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Stable degree analysis for strategy profiles of evolutionary networked games

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Abstract In this paper, we investigate the stable degree of strategy profile for evolutionary networked games by using the semi-tensor product method, and present a number of new results. First, we propose the concept of k-degree stability for strategy profiles based on a normal evolutionary networked game model. Second, using the semi-tensor product of matrices, we convert the game dynamics with "best imitate" strategy updating rule into an algebraic form. Third, based on the algebraic form of the game, we analyzed the stable degree of strategy profile, and proposed two necessary and sufficient conditions for the k-degree stability of strategy profile. Furthermore, we discuss the computation problem of the transient time within which a disturbed strategy profile can be restored, and also establish an algorithm for the verification of the stable degree of strategy profile. The study of an illustrative example shows that the new results obtained in this paper are very effective.

Keywords evolutionary networked game, strategy profiles, stable degree, semi-tensor product of matrices

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1 Introduction

Evolutionary game theory was first proposed by biologists, which originated from the study of Smith and Prices in 1973 [1]. In most of the early work on evolutionary games, a basic assumption was that individuals are mixed uniformly and interact with all other players or some players randomly [2]. However, in many practical cases, this assumption is not realistic because individuals often interact only with a subset of the population. For instance, teenagers usually imitate the fashions of their close friends, while firms, who share capital or technology, might interact and be closely connected. Such a specific feature makes that a new class of game, called the evolutionary networked game [3], was put forward. In recent years, it has been a very hot topic in the game field. Nodes of the graph standing for the topology of a networked game denote players, and edges denote interaction relationship between players. Up to now, the evolutionary networked game has been investigated widely and a great deal of excellent results have been obtained [4–6].

The key issue of evolutionary game theory is the so-called evolutionary stability [7]. An evolutionary stable strategy (ESS) is a strategy which, under the influence of natural selection, is able to withstand the

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invasion of other mutation strategy. It should be pointed out that the ESS is a static concept that does not require solving time-dependent dynamical equations. Another important issue in evolutionary games is dynamical stability [7], which is closely related to local asymptotic stability of the "replicator dynamics" [8, 9]. The biological meaning of dynamical stability is that a population's genetic composition, which can be monomorphic or polymorphic, will be restored by selection after a small disturbance. Likewise, in socio-economic environments, there may exist "faulty" behavior during the game process. For instance, some players may act strong emotionally, or their temporary agents may not completely understand the game rule. If the evolutionary equilibrium is robust to small "faulty" changes in the population's behavior, one can conclude that it is dynamically stable. It is also noted that many other kinds of evolutionary stability have been presented and studied, such as stochastic stability [10], evolutionarily stable sets [11], and local stability [12].

As well known, there are two difficulties in the study of dynamical stability in finite evolutionary networked games, in which the numbers of both players and available strategies for each player are finite. The first one is how to establish a proper neighborhood to describe local changes based on strategy choice, so that we can analyze the largest permissible mutant degree to maintain the evolutionary stability. Another one is that there are no effective mathematical tool to describe an evolutionary networked game. To our best knowledge, most of the existing results on evolutionary networked games are mainly based on experiment, computer simulation and statistics [6]. Due to the shortage of systematic tools, it is difficult to deal with the dynamic process of evolutionary networked games and obtain some really meaningful theoretical results.

Recently, a novel matrix product, namely, the semi-tensor product of matrices, has been proposed by Cheng et al. [13] and then successfully applied to express and analyze finite-valued systems. Up to now, many fundamental results have been presented for the analysis and control of Boolean networks [13–21], the coloring problem [22], the design of shifting register [23], the fault detection [24], and so on. In addition, since the dynamics of a finite game can be modeled as a logical network [25], the semi-tensor product method has also been applied to the study of game theory [25–28].

In this paper, using the semi-tensor product method, we investigate the stable degree of strategy profile for evolutionary networked games. The main contributions of this paper are as follows: (i) The semi-tensor product method is first applied to the investigation of stable degree of strategy profile, and a new theoretical framework is established via this method; (ii) Some necessary and sufficient conditions are presented for the k-degree stability of strategy profile, which are easily checked with the help of MATLAB.

The rest of this paper is organized as follows. Section 2 contains the preliminaries on evolutionary networked games and the semi-tensor product of matrices. Section 3 investigates the stable degree of strategy profile, and presents the main results of this paper. In Section 4, an illustrative example is given to support our new results, which is followed by a brief conclusion in Section 5.

2 Preliminaries

In this section, we first present the basic finite evolutionary networked game model and the concept of k-degree stability for a strategy profile, and then list some preliminaries on the semi-tensor product of matrices, which is the main tool used in this paper.

2.1 Evolutionary networked game model

A normal evolutionary networked game consists of three basic components [25, 29]:

(i) A network topological structure $(\mathcal{V}, \mathcal{E})$ with its adjacency matrix $E = (e_{ij})_{n\times n}$, where $\mathcal{V} := \{1, 2, \dots, n\}$ is the set of vertices/players, $\mathcal{E} = \{(i, j) \mid \text{there exists interaction between players } i \text{ and } j\} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. Denote by $\mathcal{N}_i$ the neighborhood of player i, where $j \in \mathcal{N}_i$ if and only if $(i, j) \in \mathcal{E}$;

(ii) A two-person symmetric game at each period $G := (S,\ A)$, where $S := \{s_1, s_2, \ldots, s_l\}$ is the common strategy set for all players, $A := (a_{ij})_{l\times l}$ is the payoff matrix, and a_{ij} denotes the payoff of a player when it chooses the strategy s_i and its opponent chooses the strategy s_j;

(iii) A strategy updating rule $F := \{f_1, f_2, \ldots, f_n\}$, according to which players with bounded rationality update their strategies at each period

$$x_i(t) = f_i(x(t-\tau), x(t-\tau+1), \ldots, x(t-1)),\ x_i(t) \in S,$$

where $x_i(t)$ denotes the strategy of player i at time t, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is the strategy profile of all players at time t, and $0 \leqslant \tau \leqslant t$ means that the strategy profile at time t depends on the strategy choice of the previous τ steps.

