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Iterative parameter estimate with batched binary-valued observations

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Abstract In this paper, we consider linear system identification with batched binary-valued observations. We constructed an iterative parameter estimate algorithm to achieve the maximum likelihood (ML) estimate. The first interesting result was that there exists at most one finite ML solution for this specific maximum likelihood problem, which was induced by the fact that the Hessian matrix of the log-likelihood function was negative definite under binary data and Gaussian system noises. The global concave property and local strongly concave property of the log-likelihood function were obtained. Under mild conditions on the system input, we proved that the ML function has a unique maximum point. The second main result was that the ML estimate was consistent under persistent excitation inputs, which infers the effectiveness of ML estimate. Finally, the proposed iterative estimate algorithm converged to a fixed vector with an exponential rate that was proved by constructing a Lyapunov function. A more interesting result was that the limit of the iterative algorithm achieved the maximization of the ML function. Numerical simulations are illustrated to support the theoretical results obtained in this paper well.

Keywords Binary-valued observation, maximum likelihood estimate, strongly convex, system identification, exponential rate

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1 Introduction

System identification with set-valued observations has been shown to have wide applications in different fields, such as networked control systems, biological networks, and communication systems [1]. The set-valued sensor poses substantial difficulties since only limited information is available to the system identification. Hence, the related results are sparse compared with its significance.

To overcome the difficulties caused by the lack of information, some assumptions are usually adopted, such as the specific periodical form [2], controllable quantization function [3], and so on. Here we consider the worst scenario where the input signals are totally independent of each other and the quantization function is fixed with only 1-bit observation.

In most cases, the consistency property of the estimation algorithm is the priority, that is, the convergence rate of the estimated parameters to the real ones when sample size tends to be infinity [3, 4]. The

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consistency problem is very important because it shows us the sample size needed to achieve an accurate estimation of the parameter. Here, we want to focus on the estimation problem from another view. With sample size fixed, how to construct an algorithm to obtain the parameter estimation as accurately as possible. Latest research shows that a set-valued system can result in a natural likelihood function, and the system identification problem can be converted to a corresponding maximum likelihood (ML) problem, which can shed light on the study of the set-valued system identification [5, 6]. Under the ML principle, we need to construct a rational likelihood function and estimate the parameter that maximizes it. The estimated parameter, which is called ML estimate, usually has many significant properties such as strong consistency [4]. The detailed advantages and more formal justification for the use of MLE is provided in [7]. In fact, some proposed estimates in previous work are ML estimate essentially [6].

In the ML field, the expectation maximization (EM) algorithm has obtained immense success [8]. The earliest reference to literature on an EM-type of algorithm is [9], which considered the estimation of parameters of a mixture of two univariate normals. And, in the following decades, many studies based on the EM algorithm were given to handle incomplete data [10–15]. Finally, in 1977, the general formulation of the EM algorithm was proposed by [16], which accelerated the broad use of the EM algorithm in many fields, such as mixture density problem [17], confined and censored Normal Data [18], neural network with hidden units [19] and so on.

As an optimization problem, the solution of the ML problem usually does not have a closed form. Even limited to the linear system with binary observations, it still cannot be resolved explicitly. To obtain the ML estimate, the EM algorithm produces a sequence of the iterative estimates $\{\hat{\theta}_t, t = 1, 2, ...\}$ to access the ML estimate. As the iteration step goes on, it is guaranteed that the log-likelihood function $\{l(\hat{\theta}_t)\}$ is non-decreasing, which brings extraordinary robustness compared to other methods such as Newton-Raphson Method and Quasi-Newton Methods [8]. Coupled with the upper bound of the log-likelihood function, there exists an l^*, which is the limit of $\{l(\hat{\theta}_t)\}$. But the convergence of the parameter estimates $\{\hat{\theta}_t\}$ cannot be concluded by the convergence of $\{l(\hat{\theta}_t)\}$ and l^* need not necessarily be the maximum value. Refs. [20, 21] analyzed the general convergence property of the EM algorithm based on a convergence theorem in point-to-set topology field and showed some counterexamples where EM algorithm fails to converge, which indicated that the convergence of EM algorithm needed to be analyzed for a specific model.

