}_N,$$

where $\mathrm{SER}_i = \exp\left(-\frac{\upsilon\sigma_i}{H_0 W}\right)$, $i = 1, 2, \ldots, N$.

It is clear that the sensor has two modes to switch. When the sensor is activated in work mode, it can send data to the remote controller with a different kind of transmission power. Otherwise, the sensor falls into sleep mode and no data are transmitted to the controller. At this time, no energy is lost and the transmission power used by the sensor is zero. Then, at each sampling time, the sensor actually has $N+1$ power states, i.e., $\sigma(k) \in \{\sigma_0, \sigma_1, \ldots, \sigma_N\}$, where $\sigma_0 = 0$ denotes that the sensor switches into sleep mode and $\{\sigma_1, \ldots, \sigma_N\}$ denote the transmission powers used by the sensor in work mode. Suppose that the chosen transmission power is driven by a random event whose probability distribution is given by $E = [e_0, e_1, \ldots, e_N]$, where $e_i = \mathrm{prob}\{\sigma(k) = \sigma_i\}$, $\sum_{i=0}^{N} e_i = 1$, $i = 0, 1, \ldots, N$. Then, the average transmission energy consumed by the sensor over an infinite time interval is given by

$$J = \lim_{k\to\infty} (1/k) \sum_{s=1}^{k} \sigma(s)h,$$

which, by the law of large numbers [13], is equal to

$$J = \sum_{i=0}^{N} e_i \sigma_i h \tag{3}$$

almost surely, where $h$ is the sampling time.

Due to the fading effect, the data packet in the transmission will be dropped. Here we model the packet dropout as a Bernoulli process [14]. To be specific, we use a binary function $\gamma(k)$ to describe whether a packet is dropped or not: $\gamma(k) = 1$ means that the measurement $x(k)$ is transmitted successfully and arrives at the controller without error; otherwise, $\gamma(k) = 0$, which indicates that the packet is dropped and the controller considers the measurement $x(k)$ to be zero. Let $\pi_i = \mathrm{prob}\{\gamma(k) = 1 | \sigma(k) = \sigma_i\}$ be the probability of a successful transmission when the transmission power level $\sigma_i$ is used. Clearly, $\pi_i = 1 - \mathrm{SER}_i$. According to (2), we have

$$\pi_i = 1 - \theta^{\sigma_i}, \tag{4}$$

where $\theta = \exp(-\upsilon/H_0 W) \in (0,\ 1)$, $i = 1, \ldots, N$.

In particular, when the sensor switches into sleep mode, then no data are transmitted; the probability of a successful transmission equals zero, i.e.,

$$\pi_0 = 1 - \theta^{\sigma_0} = 0,$$

where $\pi_0 = \mathrm{prob}\{\gamma(k) = 1 | \sigma(k) = \sigma_0\}$.

### 2.2 Problem description and the object

Assume that the plant in Figure 1 is a linear system whose dynamics are described by

$$x(k+1) = Ax(k) + Bu(k), \tag{5}$$

where $x(k) \in \mathbb{R}^n$ is the state that is directly measured by the sensor, $u(k) \in \mathbb{R}^m$ is the control input, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices. It is assumed that $A$ is unstable but the pair ($A$,$B$) is stabilizable.

The plant is to be controlled by a state feedback controller given by

$$u(k) = K\hat{x}(k), \tag{6}$$

where $K \in \mathbb{R}^{m\times n}$ is the controller gain and $\hat{x}(k)$ is the state actually received by the controller.

When a sensor is not assigned for transmission or if packet dropout occurs when the sensor gains access to the channel, the corresponding controller can either use a zeroth-order hold to produce the control signal or consider the measurement to be zero and produce a zero control signal. Here, for simplicity, we use the zero control signal strategy. According to the above discussion, $\hat{x}(k)$ can be represented as

$$\hat{x}(k) = \gamma(k)x(k). \tag{7}$$

Then, combining (5)–(7), the dynamics of the system can be rewritten as

$$x(k+1) = [A + BK\gamma(k)]x(k). \tag{8}$$

As $\gamma(k)$ is a binary function describing packet loss, system (8) is actually a switching system comprising an open-loop subsystem

$$x(k+1) = Ax(k), \quad \gamma(k) = 0, \tag{9}$$

and a closed-loop subsystem

$$x(k+1) = (A + BK)x(k), \quad \gamma(k) = 1. \tag{10}$$

Let $\alpha(k)$ denote the number of sampling periods in which system (8) is under closed-loop control over time interval $[0,\ k)$. Then, it is easy to show that $\alpha(k) = \sum_{s=0}^{k-1}\gamma(s)$. Denote $\alpha(k)/k$ as the attention rate of system (8). Then, from the law of large numbers [13], we have

$$\lim_{k\to\infty}\alpha(k)/k = \lim_{k\to\infty}\sum_{s=1}^{k}\gamma(s)/k = \mathrm{E}\{\gamma(k)\} = \mathrm{E}\{\mathrm{E}\{\gamma(k)|\sigma(k)=\sigma_i\}\} = \sum_{i=0}^{N}e_i(1-\theta^{\sigma_i}) = 1 - z \tag{11}$$

almost surely, with $z = \sum_{i=0}^{N} e_i\theta^{\sigma_i}$.