To facilitate the analysis, we denote the above evolutionary networked game by $((\mathcal{V}, \mathcal{E}),\ G,\ F)$. Suppose that the game occurs in discrete time and repeats infinitely. The game dynamics is divided into three steps, that is, at each period, (i) players only play the game G with each of their local neighbors; (ii) collect an aggregate payoff

$$p_i\Big(x_i, x_j \mid j \in \mathcal{N}_i\Big) = \sum_{j\in\mathcal{N}_i} p_{ij}(x_i, x_j),\ x_i, x_j \in S,$$

where $p_{ij} : S\times S \to \mathbb{R}$ denotes the payoff of player i playing with its neighbor j; (iii) update their strategies for the next period according to F. Actually, there are many strategy updating rules in evolutionary networked games, such as “Fermi rule” [3], “myopic best response updating rule” [5]. It is noted that the different updating rules lead to different behavior of the dynamics. As long as the strategy updating rule is given, the game dynamics is determined. In this paper, we mainly consider the “best imitate” strategy updating rule, that is, each player imitates the strategy of its neighbors including itself who gains the highest payoff.

2.2 Stable degree of strategy profile

Dynamic stability is a basic issue in the study of evolutionary game theory [7]. In the real game cases, there may exist some “faulty” behaviors. For example, some players do not act as the per given updating rule at some moment, perhaps due to a temporary interruption of communication, or an error in personal judgement. Thus, to predict or control the behavior of players, one important work is to find out if the evolutionary equilibrium is robust to small “faulty” changes in the population’s behavior, that is, local stability [12]. However, for finite evolutionary networked games, as far as we know, there are few results on the analysis of local stability. Since the numbers of players and available strategies for each player are finite, perturbations in strategies cannot be infinitesimal.

In this part, based on the basic finite evolutionary networked game model, we propose the concept for k-degree stability of strategy profile.

Definition 1. In an evolutionary networked game, a strategy profile

$$s^* \in S^n := \underbrace{S \times \cdots \times S}_{n}$$

is called k-degree stable, if the number of mutant strategies in s^* is no more than k, the iterative sequence of strategy profiles will still converge to the eventual equilibrium s^* through evolution. Otherwise, the eventual equilibrium s^* may not be played.

Actually, k-degree stability we consider refers to the sequence of iterates $\{x(t)\}$ converges to a fixed point x^* of F when it starts “near” x^*, where F is the strategy updating rule. Thus, to describe the localized perturbations of the strategy profile, we need to define a “neighborhood” in the space of strategy profiles S^n.

Definition 2. Given $s^* = (s^*_{(1)}, \ldots, s^*_{(n)}) \in S^n$, the k-step neighborhood of s^* is defined as

$$V^k_{s^*} = \{s \in S^n \mid \|s - s^*\| \leqslant k,\ s \neq s^*\}, \tag{1}$$

where $\|\cdot\|$ is the Hamming distance, which denotes the numbers of different strategies between two strategy profiles.

Denote by $x(t;s)$ the strategy profile of all players at time t starting from an initial strategy profile s. Then, by Definition 2, we have the following proposition.

Proposition 1. Strategy profile s^* is k-degree stable if and only if

(C1) $\lim_{t\to+\infty} x(t;s) = s^*$, $\forall\ s \in V_{s^*}^k \bigcup\{s^*\}$, and

(C2) there exists an $s \in V_{s^*}^{k+1}$ such that $\lim_{t\to+\infty} x(t;s) \neq s^*$.

Remark 1. The number k determines the stable degree of strategy profile. The larger the k is, the more resistent the strategy profile is against the invasion of mutants. If $k = n$, then any mutant strategy profile will globally converge to s^*. Otherwise, only a part of strategy profiles converge to s^*, which can be called "local" or "neighbor" convergence.

As we know, most evolutionary games usually assume infinite population size or are defined in continuous strategy space. The perturbations in strategy choices can be infinitesimal and the game dynamics often is described by differential equations, such as replicator dynamics [9]. The tools to study the stability in general nonlinear dynamical systems can be applied, for example, Lyapunov stability theorem. Evolutionary game dynamics in a finite population mostly is modeled as a Markov process, which is not deterministic. However, for finite evolutionary networked games, the numbers of both players and available strategies for each player are finite. Moreover, the framework we consider here is polymorphic, deterministic, pure strategy model. The methods to deal with the above problems do not seem to come into play.

The aim of this paper is to investigate the stable degree of strategy profile by using a new method, semi-tensor product approach.

2.3 Semi-tensor product of matrices

This subsection provides basic notions and results on semi-tensor product of matrices. We refer to [13] for more details. We first list some useful notations.