For binary-valued systems, the EM algorithm was introduced by [6] to estimate the model parameter and simulation results showed the convergence property of the iterative procedures. However, there are still some fundamental questions to be answered such as how to construct a convergent iterative estimate algorithm? What is the convergence rate? What kind of properties does the limit of the iteration have? Hence, its theoretical feasibility based on the specific log-likelihood function with binary data and Gaussian system noises is worth deriving.

This paper constructs an EM-typed iterative algorithm to estimate the system parameter based on batched binary data to achieve the ML estimate. At first, some properties about the ML function and ML estimate are analyzed in detail. Under mild conditions on the system input, the ML function is proved to have unique maximum point, the necessary and sufficient condition for which is given. Then, the algorithm is proved to be convergent with an exponential rate to a fixed vector, no matter where the initial value is. Finally, the point of convergence is proven to be exactly the ML estimate under batched binary-valued observations. The numeric simulations successfully proved the efficiency of our algorithm.

The rest of the paper is organized as follows: Section 2 introduces the identification problem and its corresponding ML criterion; and an iteration estimate algorithm is constructed. In Section 3 we analyze the likelihood function and obtain a sufficient and necessary condition for the existence and uniqueness of the maximum point of the likelihood function. Section 4 derives the convergence of the algorithm and obtains an exponential convergence rate. Section 5 briefly introduces how to handle the unknown threshold and error. Some results are illustrated through extensive numerical simulations in Section 6. Section 7 concludes the paper and discusses related future work.

2 Problem formulation

In this section, we introduce the system with batched binary-valued observations and we formulate the corresponding parameter identification problem. The ML estimate problem is given and an iterative algorithm is constructed.

2.1 Binary-valued system

We consider the scheme where noise enters between the n-dimensional linear system and a binary sensor. The system can be described by

$$\begin{cases} y_k = \phi_k^{\mathrm{T}}\theta + e_k, \\ s_k = I_{[y_k \leqslant C]}, \quad 1 \leqslant k \leqslant N, \end{cases} \tag{1}$$

where the data length is N, for all $k \leqslant N$, $\phi_k \in \mathbb{R}^n$ is the system input, $y_k \in \mathbb{R}$ is the unobserved system output and $\theta \in \mathbb{R}^n$ is a constant but unknown parameter vector, $s_k \in \{0, 1\}$ is the binary-valued observation generated by the comparison between the system output and a given sensor threshold $C \in \mathbb{R}$, I is the indicator function. System noise $\mathscr{E}_N = \{e_1, e_2, \ldots, e_N\}$ is assumed to be independent with a zero-mean and variance 1 Gaussian distribution.

Assumption 1. Matrix $A = \sum_{k=1}^{N} \phi_k \phi_k^{\mathrm{T}}$ is positive definite.

Remark 1. Assumption 1 is the mathematical description of persistent excitation condition under batched data environment, which is a common assumption in the research of system identification [22].

The problem of interest is to estimate the parameter θ using the binary-valued observations $\mathscr{O}_N = \{s_1, s_2, \ldots, s_N\}$ and input data $\mathscr{I}_N = \{\phi_1, \phi_2, \ldots, \phi_N\}$.

2.2 Maximum likelihood criterion

We consider the system (1), for any $k \leqslant N$, assuming that the input data ϕ_k and the parameter θ is known, then corresponding probabilities of observation $s_k = 1$ and $s_k = 0$ are as follows.

$$\begin{aligned} P\{s_k = 1|\phi_k, \theta\} &= P\{y_k \leqslant C|\phi_k, \theta\} \\ &= P\{e_k \leqslant C - \phi_k^{\mathrm{T}}\theta|\phi_k, \theta\} \\ &= F(C - \phi_k^{\mathrm{T}}\theta), \\ P\{s_k = 0|\phi_k, \theta\} &= 1 - P\{s_k = 1|\phi_k, \theta\} \\ &= 1 - F(C - \phi_k^{\mathrm{T}}\theta), \end{aligned}$$

where $F(x)$ are the cumulative distribution function (CDF) of the standard normal distribution.