The rate of switching of the plant between the open- (9) and closed-loop subsystems (10) over the time interval $[0,\ k)$ is called the chatter frequency, and is denoted by $N(k)$ in this study. Then, from the law of large numbers [13], it is easy to show that

$$\begin{aligned}\lim_{k\to\infty}N(k)/k &= \lim_{k\to\infty}(1/k)\sum_{s=0}^{k-1}\{\gamma(s)[1-\gamma(s+1)] + [1-\gamma(s)]\gamma(s+1)\}\\ &= \mathrm{E}\{\gamma(k)[1-\gamma(k+1)] + [1-\gamma(k)]\gamma(k+1)\}\\ &= 2\mathrm{E}\{\gamma(k)\} - 2E^2\{\gamma(k)\} = -2z^2 + 2z \end{aligned}\tag{12}$$

almost surely.

We need the following definition before giving our objective.

**Definition 1.** For a given probability distribution $E = [e_0, e_1, \ldots, e_N]$, system (8) is said to be exponentially almost surely (sample path) stable if there exists a scalar $\tau > 0$ such that, for any $x(0) \in \mathbb{R}^n$,

$$\mathrm{prob}\left\{\lim_{k\to\infty}(1/k)\ln||x(k)|| \leqslant -\tau\right\} = 1,$$

where the constant $\tau$ is named the stability margin [15].

The objective of this study is to find a co-design framework for the probability distribution scheme $E$ and the feedback controller (6) such that the plant can be stabilized with the largest stability margin. Specifically, our intention is to determine the distribution probabilities, $e_i$, $i = 0, 1, \ldots, N$, and controller gain, $K$, such that

(i) the plant is stabilized with the largest stability margin $\tau_{\max}$;

(ii) the average transmission energy is no larger than the given energy budget, i.e.,

$$J = \sum_{i=0}^{N} e_i\sigma_i h \leqslant \Delta, \tag{13}$$

where $\sigma_0 < \Delta < \sigma_N$.

## 3 Main results

In this section, first, sufficient conditions for exponentially almost sure stability of system (8) are given. Then, an optimal probability distribution scheme is proposed to minimize the parameter $z$, and hence, maximize the stability margin. Finally, a co-design procedure for the probability distribution scheme and a feedback controller are derived.

### 3.1 Stability analysis

For an NCS composed of subsystems (9) and (10), choose a piecewise-quadratic Lyapunov-like function candidate as follows:

$$V(k)=\begin{cases} V_{\rm o}(k)=x^{\rm T}(k)P_{\rm o}x(k), \text{ if } \gamma(k)=0,\\ V_{\rm c}(k)=x^{\rm T}(k)P_{\rm c}x(k), \text{ if } \gamma(k)=1,\end{cases} \tag{14}$$

where $0<P_{\rm o},\ P_{\rm c}\in\mathbb{R}^{n\times n}$.

Then, the exponentially almost sure stability of system (8) is given below.

**Theorem 1.** If the following conditions hold,

(1) there exist two constant scalars, $0<\eta_{\rm o}<1$ and $\eta_{\rm c}>1$, such that the positive-definite quadratic function $V(k)$ in (14) satisfies

$$\eta_{\rm o}V_{\rm o}(k+1)<V_{\rm o}(k),\quad \eta_{\rm c}V_{\rm c}(k+1)<V_{\rm c}(k); \tag{15}$$

(2) there exists a constant scalar, $\mu>1$, such that

$$V_{\rm o}(k)\leqslant\mu V_{\rm c}(k),\quad V_{\rm c}(k)\leqslant\mu V_{\rm o}(k), \tag{16}$$

for any $x(k)$;

(3) the transmission power and its distribution probabilities satisfy

$$(\ln\mu)z^2-0.5(\ln\eta_{\rm c}-\ln\eta_{\rm o}+2\ln\mu)z+0.5\ln\eta_{\rm c}>0, \tag{17}$$

where $z=\sum_{i=0}^{N}e_i\theta^{\sigma_i}$; then system (8) is exponentially almost surely stable with a stability margin equal to $\tau=(\ln\mu)z^2-0.5(\ln\eta_{\rm c}-\ln\eta_{\rm o}+2\ln\mu)z+0.5\ln\eta_{\rm c}$.