- $\mathcal{D}_k := \{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\}$, and $\mathcal{D}_k^n := \underbrace{\mathcal{D}_k \times \cdots \times \mathcal{D}_k}_{n}$.
- $\Delta_n := \{\delta_n^k \mid 1 \leqslant k \leqslant n\}$, where δ_n^k denotes the k-th column of the identity matrix I_n.
- $\mathbf{1}_n := [\underbrace{1\ \cdots\ 1}_{n}]$.
- An $n \times t$ matrix M is called a logical matrix, if $M = [\delta_n^{i_1}\ \delta_n^{i_2}\ \cdots\ \delta_n^{i_t}]$. We express M briefly as $M = \delta_n[i_1\ i_2\ \cdots\ i_t]$.
- Given a matrix $A = (a_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$, $\mathrm{Col}_i(A)$ denotes the i-th column of A, $\mathrm{Row}_i(A)$ denotes the i-th row of A, and vector $V_r(A) = (a_{11}, a_{12}, \ldots, a_{1n}, \ldots, a_{m1}, a_{m2}, \ldots, a_{mn})^{\mathrm{T}}$ is the row stacking form of A.

Next, we recall some necessary preliminaries on the semi-tensor product of matrices, which will be used in the sequel.

Definition 3. [13] The semi-tensor product of the two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}), \tag{2}$$

where $\alpha = \mathrm{lcm}(n,p)$ is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

Remark 2. It is noted that when $n = p$, the semi-tensor product of A and B becomes the conventional matrix product. Thus, it is a generalization of the conventional matrix product. We can simply call it "product" and omit the symbol "$\ltimes$" without confusion.

Proposition 2. [13] The semi-tensor product has the following properties:

(i) Let $X \in \mathbb{R}^{t\times 1}$ be a column vector and $A \in \mathbb{R}^{m\times n}$. Then $X \ltimes A = (I_t \otimes A) \ltimes X$.

(ii) Let $X \in \mathbb{R}^{m\times 1}$ and $Y \in \mathbb{R}^{n\times 1}$ be two column vectors. Then $Y \ltimes X = W_{[m,n]} \ltimes X \ltimes Y$, where

$$W_{[m,n]} = \delta_{mn}[1\ \ m+1\ \ \cdots\ \ (n-1)m+1$$

$$\begin{matrix} 2 & m+2 & \cdots & (n-1)m+2 \\ & & \cdots & \\ m & m+m & \cdots & (n-1)m+m] \end{matrix}$$

is called the swap matrix [13].

By identifying $\frac{k-i}{k-1} \sim \delta_k^i,\ i=1,\ldots,k$, we have $\mathcal{D}_k \sim \Delta_k$, where "$\sim$" denotes the two different forms of the same object.

Lemma 1. [13] Let $f:\mathcal{D}_k^n \to \mathcal{D}_k$ be a k-valued logical function. Then, there exists a unique matrix $M_f \in \mathcal{L}_{k\times k^n}$, called the structural matrix of f, such that

$$f(x_1,x_2,\ldots,x_n) = M_f \ltimes_{i=1}^n x_i, \tag{3}$$

where $\ltimes_{i=1}^n x_i = x_1 \ltimes x_2 \ltimes \cdots x_n,\ x_i \in \Delta_k,\ i=1,2,\ldots,n.$

Finally, we list the structural matrices of two logical operators, which will be used later.

Lemma 2. [13] The dummy operator $\sigma_{d,k}$ is defined as $\sigma_{d,k}(p,q)=q,\ \forall\ p,\ q\in\mathcal{D}_k$. The structural matrix of $\sigma_{d,k}$ is $E_{d,k} := [\underbrace{I_k\ I_k \cdots I_k}_{k}]$, which has the following property:

$$E_{d,k} \ltimes p \ltimes q = q \text{ and } E_{d,k} \ltimes W_{[k]} \ltimes p \ltimes q = p.$$

Lemma 3. [13] Assuming that $x_i \in \Delta_k,\ i=1,2,\ldots,n$ and $x = \ltimes_{i=1}^n x_i$. Then, $x_i = S_{i,k}^n x,\ i = 1,2,\ldots,n$, where $S_{i,k}^n = \mathbf{1}_{k^{i-1}} \otimes I_k \otimes \mathbf{1}_{k^{n-i}}, i=1,2,\ldots,n.$

3 Main results

In this section, we investigate the stable degree of strategy profile in evolutionary networked games, and present the main results of this paper. Subsection 3.1 converts the game dynamics into an algebraic form, based on which some necessary and sufficient conditions for the k-degree stability are obtained in Subsection 3.2.

3.1 Algebraic formulation

In order to analyze the stable degree of strategy profile, this part establishes the algebraic formulation of evolutionary networked game $((\mathcal{V},\mathcal{E}),\ G,\ F)$ via the following two steps.