With the above conditional probability, we can construct a likelihood function to represent the overall probability of observation data $\mathscr{O}_N$ given the input data $\mathscr{I}_N$ and parameter θ,

$$\begin{aligned} L(\theta) &= P\{\mathscr{O}_N|\mathscr{I}_N, \theta\} \\ &= \prod_{k=1}^{N} P\{s_k|\phi_k, \theta\} \\ &= \prod_{\{k:s_k=1\}} P\{s_k = 1|\phi_k, \theta\} \cdot \prod_{\{k:s_k=0\}} P\{s_k = 0|\phi_k, \theta\} \\ &= \prod_{\{k:s_k=1\}} F(C - \phi_k^{\mathrm{T}}\theta) \cdot \prod_{\{k:s_k=0\}} \big(1 - F(C - \phi_k^{\mathrm{T}}\theta)\big). \end{aligned}$$

In practical applications, we prefer log-likelihood function $l(\theta)$ which is the logarithmic transformation of the likelihood function $L(\theta)$,

$$l(\theta) = \log(L(\theta))$$

$$= \sum_{\{k:s_k=1\}} \log[F(C-\phi_k^{\mathrm{T}}\theta)] + \sum_{\{k:s_k=0\}} \log[1-F(C-\phi_k^{\mathrm{T}}\theta)]$$

$$= \sum_{k=1}^{N} \Big[\log[F(C-\phi_k^{\mathrm{T}}\theta)]I_{[s_k=1]} + \log[1-F(C-\phi_k^{\mathrm{T}}\theta)]I_{[s_k=0]} \Big]. \tag{2}$$

The corresponding ML estimate is the parameter that maximizes the log-likelihood function:

$$\hat{\theta} = \arg\max_{\theta} l(\theta). \tag{3}$$

Remark 2. Since log-likelihood function and related ML estimate are relevant to N observations, $l_N(\theta)$ and $\hat{\theta}_N$ are more accurate representations. In this paper, for the convenience of description, symbols $l(\theta)$ and $\hat{\theta}$ are employed in the case of no conflict.

2.3 Explicit solution of the ML problem

As for some specific binary-valued systems, the explicit solution of ML problem (3) exists. For example, Ref. [2] considered a case where the input data $\mathscr{I}_N = \{\phi_1, \phi_2, \ldots, \phi_N\}$ has periodical form shown in Eq. (4):

$$\begin{cases} \{\phi_k, k \geqslant 1\} \text{ is } n\text{-periodic, i.e. } \phi_k = \phi_{k+n}; \\ N \text{ is divisible by } n, \text{and } N/n = \bar{N}; \\ \Phi_1 = (\phi_1, \phi_2, \ldots, \phi_n)^{\mathrm{T}} \text{ is reversible.} \end{cases} \tag{4}$$

In this case, the log-likelihood function is converted to

$$l(\theta) = \sum_{i=1}^{n} \left\{ N_{i1} \log[F(C-\phi_i^{\mathrm{T}}\theta)] + N_{i0} \log[1-F(C-\phi_i^{\mathrm{T}}\theta)] \right\},$$

where, $N_{i1} = \sum_{j=0}^{L-1} s_{i+nj}$ represents the number of observations whose $\phi_k = \phi_i$ and $s_k = 1$, and $N_{i0} = \sum_{j=0}^{L-1}(1 - s_{i+nj})$ represents the number of observations whose $\phi_k = \phi_i$ and $s_k = 0$. Hence, $\forall i \leqslant n, N_{i1} + N_{i0} = \bar{N}$. The corresponding gradient vector is

$$\nabla l(\theta) = \sum_{i=1}^{n} \left\{ \Big[\frac{-N_{i1}}{F(C-\phi_i^{\mathrm{T}}\theta)} + \frac{N_{i0}}{1-F(C-\phi_i^{\mathrm{T}}\theta)} \Big] f(C-\phi_i^{\mathrm{T}}\theta)\phi_i \right\}.$$

Define

$$R = \big(C - F^{-1}(N_{11}/\bar{N}), C - F^{-1}(N_{21}/\bar{N}), \ldots, C - F^{-1}(N_{n1}/\bar{N})\big)^{\mathrm{T}},$$

the explicit solution of ML problem $\nabla l(\hat{\theta}) = 0$ is

$$\hat{\theta} = \Phi_1^{-1} R. \tag{5}$$

Remark 3. The algorithm of [2] is essentially the ML estimate (5).