*Proof.* Without loss of generality, we assume that system (8) operates in an open-loop subsystem (9) during $[k_{2j},k_{2j+1})$ and in a closed-loop subsystem (10) during $[k_{2j+1},k_{2j+2})$, where $j=0,\ 1,\ldots,\ k_0=0$. Namely,

$$\gamma(k)=\begin{cases}0,\ \ k\in[k_{2j},\ k_{2j+1}),\\ 1,\ \ k\in[k_{2j+1},\ k_{2j+2}),\end{cases}$$

Choose a piecewise-quadratic Lyapunov-like function candidate as (14). For any $k>0$, it holds from (15) that

$$V(k)<\begin{cases}\eta_{\rm o}^{-(k-k_{2j})}V_{\rm o}(k_{2j}), & \text{if } k\in[k_{2j},\ k_{2j+1}),\\ \eta_{\rm c}^{-(k-k_{2j+1})}V_{\rm c}(k_{2j+1}), & \text{if } k\in[k_{2j+1},\ k_{2j+2}).\end{cases} \tag{18}$$

Therefore, if $k\in[\,k_{2j+1},\ k_{2j+2})$, it follows from (16) and (18) that

$$\begin{aligned} V(k)&<\eta_{\rm c}^{-(k-k_{2j+1})}V_{\rm c}(k_{2j+1})\leqslant\mu\eta_{\rm c}^{-(k-k_{2j+1})}V_{\rm c}(k_{2j+1}^-)\\ &<\mu\eta_{\rm c}^{-(k-k_{2j+1})}\eta_{\rm o}^{-(k_{2j+1}-k_{2j})}V_{\rm o}(k_{2j})\\ &\leqslant\cdots\\ &<\mu^{N(k)}\eta_{\rm c}^{-\alpha(k)}\eta_{\rm o}^{-k+\alpha(k)}V(0),\end{aligned}$$

where $k_{2j+1}^-$ denotes the time instant that is immediately before $k_{2j+1}$. Similarly, for $k\in[k_{2j},\ k_{2j+1})$, we have that

$$V(k)<\mu^{N(k)}\eta_{\rm c}^{-\alpha(k)}\eta_{\rm o}^{-k+\alpha(k)}V(0). \tag{19}$$

If a quadratic form is considered in the piecewise-Lyapunov-like function (14), then

$$V(k) \geqslant ||x(k)||^2/\max\{||P_{\rm o}^{-1}||,\ ||P_{\rm c}^{-1}\}; \tag{20}$$

on the other hand,

$$V(0) \leqslant \max\{||P_{\rm o}||,\ ||P_{\rm c}||\}||x(0)||^2. \tag{21}$$

Combining (19)–(21), we know that

$$||x(k)||^2 \leqslant a\mu^{N(k)}\eta_{\rm c}^{-\alpha(k)}\eta_{\rm o}^{-k+\alpha(k)}||x(0)||^2,$$

where $a = \max\{||P_{\rm o}^{-1}||,\ ||P_{\rm c}^{-1}||\} \times \max\{||P_{\rm o}||,\ ||P_{\rm c}||\}$.

Then, it follows that

$$\begin{aligned}\lim_{k\to\infty}(1/k)\ln||x(k)|| &\leqslant 0.5\left[\lim_{k\to\infty}(1/k)\ln a + \ln\mu\lim_{k\to\infty}N(k)/k - (\ln\eta_{\rm c}-\ln\eta_{\rm o})\lim_{k\to\infty}\alpha(k)/k - \ln\eta_{\rm o}\right]\\ &\quad+\lim_{k\to\infty}(1/k)\ln||x(0)||\\ &= 0.5\left[\ln\mu\lim_{k\to\infty}N(k)/k - (\ln\eta_{\rm c}-\ln\eta_{\rm o})\lim_{k\to\infty}\alpha(k)/k - \ln\eta_{\rm o}\right].\end{aligned}$$

From (11) and (12), we have

$$\begin{aligned}\lim_{k\to\infty}(1/k)\ln||x(k)|| &\leqslant 0.5[\ln\mu(-2z^2+2z) - (\ln\eta_{\rm c}-\ln\eta_{\rm o})(1-z) - \ln\eta_{\rm o}]\\ &= -(\ln\mu)z^2 + 0.5(\ln\eta_{\rm c}-\ln\eta_{\rm o}+2\ln\mu)z - 0.5\ln\eta_{\rm c} = -\tau\end{aligned}$$

almost surely, which means that the above inequality holds with a probability of one, i.e.,

$$\text{prob}\left\{\lim_{k\to\infty}(1/k)\ln||x(k)|| \leqslant -\tau\right\} = 1,$$

where $z = \sum_{i=0}^{N} e_i\theta^{\sigma_i}$ and $\tau = (\ln\mu)z^2 - 0.5(\ln\eta_{\rm c}-\ln\eta_{\rm o}+2\ln\mu)z + 0.5\ln\eta_{\rm c}$.

From (17), we know that $\tau > 0$. Then, according to Definition 1, we know that system (8) is exponentially almost surely stable within the stability margin $\tau = (\ln\mu)z^2 - 0.5(\ln\eta_{\rm c}-\ln\eta_{\rm o}+2\ln\mu)z + 0.5\ln\eta_{\rm c}$. This completes the proof.

**Remark 1.** Theorem 1 provides a practically testable condition for the exponential stability of system (8), as almost sure (sample path) behavior can be easily observed in practice. Unlike traditional methods of dealing with switching systems [16], the limiting but not instantaneous characteristic of the attention rate and chatter frequency is discussed in this study, and thus, a probability condition is obtained. The analysis of system stability is for the first time connected with the transmission power and its probability distribution.