Firstly, convert the average payoff function of each player into an algebraic form. By identifying the i-th strategy $s_i \sim \delta_l^i$, we have $S \sim \Delta_l$, where $s_i \sim \delta_l^i$ means that the strategy $s_i \in S$ is equivalent to $\delta_l^i \in \Delta_l,\ i=1,\ldots,l$. Then, the average payoff function of player i can be expressed as

$$\begin{aligned} &\frac{1}{|\mathcal{N}_i|} p_i\Big(x_i(t), x_j(t) \mid j \in \mathcal{N}_i\Big) \\ &\quad = \frac{1}{|\mathcal{N}_i|} V_r^{\mathrm{T}}(A) \sum_{j\in\mathcal{N}_i} x_i(t)x_j(t) \\ &\quad = \frac{1}{|\mathcal{N}_i|} V_r^{\mathrm{T}}(A) \sum_{j\in\mathcal{N}_i} \Gamma_{ij}x(t) := M_{p_i}x(t), \end{aligned}$$

where $|\mathcal{N}_i|$ is the number of neighbors of player i, $M_{p_i} \in \mathbb{R}^{1\times l^n}$ is the structural matrix of the average payoff function, $x_i(t) \in \Delta_l$ is the strategy of player i at time t, $x(t) = \ltimes_{i=1}^n x_i(t) \in \Delta_{l^n}$ is the strategy profile at time t, and

$$\Gamma_{ij} = \begin{cases} (E_{d,l})^{n-2} W_{[l^j,l^{n-j}]} W_{[l^i,l^{j-i-1}]}, & i<j, \\ W_{[l]}(E_{d,l})^{n-2} W_{[l^i,l^{n-i}]} W_{[l^j,l^{i-j-1}]}, & i>j. \end{cases}$$

Denote

$$M = [M_{p_1}^{\mathrm{T}}, M_{p_2}^{\mathrm{T}}, \ldots, M_{p_n}^{\mathrm{T}}]^{\mathrm{T}} := [m_{ij}]_{n\times l^n}.$$

It is easy to see that m_{ij} represents the payoff of player i when the strategy profile at time t is $\delta_{l^n}^j$.

Secondly, according to the "best imitate" strategy updating rule, we determine the strategy profile of the next period. For each possible strategy profile $x(t) = \delta_{l^n}^\alpha$ $(1 \leqslant \alpha \leqslant l^n)$ at time t, we consider the α-th column of matrix M. For each player i, find the set of neighbors including itself who has the highest payoff, and denote the set by

$$Q_i = \Big\{ j_\alpha \mid m_{j_\alpha \alpha} \geqslant m_{j\alpha},\ \forall j \in \mathcal{N}_i \cup \{i\} \Big\}. \tag{4}$$

If $i \notin Q_i$, then the i-th player chooses its strategy at time $t+1$ according to the following rule:

$$x_i(t+1) = x_{j_\alpha}(t), \tag{5}$$

where $j_\alpha = \max\{i \mid i \in Q_i\}$. Otherwise, considering the cost of strategy transformation, the strategy of player i remains unchanged, i.e., $x_i(t+1) = x_i(t)$. Based on Lemma 3, we have $x_i(t+1) = L_i x(t)$, where

$$\mathrm{Col}_\alpha(L_i) = \begin{cases} S_{j_\alpha,l}^n \delta_{l^n}^\alpha, & i \notin Q_i, \\ S_{i,l}^n \delta_{l^n}^\alpha, & i \in Q_i. \end{cases}$$

Then, the algebraic form of the game dynamics is

$$x(t+1) = Lx(t), \tag{6}$$

where $x(t) = \ltimes_{i=1}^n x_i(t) \in \Delta_{l^n}$ and $\mathrm{Col}_i(L) = \mathrm{Col}_i(L_1) \ltimes \cdots \ltimes \mathrm{Col}_i(L_n),\ i \in N = \{1, 2, \dots, n\}$.

Remark 3. The "best imitate" strategy updating rule is one of the typical strategy updating rules used very frequently in evolutionary networked games. For many other strategy updating rules, similar ideas can be applied to convert the game dynamics into an algebraic form. In [29], for the unconditional imitation with equal probability updating rule, it may involve the mixed strategy case, which is not in the scope of this paper and will be a topic of our future research.

Remark 4. It is worth noting that all the characteristics of the game $((\mathcal{V}, \mathcal{E}),\ G,\ F)$ can be revealed completely by the game transition matrix L in (6). Thus, one can analyze the dynamic process of the game by investigating the properties of L.

Remark 5. Computational complexity is still a challenging problem in the algorithm. It should be pointed out that L is a sparse matrix in which most of the elements are zero. Though the dimension is $l^n \times l^n$, it does not need a lot of storage spaces and can be calculated relatively fast by a computer.

3.2 Stable degree analysis of strategy profile

In this subsection, based on the algebraic formulation of game dynamics, we analyze the stable degree of strategy profile, and present two necessary and sufficient conditions for the k-degree stability.

Using the vector form of logical variables, we have $s \sim \delta_{l^n}^j$, where $s \in S^n$ and $j \in \{1, 2, \dots, l^n\}$. Then, given a strategy profile $s^* = \delta_{l^n}^{j_0}$, the k-step neighborhood of s^* can be rewritten as

$$V_{s^*}^k = \left\{ s = \delta_{l^n}^j \;\middle|\; n - \sum_{i=1}^n V_r^{\mathrm{T}}(I_l) S_{i,l}^n (I_{l^n} \otimes S_{i,l}^n) \delta_{l^n}^{j_0} \delta_{l^n}^j \leqslant k, j \neq j_0 \right\}, \tag{7}$$

where $V_r(I_l)$ is the row stacking form of the identity matrix I_l. For statement ease, we define a set as $\widehat{V}_{s^*}^k = \{j \mid \delta_{l^n}^j \in V_{s^*}^k\}$.