2.4 Iterative estimate algorithm

If we relax restrictions on the input data $\mathscr{I}_N$, the explicit solution of ML problem usually does not exist. In this section, we propose an iterative estimate algorithm to obtain the ML estimate.

The proposed iterative algorithm employs the basic idea of the EM algorithm. Hence, we first briefly introduce the EM algorithm. Given the estimate at the t-th iteration $\hat{\theta}_t$, EM algorithm is able to construct function $l(\theta|\hat{\theta}_t)$ based on specific conditional expectations. And, function $l(\theta|\hat{\theta}_t)$ satisfies the following two properties:

(i) $l(\theta|\hat{\theta}_t) \leqslant l(\theta)$ holds for all θ,

(ii) $l(\hat{\theta}_t|\hat{\theta}_t) = l(\hat{\theta}_t)$.

Then, calculate $\hat{\theta}_{t+1}$ that maximize the function $l(\theta|\hat{\theta}_t)$ as the estimate of $(t+1)$-th iteration. We can see that

$$l(\hat{\theta}_{t+1}) \geqslant l(\hat{\theta}_{t+1}|\hat{\theta}_t) = \max l(\theta|\hat{\theta}_t) \geqslant l(\hat{\theta}_t|\hat{\theta}_t) = l(\hat{\theta}_t).$$

This guarantees the non-decreasing property of iterative algorithm for log-likelihood function, which provides extraordinary robustness. The construction process of function $l(\theta|\hat{\theta}_t)$ is the E-step and maximization process is the M-step.

Back to the binary-valued model, the E-step provides the following $l(\theta|\hat{\theta}_t)$ with the quadratic form:

$$\begin{aligned} l(\theta|\hat{\theta}_t) = & -\frac{1}{2}\theta^{\mathrm{T}}\Big(\sum_{k=1}^{N}\phi_k\phi_k^{\mathrm{T}}\Big)\theta + \Bigg[\Big(\sum_{k=1}^{N}\phi_k\phi_k^{\mathrm{T}}\Big)\hat{\theta}_t - \Big(\sum_{k=1}^{N}\phi_k\cdot f(C-\phi_k^{\mathrm{T}}\hat{\theta}_t) \\ & \times\Big[\frac{I_{[s_k=1]}}{F(C-\phi_k^{\mathrm{T}}\hat{\theta}_t)} - \frac{I_{[s_k=0]}}{1-F(C-\phi_k^{\mathrm{T}}\hat{\theta}_t)}\Big]\Big)\Bigg]^{\mathrm{T}}\theta + l_1(\hat{\theta}_t), \end{aligned}$$

where $l_1(\hat{\theta}_t)$ is the part independent from θ.

Under Assumption 1, the iterative algorithm is as follows:

$$\begin{aligned} \hat{\theta}_{t+1} &= \arg\max_{\theta} l(\theta|\hat{\theta}_t) \\ &= \hat{\theta}_t - \Big(\sum_{k=1}^{N}\phi_k\phi_k^{\mathrm{T}}\Big)^{-1}\Big(\sum_{k=1}^{N}\phi_k\cdot f(C-\phi_k^{\mathrm{T}}\hat{\theta}_t)\Big[\frac{I_{[s_k=1]}}{F(C-\phi_k^{\mathrm{T}}\hat{\theta}_t)} - \frac{I_{[s_k=0]}}{1-F(C-\phi_k^{\mathrm{T}}\hat{\theta}_t)}\Big]\Big). \end{aligned} \tag{6}$$

The above algorithm can be proved to converge to the ML estimate (3) with an exponential rate in the rest of this paper. As the initial issue, the existence and uniqueness of the solution of (3) is needed to be checked, which assures that the iterative algorithm based on MLE is effective.