**Corollary 1.** Suppose that conditions (1) and (2) in Theorem 1 hold. Then, system (8) is exponentially almost surely stable if the transmission power levels and their probability distribution satisfy

$$\sum_{i=0}^{N} e_i\theta^{\sigma_i} < z_1,$$

where $z_1 = [0.5(\ln\eta_{\rm c}-\ln\eta_{\rm o}+2\ln\mu) - \sqrt{0.25(\ln\eta_{\rm c}-\ln\eta_{\rm o}+2\ln\mu)^2 - 2\ln\eta_{\rm c}\ln\mu}]/2\ln\mu$.

*Proof.* Let $\tau > 0$; then, we have $z < z_1$ or $z > z_2$, where $z_2 = 0.5(\ln\eta_{\rm c}-\ln\eta_{\rm o}+2\ln\mu)/2\ln\mu + \sqrt{0.25(\ln\eta_{\rm c}-\ln\eta_{\rm o}+2\ln\mu)^2 - 2\ln\eta_{\rm c}\ln\mu}/2\ln\mu$. It is easy to know that $0 < z \leqslant 1$, so $z > z_2 > 1$ is out of the feasibility region and should be deleted. Then, the feasible region of $z$ such that $\tau > 0$ is given by $z \in (0,\ 1] \cap (-\infty, z_1)$. Let $\tau < 0$; then, we have $z_1 < z < z_2$. Note that $\tau(z=1) = 0.5\ln\eta_{\rm o} < 0$. Thus, $z = 1 \in (z_1, z_2)$, which means that $z_1 < 1$. Then, the feasible region of the variable$z$ such that $\tau > 0$ is reduced to $z \in (0, z_1)$. In other words, if $0 < z < z_1$, then we can find that $\tau > 0$, i.e., condition (3) in Theorem 1 holds. This completes the proof.

**Corollary 2.** Suppose that conditions (1) and (2) in Theorem 1 hold. If there is only one kind of transmission power level, i.e., $N = 1$ and $p_1 = 1$, then the problem in this study is reduced to a pure packet-dropout problem. From condition 3) in Theorem 1, we know that system (8) is exponentially almost surely stable if the packet-dropout rate $\bar{z}$ satisfies

$$(\ln \mu)\bar{z}^2 - 0.5(\ln \eta_c - \ln \eta_o + 2\ln \mu)\bar{z} + 0.5\ln \eta_c > 0.$$

By using the wireless channel model in (2) and from Corollary 1, we can obtain a stability condition for transmission power as $\sigma > \ln z_1 / \ln \theta$, which means that system (8) is exponentially almost surely stable if the transmission power is larger than $\ln z_1 / \ln \theta$.

### 3.2 Optimal probability distribution scheme

From the definition of $\tau = (\ln \mu)z^2 - 0.5(\ln \eta_c - \ln \eta_o + 2\ln \mu)z + 0.5\ln \eta_c$ and by the derivation method, it is easy to show that the stability margin $\tau$ decreases with the variable $z$ over $z \in (0, z_1)$, and smaller $z$ leads to a larger stability margin $\tau$, i.e., a better system performance. To obtain the largest stability margin $\tau_{\max}$, we need to find the minimal $z_{\min}$. To be specific, we are interested in finding a probability distribution scheme $E = [e_0, e_1, \ldots, e_N]$ that solves the following optimization problem.

**Problem 1.**

$$\min_{E} \quad z = \sum_{i=0}^{N} e_i \theta^{\sigma_i} \quad \text{s.t.} \quad \sum_{i=0}^{N} e_i \sigma_i h \leqslant \Delta, \quad \sum_{i=0}^{N} e_i = 1,$$

where $\sum_{i=0}^{N} e_i \sigma_i h \leqslant \Delta$ denotes the transmission energy constraint and $\sum_{i=0}^{N} e_i = 1$ denotes the probability distribution condition.

A necessary condition for an optimal probability distribution scheme for Problem 1 is given as bellow.

**Lemma 1.** Let $E^* = \{e_0^*, e_1^*, \ldots, e_N^*\}$ be an optimal scheme to Problem 1. Then,

$$\sum_{i=0}^{N} e_i^* \sigma_i h = \Delta. \tag{22}$$

*Proof.* see Appendix A.

Based on Lemma 1, the following result characterizes the optimal probability distribution schedule.

**Theorem 2.** The optimal solution to Problem 1 is given by the solution to the following set of equations:

$$\begin{cases} \partial f(e_0, e_1, \ldots, e_N, \lambda_1, \lambda_2)/\partial e_0 = 0, \\ \partial f(e_0, e_1, \ldots, e_N, \lambda_1, \lambda_2)/\partial e_1 = 0, \\ \quad\quad\quad \cdots \\ \partial f(e_0, e_1, \ldots, e_N, \lambda_1, \lambda_2)/\partial e_N = 0, \\ \partial f(e_0, e_1, \ldots, e_N, \lambda_1, \lambda_2)/\partial \lambda_1 = 0, \\ \partial f(e_0, e_1, \ldots, e_N, \lambda_1, \lambda_2)/\partial \lambda_2 = 0, \end{cases}$$

where

$$f = \sum_{i=0}^{N} e_i \theta^{\sigma_i} + \lambda_1 \left( \sum_{i=0}^{N} e_i - 1 \right)^2 + \lambda_2 \left( \sum_{i=0}^{N} e_i \sigma_i h - \Delta \right)^2.$$

Corollaries 3 and 4 give the simplest cases of optimal solutions to problem 1 when $N= 1$ and $N= 2$.