Since the calculation of $V_{s^*}^k$ is fundamental for our main result, we would like to find a simple formula to get the k-step neighborhood of a strategy profile. First, we consider a special strategy profile $s^* = \delta_{l^n}^1$, that is, all players choose the strategy s_1.

Construct a sequence of row vectors $\{\Gamma_i | i = 1, 2, \dots\}$ as

$$\Gamma_1 = [\underbrace{0\ 1\ 1\ \cdots\ 1}_{l}] \in \mathbb{R}^{1\times l},$$

$$\Gamma_i = [\underbrace{\Gamma_{i-1} \vdots \Gamma_{i-1} + \mathbf{1}_{l^{i-1}} \vdots \Gamma_{i-1} + \mathbf{1}_{l^{i-1}} \vdots \cdots \vdots \Gamma_{i-1} + \mathbf{1}_{l^{i-1}}}_{l}] \in \mathbb{R}^{1\times l^i},\ i = 2, 3, \dots,$$

where $\mathbf{1}_n := [\underbrace{1\ 1\ \cdots\ 1}_{n}]$.

By this construction, it is easy to obtain the following lemma.

Lemma 4. Consider the special strategy profile $s^* = \delta_{l^n}^1$. Then, the k-step neighborhood of s^* can be calculated by

$$V_{s^*}^k = \{\delta_{l^n}^j \mid \mathrm{Col}_j(\Gamma_n) \leqslant k\}.$$

Remark 6. For a general strategy profile $s^* = \delta_{l^n}^{j_0} = \delta_l^{j_1}\delta_l^{j_2}\cdots\delta_l^{j_n}$, we can use a coordinate transformation $y_i = T_i x_i$, where $T_i = \delta_l[\, l-j_i+2\ \ l-j_i+3\ \cdots\ l\ 1\ 2\ \cdots\ l-j_i+1]$, under which the strategy profile $s^* = \delta_{l^n}^{j_0}$ becomes $\widetilde{s}^* = \delta_{l^n}^1$. Meanwhile, the algebraic form of the game dynamics (6) is transformed into

$$y(t+1) = TLT^{-1}y(t),$$

where $T = T_1(I_l \otimes T_2)(I_{l^2} \otimes T_3)\cdots(I_{l^{n-1}} \otimes T_n)$ and $y(t) = \ltimes_{i=1}^n y_i(t) \in \Delta_{l^n}$. Thus, Lemma 4 can also be applied to a general case.

In the following, we give a simple example to illustrate how to calculate $V_{s^*}^k$.

Example 1. Suppose $n=4,\ l=2,\ s^* = \delta_{16}^1$. First, construct $\Gamma_i,\ i=1,2,3,4$ as follows

$$\Gamma_1 = [0\ 1],\ \Gamma_2 = [0\ 1\ 1\ 2],\ \Gamma_3 = [0\ 1\ 1\ 2\ 1\ 2\ 2\ 3],$$

$$\Gamma_4 = [0\ 1\ 1\ 2\ 1\ 2\ 2\ 3\ 1\ 2\ 2\ 3\ 2\ 3\ 3\ 4].$$

By Lemma 4, it is easy to obtain that $\widehat{V}_{s^*}^1 = \{2,3,5,9\}$, $\widehat{V}_{s^*}^2 = \{2,3,4,5,6,7,9,10,11,13\}$.

Next, we analyze the stable degree of strategy profile s^*. For $t \in \mathbb{Z}_+$ and the given strategy profile $s^* = \delta_{l^n}^{j_0}$, let $R_t(j_0)$ be a series of sets generated inductively by

$$\begin{aligned}
&R_1(j_0) = \{j \mid \mathrm{Col}_j(L) = \delta_{l^n}^{j_0}\},\\
&R_2(j_0) = \{j \mid \mathrm{Col}_j(L) = \delta_{l^n}^{j_1},\ \forall\ j_1 \in R_1(j_0)\backslash\{j_0\}\},\\
&R_t(j_0) = \{j \mid \mathrm{Col}_j(L) = \delta_{l^n}^{j_{t-1}},\ \forall\ j_{t-1} \in R_{t-1}(j_0)\},\quad t \geqslant 3,
\end{aligned}$$

where $j \in R_t(j_0)$ indicates that the strategy profile $\delta_{l^n}^j$ can reach s^* at the t-th step under the given strategy updating rule of the game. Then, based on the algebraic formulation (6) of game dynamics, we have the following result.

Theorem 1. Consider the evolutionary networked game $((\mathcal{V},\mathcal{E}),\ G,\ F)$ with its algebraic form (6). Then, strategy profile $s^* = \delta_{l^n}^{j_0}$ is k-degree stable if and only if

$$\begin{cases} j_0 \in R_1(j_0),\\ \widehat{V}_{s^*}^k \subseteq \bigcup_{t=1}^{l^n} R_t(j_0),\\ \widehat{V}_{s^*}^{k+1} \not\subseteq \bigcup_{t=1}^{l^n} R_t(j_0). \end{cases} \tag{8}$$

Proof. (Sufficiency) Assuming that Eq. (8) is satisfied, we only need to prove (C1) and (C2) hold (see Proposition 1).