3 Existence and uniqueness of the ML estimate

In this section, we explore the properties of ML estimate by analyzing the log-likelihood function (2) and prove the existence and uniqueness of ML estimate. The main reason is that the special likelihood function with binary-valued observations is concave.

Denote $f(x) = F'(x)$ as the probability dense function (PDF) of the standard normal distribution. Then, we can see that

Lemma 1. Function

$$p(x) = -\frac{\mathrm{d}^2\log[F(x)]}{\mathrm{d}x^2} = \frac{f(x)xF(x)+f^2(x)}{F^2(x)}$$

is a strictly decreasing function and $0 < p(x) < 1$ for $x \in (-\infty, \infty)$.

Proof. The detailed proof is in Appendix A.1.

Remark 4. The result of Lemma 1 is not always valid for other distributions. Given below are two examples: (1) If $F_\sigma(x)$ is the CDF of normal distribution with mean 0 and variance σ^2, then $p_\sigma(x) = -\frac{\mathrm{d}^2\log[F_\sigma(x)]}{\mathrm{d}x^2} \in (0, 1/\sigma^2)$. (2) If $F_{t,d}(x)$ is the CDF of t distribution with degrees of freedom d, then $p_{t,d}(x) = -\frac{\mathrm{d}^2\log[F_{t,d}(x)]}{\mathrm{d}x^2} > 0$ does not always hold. In Appendix A.1, the examples (2) are illustrated in a graph.

Remark 5. The symmetrical properties of normal distribution is widely used in this paper, that is, $f(-x) = f(x), F(-x) = 1 - F(x)$.

The non-negative property of $p(x)$ induces the concaveness of the log-likelihood function, which is described in the following lemma.

Lemma 2. Under Assumption 1, log-likelihood function $l(\theta)$ given in (2) is a concave function on $\mathbb{R}^n$. Furthermore, given any $r > 0$, $l(\theta)$ is a strongly concave function on the set $\mathscr{S} = \{\theta : \|\theta\| \leqslant r\}$.

Proof. Based on Assumption 1, $A = \sum_{k=1}^{N} \phi_k \phi_k^{\mathrm{T}} > 0$. Hence, there exists a minimal eigen value $\lambda_1 = \lambda_{\min}(A)$ such that $A \geqslant \lambda_1 I$.

Calculate the gradient vector and Hessian matrix of the log-likelihood function:

$$\nabla l(\theta) = \sum_{k=1}^{N} \left[\left(\frac{-f(C-\phi_k^{\mathrm{T}}\theta)}{F(C-\phi_k^{\mathrm{T}}\theta)} I_{[s_k=1]} + \frac{f(C-\phi_k^{\mathrm{T}}\theta)}{1-F(C-\phi_k^{\mathrm{T}}\theta)} I_{[s_k=0]} \right) \phi_k \right],$$

$$\begin{aligned} \nabla^2 l(\theta) = -\sum_{k=1}^{N} \Bigg[\Bigg(& \frac{f(x)xF(x)+f^2(x)}{F^2(x)}\Big|_{x=C-\phi_k^{\mathrm{T}}\theta} \cdot I_{[s_k=1]} \\ & + \frac{f(x)xF(x)+f^2(x)}{F^2(x)}\Big|_{x=\phi_k^{\mathrm{T}}\theta-C} \cdot I_{[s_k=0]} \Bigg) \phi_k \phi_k^{\mathrm{T}} \Bigg]. \end{aligned}$$

Denote $p(x) = \frac{f(x)xF(x)+f^2(x)}{F^2(x)}$ and rewrite the Hessian matrix as follows:

$$\nabla^2 l(\theta) = -\sum_{k=1}^{N} \Big[\big(p(C-\phi_k^{\mathrm{T}}\theta) \cdot I_{[s_k=1]} + p(\phi_k^{\mathrm{T}}\theta - C) \cdot I_{[s_k=0]} \big) \phi_k \phi_k^{\mathrm{T}} \Big].$$