**Corollary 3.** If $N= 1$, i.e., there is only one kind of transmission power, then the sensor only needs to decide when to send data to the remote controller. According to Theorem 2, the optimal probability distribution scheme is given by

$$e_0^* = 1 - \Delta/(\sigma_1 h) \quad \text{and} \quad e_1^* = \Delta/(\sigma_1 h),$$

where $e_0^*$ and $e_1^*$ denote the probabilities of the sensor being in sleep mode and in work mode, respectively.

**Corollary 4.** If $N = 2$, the sensor has two transmission power levels: high, $\sigma_1$, and low, $\sigma_2$. Then, according to Theorem 2, the optimal solution is given by

$$e_0^* = 0, \quad e_1^* = \frac{\Delta/h - \sigma_2}{\sigma_1 - \sigma_2} \quad \text{and} \quad e_2^* = \frac{\sigma_1 - \Delta/h}{\sigma_1 - \sigma_2},$$

where $e_0^*$ denotes the probability of the sensor in sleep mode, and $e_1^*$ and $e_2^*$ denote the probability of the sensor using transmission powers $\sigma_1$ and $\sigma_2$ in work mode, respectively.

### 3.3 Controller design

When Problem 1 is solved, then the remaining work is to find a controller gain such that the system can be stabilized within the largest stability margin. Let $E^* = \{e_0^*, e_1^*, \ldots, e_N^*\}$ denote the optimal solution to Problem 1; then, based on Theorem 1, we can give a design procedure for controller gain, which is obtained using the following lemma.

**Lemma 2.** The following conditions are equivalent:

(i) there exist matrices $0 < P_{\rm c} \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{m\times n}$ such that

$$\eta_{\rm c}(A + BK)^{\rm T} P_{\rm c}(A + BK) - P_{\rm c} < 0;$$

(ii) there exist matrices $0 < X_{\rm c} \in \mathbb{R}^{n\times n}$ and $L_{\rm c} \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} -X_{\rm c} & \sqrt{\eta_{\rm c}}(AX_{\rm c} + BL_{\rm c})^{\rm T} \\ \sqrt{\eta_{\rm c}}(AX_{\rm c} + BL_{\rm c}) & -X_{\rm c} \end{bmatrix} < 0,$$

where $P_{\rm c} = X_{\rm c}^{-1}$, $K = L_{\rm c}P_{\rm c}$.

**Theorem 3.** System (8) can be stabilized by a state feedback controller (6) if there exist positive scalars $0 < \eta_{\rm o} < 1$, $\eta_{\rm c} > 1$, and $\mu > 1$ and matrices $0 < P_{\rm o} \in \mathbb{R}^{n\times n}$, $0 < X_{\rm c} \in \mathbb{R}^{n\times n}$, and $L_{\rm c} \in \mathbb{R}^{m\times n}$ such that

$$\eta_{\rm o} A^{\rm T} P_{\rm o} A - P_{\rm o} < 0, \tag{23}$$

$$\begin{bmatrix} -X_{\rm c} & \sqrt{\eta_{\rm c}}(AX_{\rm c} + BL_{\rm c})^{\rm T} \\ \sqrt{\eta_{\rm c}}(AX_{\rm c} + BL_{\rm c}) & -X_{\rm c} \end{bmatrix} < 0, \tag{24}$$

$$P_{\rm o} \leqslant \mu P_{\rm c}, \quad P_{\rm c} \leqslant \mu P_{\rm o}, \tag{25}$$

$$(\ln \mu)(z^*)^2 - 0.5(\ln \eta_{\rm c} - \ln \eta_{\rm o} + 2\ln \mu)z^* + 0.5 \ln \eta_{\rm c} > 0, \tag{26}$$

where $P_{\rm c}X_{\rm c} = I$ and $z^* = \sum_{i=0}^N e_i^* \theta^{\sigma_i}$. The controller gain is given by

$$K = L_{\rm c}P_{\rm c}. \tag{27}$$

*Proof.* From (23), it is easy to show that

$$\begin{aligned} \eta_{\rm o}V_{\rm o}(k+1) - V_{\rm o}(k) &= \eta_{\rm o}x^{\rm T}(k+1)P_{\rm o}x(k+1) - x^{\rm T}(k)P_{\rm o}x(k) \\ &= x^{\rm T}(k)(\eta_{\rm o}A^{\rm T}P_{\rm o}A - P_{\rm o})x(k) < 0, \end{aligned}$$