Since $j_0 \in R_1(j_0)$, one can obtain $Ls^* = \mathrm{Col}_{j_0}(L) = \delta_{l^n}^{j_0}$. Hence, $\delta_{l^n}^{j_0}$ is a fixed point of (6), that is, once the strategy profile $s^* = \delta_{l^n}^{j_0}$ is chosen, it will remain unchanged forever under the strategy updating rule of the game. Moreover, for any $j \in \widehat{V}_{s^*}^k$, since

$$\widehat{V}_{s^*}^k \subseteq \bigcup_{t=1}^{l^n} R_t(j_0),$$

we have $j \in \bigcup_{t=1}^{l^n} R_t(j_0)$. Then there exists a positive integer $\widetilde{t} \leqslant l^n$, such that $j \in R_{\widetilde{t}}(j_0)$, which means that strategy profile $\delta_{l^n}^j$ will converge to s^* at the $\widetilde{t}$-th step. Based on the above discussion, we conclude that (C1) holds.

Moreover, from the condition $\widehat{V}_{s^*}^{k+1} \not\subseteq \bigcup_{t=1}^{l^n} R_t(j_0)$, one can see that there exists a $j \in \widehat{V}_{s^*}^{k+1}$ such that $j \notin \bigcup_{t=1}^{l^n} R_t(j_0)$. Now, we show that $\lim_{t\to+\infty} x(t;\delta_{l^n}^j) \neq s^*$. In fact, if $\lim_{t\to+\infty} x(t;\delta_{l^n}^j) = s^*$, then, there exists a $\tau \geqslant l^n+1$, such that $x(\tau;\delta_{l^n}^j) = s^*$. However, because there are only l^n different strategy profiles in the game, there must exist two strategy profiles satisfying $x(\tau_1;\delta_{l^n}^j) = x(\tau_2;\delta_{l^n}^j)$, $\tau_1,\tau_2 \leqslant \tau$. In this case, a cycle of strategy profiles or another fixed point appears, which is a contradiction to the fact that s^* is an only local fixed point of (6). Therefore, (C2) is satisfied and the sufficiency is completed.

(Necessity) Suppose that strategy profile $s^* = \delta_{l^n}^{j_0}$ is k-degree stable. From Definition 1, it is easy to see that $j_0 \in R_1(j_0)$.

On the other hand, similar to the proof of sufficiency, we know that for an arbitrary strategy profile s, if $\lim_{t\to+\infty} x(t;s) = s^*$, then the strategy profile s will converge to s^* within l^n steps. Therefore, the set $\{\delta_{l^n}^j \mid j \in \bigcup_{t=1}^{l^n} R_t(j_0)\}$ contains all the strategy profiles that can converge to s^* through evolution. Thus, the conditions (C1) and (C2) guarantee that

$$\widehat{V}_{s^*}^{k} \subseteq \bigcup_{t=1}^{l^n} R_t(j_0) \text{ and } \widehat{V}_{s^*}^{k+1} \not\subseteq \bigcup_{t=1}^{l^n} R_t(j_0),$$

and the proof is completed.

From Theorem 1, to verify the stable degree, we need to calculate l^n sets. However, for a given strategy profile, some computations are redundant since the mutant strategy profile may be restored in less than l^n steps. To reduce the computation, we now consider the transient time, i.e., the longest time, needed to restore the mutant strategy profile $\delta_{l^n}^{j_0}$ through evolution and then remain unchanged.

First, construct a sequence of row vectors $\xi_1,\ldots,\xi_t,\ldots$ by the following equation:

$$\xi_t = \xi_{t-1}L = \xi_0 L^t,\ t \geqslant 1, \tag{9}$$

where $\xi_0 = (\delta_{l^n}^{j_0})^{\mathrm{T}}$, and $\xi_t = [\xi_{t,1},\ldots,\xi_{t,l^n}] \in \mathcal{B}_{1\times l^n}$.

Second, define an order, denoted by "$\succeq$", on the m-dimensional vector space $\mathbb{R}^m$ as follows:

$$(x_1,x_2,\ldots,x_m) \succeq (y_1,y_2,\ldots,y_m),$$

if and only if $x_i \geqslant y_i,\ i=1,2,\ldots,m$. Similarly, orders "$\preceq$" and "$\prec$" denote $x_i \leqslant y_i$ and $x_i < y_i$, respectively. Based on the above, we have the following proposition.

Proposition 3. Suppose that $\xi_{1,j_0} = 1$, then for any $t \geqslant 1$, we have $\xi_{t+1} \succeq \xi_t$, that is, $\xi_{t+1,j} \geqslant \xi_{t,j}$, $j = 1,2,\ldots,l^n$.

Proof. We prove it by mathematical induction. First, we show that $\xi_2 \succeq \xi_1$. Note that $\xi_{1,j_0} = 1$, then it follows that

$$\begin{aligned}\xi_2 &= \xi_1 L = \sum_{i=1}^{l^n} \xi_{1,i}\mathrm{Row}_i(L)\\ &= \mathrm{Row}_{j_0}(L) + \sum_{i=1,i\neq j_0}^{l^n} \xi_{1,i}\mathrm{Row}_i(L)\\ &= \xi_1 + \sum_{i=1,i\neq j_0}^{l^n} \xi_{1,i}\mathrm{Row}_i(L) \succeq \xi_1,\end{aligned}$$

which implies that the conclusion is true for $t=1$.

Assuming that $\xi_t \succeq \xi_{t-1},\ t\geqslant 2$, we consider the case of $t+1$. In this case, one can obtain

$$\begin{aligned}\xi_{t+1} &= \xi_t L = \sum_{i=1}^{l^n} \xi_{t,i}\mathrm{Row}_i(L)\\ &\succeq \sum_{i=1}^{l^n} \xi_{t-1,i}\mathrm{Row}_i(L) = \xi_{t-1}L = \xi_t,\end{aligned}$$

that is, $\xi_{t+1} \succeq \xi_t$. By induction, the conclusion holds.