Lemma 1 presents the monotonicity and boundedness of function $p(x)$. Hence, $\nabla^2 l(\theta) \leqslant 0$ can be directly concluded through $p(x) > 0$, which infers the concave property of $l(\theta)$. If limited on the set $\mathscr{S} = \{\theta : \|\theta\| \leqslant r\}$, the boundedness of θ guarantees that $\forall k \leqslant N$, there exists $\epsilon_{r,\Phi,k} > 0$ such that

$$\big(p(C-\phi_k^{\mathrm{T}}\theta) I_{[s_k=1]} + p(\phi_k^{\mathrm{T}}\theta - C) I_{[s_k=0]} \big) \geqslant \epsilon_{r,\Phi,k}.$$

Denote $\epsilon_{r,\Phi} = \min_{k \leqslant N}\{\epsilon_{r,\Phi,k}\}$, then $\nabla^2 l(\theta) \leqslant -\epsilon_{r,\Phi} \sum_{k=1}^{N} \phi_k \phi_k^{\mathrm{T}} = -\epsilon_{r,\Phi} A$. Coupled with $A \geqslant \lambda_1 I$, we can see that $\nabla^2 l(\theta) \leqslant -\lambda_1 \epsilon_{r,\Phi} I$, which infers the strongly concave property of the function $l(\theta)$ on the set $\mathscr{S}$.

Remark 6. Based on Remark 4, binary-valued system with t distribution error does not have the concave log-likelihood function. Hence, the property is not general for binary-valued model.

According to the concave property of the log-likelihood function, we can prove the uniqueness of ML estimate first.

Theorem 1. Under Assumption 1, log-likelihood function $l(\theta)$ given in (2) has at most one maximum point.

Proof. Assume there exist two maximum points θ_1 and θ_2. Let $r_1 = \max(\|\theta_1\|, \|\theta_2\|)$, Lemma 2 proves the strongly concave property of $l(\theta)$ on the set $\mathscr{S} = \{\theta, \|\theta\| \leqslant r_1\}$, which makes it impossible to come up two different maximum points on the set $\mathscr{S}$. The contradiction shows that there is at most one maximum point for $l(\theta)$.

To reveal the condition that ML estimate exists. Some novel conditions are given.

Definition 1. Denote

$$\Psi = \big(\phi_1 (I_{[s_1=0]} - I_{[s_1=1]}), \ldots, \phi_N (I_{[s_N=0]} - I_{[s_N=1]}) \big)$$

as the integrated matrix that combines the information of both input data $\mathscr{I}_N = \{\phi_1, \phi_2, \ldots, \phi_N\}$ and binary-valued observation $\mathscr{O}_N = \{s_1, s_2, \ldots, s_N\}$.

Definition 2. Given input $\mathscr{I}_N$ and binary-valued observations $\mathscr{O}_N$, if there exists non-zero vector $\gamma \in \mathbb{R}^n$ such that $\Psi^{\mathrm{T}}\gamma \geqslant 0$, then the data $(\mathscr{I}_N, \mathscr{O}_N)$ is called ineffective, otherwise it is called effective.

Lemma 3. Under Assumption 1, if data $(\mathscr{I}_N, \mathscr{O}_N)$ is effective, then $\forall\, b \in \mathbb{R}$, the set $\mathscr{S}_b = \{\theta : l(\theta) \geqslant b\}$ is a bounded set.

Proof. The detailed proof is in Appendix A.2.

Based on the previous discussion, we can give an explicit description for the existence and uniqueness of ML estimate.

Theorem 2. Under Assumption 1, the log-likelihood function $l(\theta)$ given in (2) has a unique maximum point if and only if that data $(\mathscr{I}_N, \mathscr{O}_N)$ is effective.

Proof. We prove the theorem from two directions.

Sufficiency. Given any θ_1, the global maximum point of $l(\theta)$ is on the set $\mathscr{S}_{l(\theta_1)} = \{\theta : l(\theta) \geqslant l(\theta_1)\}$. From Lemma 3, that data $(\mathscr{I}_N, \mathscr{O}_N)$ is effective to conclude that $\mathscr{S}_{l(\theta_1)}$ is a bounded set, which infers the existence of maximum point of $l(\theta)$. Coupled with the result of Theorem 1, there is exactly one maximum point for $l(\theta)$.