which means that

$$\eta_{\rm o}V_{\rm o}(k+1) < V_{\rm o}(k). \tag{28}$$

On the other hand, from (24) and Lemma 2, we have

$$\eta_{\rm c}(A + BK)^{\rm T} P_{\rm c}(A + BK) - P_{\rm c} < 0,$$

which means that

$$\begin{aligned} \eta_{\rm c}V_{\rm c}(k+1) - V_{\rm c}(k) &= \eta_{\rm c}x^{\rm T}(k+1)P_{\rm c}x(k+1) - x^{\rm T}(k)P_{\rm c}x(k) \\ &= \eta_{\rm c}x^{\rm T}(k)[(A+BK)^{\rm T}P_{\rm c}(A+BK) - P_{\rm c}]x(k) < 0, \end{aligned}$$

i.e.,

$$\eta_{\rm c} V_{\rm c}(k+1) < V_{\rm c}(k), \tag{29}$$

where $P_{\rm c}X_{\rm c} = I$.

Therefore, we can conclude that condition (1) in Theorem 1 is satisfied if Eqs. (23) and (24) hold.

It follows from (25) that

$$P_{\rm o} \leqslant \mu P_{\rm c}, \quad P_{\rm c} \leqslant \mu P_{\rm o},$$

which implies that $V_{\rm o}(k) \leqslant \mu V_{\rm c}(k)$ and $V_{\rm c}(k) \leqslant \mu V_{\rm o}(k)$, i.e., condition (2) in Theorem 1 holds.

Then, combining (26), (28), and (29) and using Theorem 1, system (8) can be exponentially stabilized by state-feedback controller (6) with a stability margin equal to $\tau = (\ln\mu)(z^*)^2 - 0.5(\ln\eta_{\rm c} - \ln\eta_{\rm o} + 2\ln\mu)z^* + 0.5\ln\eta_{\rm c}$. This completes the proof.

Note that $\tau = (\ln\mu)(z^*)^2 - 0.5(\ln\eta_{\rm c} - \ln\eta_{\rm o} + 2\ln\mu)z^* + 0.5\ln\eta_{\rm c} = (\ln\mu)z^*(z^*-1) + 0.5\ln\eta_{\rm c}(1-z^*) + 0.5z^*\ln\eta_{\rm o}$ and $0 < z^* < 1$. It is easy to show that larger $\eta_{\rm o}$ and $\eta_{\rm c}$ and smaller $\mu$ will lead to a larger stability margin, $\tau$, i.e., a better system performance. Thus, the largest stability margin can be achieved by maximizing the parameters $\eta_{\rm o}$ and $\eta_{\rm c}$ and minimizing the parameter $\mu$. However, the parameters $\eta_{\rm o}$, $\eta_{\rm c}$, and $\mu$ are coupled by the conditions (23)–(25), making it difficult for us to find the maximum values of $\eta_{\rm o}$ and $\eta_{\rm c}$ and the minimum value of $\mu$ simultaneously. So in this study, we must simplify our aim to find a sub-optimal stability margin with maximal $\eta_{\rm o}$ and $\eta_{\rm c}$ and a feasible $\mu$. The maximization of $\eta_{\rm o}$ and $\eta_{\rm c}$ can be realized by the following procedures using conditions (23) and (24):

(a) Maximize $\eta_{\rm o}$ s.t. $\eta_{\rm o} A^{\rm T} P_{\rm o} A < P_{\rm o}$;

(b) Maximize $\eta_{\rm c}$ s.t.

$$\begin{bmatrix} -X_{\rm c} & \sqrt{\eta_{\rm c}}(AX_{\rm c}+BL_{\rm c})^{\rm T} \\ \sqrt{\eta_{\rm c}}(AX_{\rm c}+BL_{\rm c}) & -X_{\rm c} \end{bmatrix} < 0.$$

In fact, by following the above two steps, we can obtain not only $\eta_{\rm o}^{\max}$ and $\eta_{\rm c}^{\max}$ but also the corresponding controller gain, given by $K = L_{\rm c}X_{\rm c}^{-1}$.

With the above-obtained matrices $P_{\rm o}$ and $P_{\rm c} = X_{\rm c}^{-1}$ in steps (a) and (b), we can find a feasible $\mu = \max\{\lambda(P_{\rm o}P_{\rm c}^{-1}), \lambda(P_{\rm c}P_{\rm o}^{-1})\}$ subject to condition (25).

Finally, based on the above discussion, we can now outline a co-design procedure for a probability-distribution scheme of power scheduling and controller design as Algorithm 1.