From Proposition 3 and the fact that $L \in \mathcal{L}_{l^n \times l^n}$, there must exist an integer τ ($0 \leqslant \tau \leqslant l^n$) such that $\xi_\tau = \xi_{\tau+1}$. Let

$$\tau_0 = \min\{\tau \mid \xi_\tau = \xi_{\tau+1}\}. \tag{10}$$

It is obvious that $\tau_0 \leqslant l^n$. Then, we have the following result.

Theorem 2. For the given strategy profile $s^* = \delta_{l^n}^{j_0}$, τ_0 defined in (10) is just the transient time.

Proof. First, we define a series of sets as follows:

$$\Omega_t = \{i \mid \xi_{t,i} \neq 0,\ i = 1, 2, \ldots, l^n\},\ t = 1, 2, \ldots.$$

Considering that strategy profile s^* remains unchanged under the strategy updating rule, one can see that $L\delta_{l^n}^{j_0} = \delta_{l^n}^{j_0}$, which yields $\xi_{1,j_0} = 1$. Moreover, since $\xi_0 = (\delta_{l^n}^{j_0})^{\mathrm{T}}$ and $\xi_1 = \xi_0 L = \mathrm{Row}_{j_0}(L)$, it is easy to obtain

$$L\delta_{l^n}^{i_1} = \delta_{l^n}^{j_0},\ \forall\ i_1 \in \Omega_1,$$

which implies that $\left\{\delta_{l^n}^{i_1} | i_1 \in \Omega_1\right\}$ contains all the strategy profiles, which reach s^* in one step.

Similarly, from the equation $\xi_2 = \xi_1 L = \xi_0 L^2$, we can obtain

$$L^2\delta_{l^n}^{i_2} = \delta_{l^n}^{j_0},\ \forall\ i_2 \in \Omega_2.$$

Therefore, $\left\{\delta_{l^n}^{i_2} | i_2 \in \Omega_2\right\}$ contains all the strategy profiles, which reach s^* at the second step.

Generally speaking, the column indexes of nonzero elements in $\xi_t,\ t \geqslant 3$, corresponds to strategy profiles that can reach s^* at the t-th step, that is, $L^t\delta_{l^n}^{i_t} = \delta_{l^n}^{j_0},\ \forall\ i_t \in \Omega_t$. Combining this and the definition of τ_0, when $t = \tau_0$, all the strategy profiles that can reach s^* under the strategy updating rule are obtained. Hence, τ_0 defined in (10) is the transition time.

From Theorem 2, the following result is obvious.

Corollary 1. $s^* = \delta_{l^n}^{j_0}$ is k-degree stable if and only if

$$\eta_k \preceq \xi_{\tau_0} \prec \eta_{k+1},$$

where τ_0 is the transient time, and $\eta_k = \sum_{i \in \widehat{V}_{s^*}^k} (\delta_{l^n}^i)^{\mathrm{T}}$, $1 \leqslant k \leqslant l^n$.

It should be pointed out that when the number of mutant strategies in s^* is no more than k, s^* will be restored within τ_0^* steps, where $\tau_0^* = \min\{\tau \mid \xi_\tau \succeq \eta_k\}$. With this, we have the following corollary.

Corollary 2. If strategy profile $s^* = \delta_{l^n}^{j_0}$ is k-degree stable, then $\mathrm{Col}_j(L^{\tau_0^*}) = \delta_{l^n}^{j_0},\ \forall\ j \in \widehat{V}_{s^*}^k$.

Based on Corollary 1, we establish an algorithm to calculate the stable degree of strategy profile $s^* = \delta_{l^n}^{j_0}$.

Algorithm 1

(1) Setting $\xi_0 = (\delta_{l^n}^{j_0})^{\mathrm{T}}$, calculate the vector ξ_1 and judge whether $\xi_{1,j_0} = 1$. If $\xi_{1,j_0} \neq 1$, then stop the computation. Otherwise, go to next step;

(2) Compute $\xi_t,\ 2 \leqslant t \leqslant \tau_0$, by the equation (9), where τ_0 is defined in (10);

(3) Calculate the stable degree of strategy profile s^* as follows:

$$k = \min\left\{\alpha \;\middle|\; n - \sum_{i=1}^{n} V_r^{\mathrm{T}}(I_l) S_{i,l}^n (I_{l^n} \otimes S_{i,l}^n) \delta_{l^n}^{j_0} \delta_{l^n}^{j} = \alpha,\ \xi_{\tau_0,j} = 0,\ 1 \leqslant j \leqslant l^n\right\} - 1.$$

4 An illustrative example

In this section, we present an illustrative example to show how to use our main results to investigate the stable degree of strategy profile for evolutionary networked games.

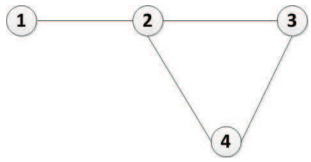

Figure 1 The network of the game in Example 2.

Example 2. Consider an evolutionary networked game, described as follows:

- its network topological structure is shown in Figure 1;
- its player set is $N = \{1, 2, 3, 4\}$, and the basic game between two players is Stag-hunt Game [30], where the strategy set is {“stag-hunt”, “hare-hunt”} and the corresponding payoff matrix is

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix};$$

- the adopted strategy updating rule is “best imitate” strategy updating rule.

From the payoff matrix A of Stag-hunt Game, we can see that each player will gain the maximum benefit when all the players adopt “stag-hunt” strategy. In the following, we use the results obtained in this paper to analyze the stable degree of the strategy profile.

First, based on the results presented in Section 3.2, we convert the game dynamics into an algebraic form. Using the vector form of logical variables, by identifying the strategy“stag-hunt”$\sim \delta_2^1$, and “hare-hunt”$\sim \delta_2^2$, the payoff function of each player can be calculated in the following way:

$$\begin{aligned}
p_1(x(t)) &= V_r^{\mathrm{T}}(A)x_1(t)x_2(t) = V_r^{\mathrm{T}}(A)(E_d)^2 W_{[4]}x(t), \\
p_2(x(t)) &= V_r^{\mathrm{T}}(A)[x_2(t)x_1(t) + x_2(t)x_3(t) + x_2(t)x_4(t)] \\
&= V_r^{\mathrm{T}}(A)\big[W_{[2]}(E_d)^2 W_{[4]} + (E_d)^2 W_{[8,2]} + (E_d)^2 W_{[4,2]}\big]x(t), \\
p_3(x(t)) &= V_r^{\mathrm{T}}(A)[x_3(t)x_2(t) + x_3(t)x_4(t)] = V_r^{\mathrm{T}}(A)\big[W_{[2]}(E_d)^2 W_{[8,2]} + (E_d)^2\big]x(t), \\
p_4(x(t)) &= V_r^{\mathrm{T}}(A)[x_4(t)x_2(t) + x_4(t)x_3(t)] = V_r^{\mathrm{T}}(A)\big[W_{[2]}(E_d)^2 W_{[4,2]} + W_{[2]}(E_d)^2\big]x(t),
\end{aligned}$$

where $x(t) = \ltimes_{i=1}^4 x(t) \in \Delta_{16}$. It is easy to obtain the average payoff matrix as

$$M = \begin{bmatrix} 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 3 & \frac{3}{2} & 1 & 1 & \frac{3}{2} & 0 & 1 & 1 & 3 & \frac{3}{2} & 1 & 1 & \frac{3}{2} & 0 & 1 & 1 \\ 3 & 1 & \frac{3}{2} & 1 & \frac{3}{2} & 1 & 0 & 1 & 3 & 1 & \frac{3}{2} & 1 & \frac{3}{2} & 1 & 0 & 1 \end{bmatrix}.$$

Thus, the algebraic form of game dynamics is given as

$$x(t+1) = Lx(t), \tag{11}$$

where $L = \delta_{16}[1\ 1\ 1\ 4 \vdots 9\ \ 16\ \ 16\ \ 16 \vdots 1\ \ 9\ \ 9\ \ 16 \vdots 9\ \ 16\ \ 16\ \ 16]$.

Next, we investigate the stable degree of strategy profile $s^* = \delta_{16}^1$, which stands for that all players choose the “stag-hunt” strategy. Since $\mathrm{Col}_1(L) = \delta_{16}^1$, s^* is a fixed point of (11), with which we obtain $R_1(1) = \{1, 2, 3, 9\}, R_2(1) = \{5, 10, 11, 13\}$, and $R_t(1) = \emptyset,\ 3 \leqslant t \leqslant 16$.

It is noted that the 1-step neighborhood and 2-step neighborhood of s^* are $\widehat{V}_{s^*}^1 = \{2, 3, 5, 9\}$ and $\widehat{V}_{s^*}^2 = \{4, 6, 7, 10, 11, 13\} \bigcup \widehat{V}_{s^*}^1$, respectively. Thus, we have

$$\widehat{V}_{s^*}^1 \subseteq \bigcup_{t=1}^{16} R_t(1) \text{ and } \widehat{V}_{s^*}^2 \not\subseteq \bigcup_{t=1}^{16} R_t(1).$$

Hence, the stable degree of s^* is $k = 1$, which implies that strategy profile s^* will be chosen again through evolution and remain unchanged when the number of mutant strategies in s^* is no more than one.

Finally, we calculate the transient time. Letting $\xi_0 = (\delta_{16}^1)^{\mathrm{T}}$ and by (9), we have $\xi_1 = [0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$, $\xi_2 = [0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0]$, and $\xi_3 = \xi_2$, which implies that the transient time of s^* is $\tau_0 = 2$. Thus, it concludes that when a mutation arises in s^*, it will never be chosen again by players if s^* is not restored within two steps.

5 Conclusion

In this paper, we have investigated the stable degree of strategy profile for evolutionary networked games via the semi-tensor product method. The concept of k-degree stability of strategy profile is proposed based on a normal evolutionary networked game model. Using semi-tensor product of matrices, the algebraic form of the game dynamics with "best imitate" strategy updating rule is established, based on which and by defining "neighborhood" in the space of strategy profiles, two necessary and sufficient conditions have been obtained for the k-degree stability of strategy profiles. Furthermore, a method of calculating the transient time is also developed for disturbed strategy profiles, and an algorithm has been established to compute the stable degree for a given strategy profile. The illustrative example provided has shown that the new results obtained in this paper are effective.

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Conflict of interest The authors declare that they have no conflict of interest.

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