**Algorithm 1** (Co-design of power allocation probability distribution and controller gain)

1. Using Theorem 2 to find the optimal probability distribution, $E^* = \{e_0^*, e_1^*, \ldots, e_N^*\}$.
2. Find $\eta_{\rm o}^{\max}$ and matrix $0 < P_{\rm o} \in \mathbb{R}^{n\times n}$ by solving the problem in procedure (a).
3. Find $\eta_{\rm c}^{\max}$ and matrix $0 < P_{\rm c} \in \mathbb{R}^{n\times n}$ by solving the problem in procedure ($b$'). Matrix $P_{\rm c}$ can be obtained by $P_{\rm c} = X_{\rm c}^{-1}$.
4. Calculate the controller gain, i.e., $K = L_{\rm c}X_{\rm c}^{-1}$.
5. Let $\mu = \max\{\lambda(P_{\rm o}P_{\rm c}^{-1}), \lambda(P_{\rm c}P_{\rm o}^{-1})\}$.
6. Then, the stability margin is given by $\tau = (\ln\mu)(z^*)^2 - 0.5(\ln\eta_{\rm c}^{\max} - \ln\eta_{\rm o}^{\max} + 2\ln\mu)z^* + 0.5\ln\eta_{\rm c}^{\max}$, where $z^* = \sum_{i=0}^{N} e_i^*\theta^{\sigma_i}$.

**Remark 2.** From Lemma 2 in [17], we know that the linear matrix inequality (24) holds if and only if $\rho(A+BK) < \eta_{\rm c}^{-1}$. Thus, if the plant is totally controllable, then the eigenvalue of matrix $A+BK$ can be designed arbitrarily and we can always find a sufficiently large $\eta_{\rm c}$ such that the inequalities (24) and (26) hold at the same time, no matter what values of $\eta_{\rm o}$ and $\mu$ are used. In other words, if the plant is totally controllable, then we can always find a controller to stabilize the plant with a given energy budget. On the other hand, if the plant is controllable but not totally controllable, such $\eta_{\rm c}$ may not exist and inequality (17) cannot be guaranteed.

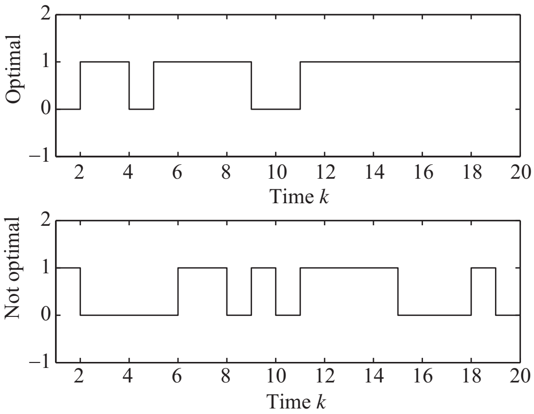


**Figure 2** Communication state.

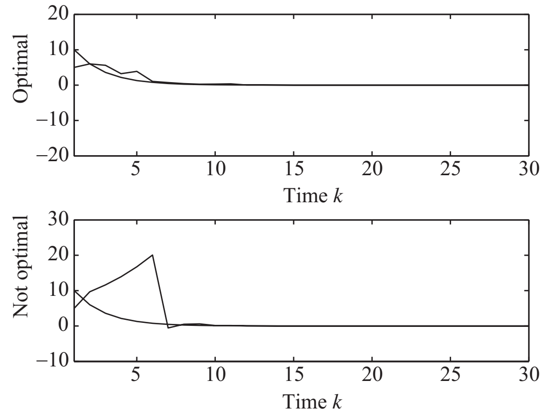


**Figure 3** System state.

## 4 Simulation

Consider a networked system whose dynamics are given by

$$x(k+1) = \begin{bmatrix} 0.6 & 0 \\ 0 & 1.2 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k).$$

It is assumed that the packet is transmitted to the controller via a wireless fading channel with $\theta = 0.2$. When the sensor is activated, there are three kinds of transmission power levels to be chosen, i.e., $\sigma_1 = 0.4$, $\sigma_2 = 0.6$, and $\sigma_3 = 1.2$. Then, $\sigma(k) \in \{0,\ 0.4,\ 0.6,\ 1.2\}$. The average transmission energy is given by $\Delta = 0.8$, where the sampling time is $h = 1$. Then, according to Theorem 2, the optimal solution to Problem 1 is given by $E^* : (0,\ 0,\ 2/3,\ 1/3)$.

For comparison, let us consider another probability distribution as $E_1 : (0,\ 2/3,\ 0,\ 1/3)$, which also satisfies the average transmission energy constraint.

Let $\eta_{\rm c} = 2.7$ and $\eta_{\rm o} = 0.69$. By computation, we can obtain

$$P_{\rm o} = \begin{bmatrix} 1.8179 & 0 \\ 0 & 1.0012 \end{bmatrix}, \quad P_{\rm c} = \begin{bmatrix} 1.0573 & 0 \\ 0 & 1.0455 \end{bmatrix}, \quad K = [\,0\ \ -1.1971\,].$$

Choose $\mu = 1.72$. Then, the maximal stability margin is given by $\tau_{\max} = 0.178$. The stability margin $\tau_1$ under scheduling policy $E_1$ is given by $\tau_1 = 0.094 < \tau_{\max}$.

The communication states under the two scheduling policies are given as in Figure 2, where 0 on the $y$-axis indicates that the packet dropout has occurred or that the sensor is in sleep mode; 1 on the $y$-axis indicates that the measurement has been transmitted to the remote controller successfully. Figure 2 shows that the number of systems in the open loop of policy $E^*$ is less than that of policy $E_1$, and that the switching number between the open and closed loops of policy $E^*$ is less than that of policy $E_1$. As shown in [18], if the total activation time of the unstable subsystem (i.e., open-loop subsystem) is relatively small compared with that of the stable subsystem (i.e., the closed-loop system) and the switching number is relatively small, then the exponential stability of the system can be guaranteed. From this theory, the performance of a system under policy $E^*$ should be better than that of policy $E_1$.

The states of the system under the two scheduling policies are given in Figure 3, which shows that the overshoot and stabilization times of policy $E^*$ are less than those of policy $E_1$. Thus, the performance of a system under policy $E^*$ is really better than that under policy $E_1$, which verifies our results.

## 5 Conclusion and future work

This study investigates NCSs with transmission energy constraints. We implement a Bernoulli process to describe the data packet dropouts due to channel fading, where a successful reception probability depends upon the transmission power level used. A new random power-scheduling policy is proposed, which makes it possible to connect the stability conditions with the transmission power and its probability distribution. We derive an explicit expression for the stability margin of NCSs and show how to compute the optimal power probability distribution scheme to maximize the stability margin. A co-design procedure of a feedback controller and probability distribution scheme is proposed in the last part of this study.

In the future, a challenge will be to extend the results to systems with more than one transmission node. Then, we need to decide when to schedule which node to send the data, and at what power level. It is also of interest to find an optimal power allocation that can stabilize the systems with the smallest amount of energy consumption.

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**Conflict of interest** The authors declare that they have no conflict of interest.

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## Appendix A Proof of Lemma 1

We prove this lemma by contradiction. Suppose that $E = [e_0, e_1, \ldots, e_N]$ is an optimal scheme for Problem 1 with $\sum_{i=0}^{N} e_i = 1$ and $\sum_{i=0}^{N} e_i\sigma_i h < \Delta$. Note that

$$\left(\Delta - \sum_{i=0}^{N} e_i\sigma_i h\right)/\sigma_N h = \delta > 0. \tag{A1}$$

Then, we can construct a new scheme $\hat{E} = [\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_N]$ based on $E$ according to the follow steps:

Step 1. Let $b_i(0) = e_i$, $y(0) = \delta$, $i = 0, 1, \ldots, N$, $m = 0$, and $k = 0$.

Step 2. If $b_m(k) \geqslant y(k)$, then set $\hat{e}_i = \begin{cases} b_i(k) + y(k), & i = N, \\ b_i(k) - y(k), & i = m, \\ b_i(k), & \text{otherwise}, \end{cases}$ and exit. Otherwise, go to Step 3.

Step 3. Let $b_i(k+1) = \begin{cases} b_i(k) + b_m(k), & i = N, \\ b_i(k) - b_m(k), & i = m, \\ b_i(k), & \text{otherwise}, \end{cases}$ $y(k+1) = \frac{\Delta - \sum_{i=0}^{N} b_i(k+1)\sigma_i h}{(\sigma_N - \sigma_{m+1})h}$, $m = m+1$, $k = k+1$ and go to Step 2.

Thus, we have

$$\sum_{i=0}^{N} \hat{e}_i = 1 \quad \text{and} \quad \sum_{i=0}^{N} \hat{e}_i\sigma_i h = \Delta, \tag{A2}$$

where $\begin{cases} \hat{e}_i > e_i, i = N, \\ \hat{e}_i = e_i, i = m+1, \ldots, N-1, \\ \hat{e}_i < e_i, i = 0, 1, \ldots, m. \end{cases}$

Since $\sum_{i=0}^{N} e_i = \sum_{i=0}^{N} \hat{e}_i = 1$, we have

$$\hat{e}_N - e_N = \sum_{i=0}^{m} (e_i - \hat{e}_i). \tag{A3}$$

Note that $\sigma_0 < \sigma_1 < \cdots < \sigma_m < \sigma_N$ and $0 < \theta < 1$; combining these with (A2), it is straightforward to show that

$$\begin{aligned} z(\hat{E}) - z(E) &= \sum_{i=0}^{N} \hat{e}_i\theta^{\sigma_i} - \sum_{i=0}^{N} e_i\theta^{\sigma_i} = (\hat{e}_N - e_N)\theta^{\sigma_N} - \sum_{i=0}^{m} (e_i - \hat{e}_i)\theta^{\sigma_i} \\ &< (\hat{e}_N - e_N)\theta^{\sigma_N} - \theta^{\sigma_N} \sum_{i=0}^{m} (e_i - \hat{e}_i) = \theta^{\sigma_N}\left[(\hat{e}_N - e_N) - \sum_{i=0}^{m} (e_i - \hat{e}_i)\right] = 0, \end{aligned} \tag{A4}$$

i.e., $z(\hat{E}) < z(E)$, which means that $E$ cannot be optimal. Therefore, any optimal solution to Problem 1 needs to satisfy (22). This completes the proof.