Necessity. If $(\mathscr{I}_N, \mathscr{O}_N)$ is ineffective, there is a non-zero vector $\gamma \in \mathbb{R}^n$ such that $\Psi^{\mathrm{T}}\gamma \geqslant 0$. Because $\Psi\Psi^{\mathrm{T}} = \sum_{k=1}^{N} \phi_k\phi_k^{\mathrm{T}}$, Assumption 1 rejects that $\Psi^{\mathrm{T}}\gamma = 0$. Hence, there is at least one positive component for vector $\Psi^{\mathrm{T}}\gamma$.

Given any parameter θ, we define a scalar function $h_{\theta,\gamma}(r)$ as follows:

$$\begin{aligned} h_{\theta,\gamma}(r) &= l(\theta + r\gamma) \\ &= \sum_{k=1}^{N} \Big[\log[F(C - \phi_k^{\mathrm{T}}(\theta + r\gamma))]I_{[s_k=1]} + \log[F(-C + \phi_k^{\mathrm{T}}(\theta + r\gamma))]I_{[s_k=0]} \Big] \\ &= \sum_{k=1}^{N} \Big[\log[F(-\phi_k^{\mathrm{T}}\gamma r + C - \phi_k^{\mathrm{T}}\theta)]I_{[s_k=1]} + \log[F(\phi_k^{\mathrm{T}}\gamma r + C + \phi_k^{\mathrm{T}}\theta)]I_{[s_k=0]} \Big]. \end{aligned}$$

Note that the k-th component of $\Psi^{\mathrm{T}}\gamma$ is

$$(\Psi^{\mathrm{T}}\gamma)_k = -\phi_k^{\mathrm{T}}\gamma I_{[s_k=1]} + \phi_k^{\mathrm{T}}\gamma I_{[s_k=0]},$$

which is consistent with the coefficient of r in $h_{\theta,\gamma}(r)$. The property of $\Psi^{\mathrm{T}}\gamma$, which is described before, and the strictly increasing property of functions $F(.)$ and $\log(.)$ conclude that $h_{\theta,\gamma}(r)$ is a strictly increasing function.

Suppose θ^* is a maximum value point, $h_{\theta^*,\gamma}(r) = l(\theta^* + r\gamma)$ should be non-increasing at $r = 0$, which is contradictory with the strictly increasing property of $h_{\theta^*,\gamma}(r)$. So, there does not exist any finite maximum point for $l(\theta)$.

Below are some examples to illustrate how to prove the effectiveness and ineffectiveness of data.

Example 1. Assuming the model dimension $n = 1$, all the input data $\{\phi_k, k \leqslant N\}$ are positive. The observation $s_k = 0$ holds for any $k \leqslant N$. Then the integrated matrix is

$$\begin{aligned} \Psi &= (\phi_1(I_{[s_1=0]} - I_{[s_1=1]}), \ldots, \phi_N(I_{[s_N=0]} - I_{[s_N=1]})) \\ &= (\phi_1, \phi_2, \ldots, \phi_N). \end{aligned}$$

The data obviously is ineffective because there exists $\gamma = 1$, such that $\Psi^{\mathrm{T}}\gamma \geqslant 0$.

Example 2. Assuming the model dimension $n = 1$, all the input data $\{\phi_k, k \leqslant N\}$ are positive. There exists $k_1 \neq k_2$ such that observation $s_{k_1} = 0$ and $s_{k_2} = 1$. Then the integrate matrix is as follows:

$$\Psi = (\phi_1(I_{[s_1=0]} - I_{[s_1=1]}), \ldots, \phi_{k_1}, \ldots, -\phi_{k_2}, \ldots, \phi_N(I_{[s_N=0]} - I_{[s_N=1]})).$$

Given any $\gamma \neq 0$, $(\Psi^{\mathrm{T}}\gamma)_{k_1} = \phi_{k_1}\gamma, (\Psi^{\mathrm{T}}\gamma)_{k_2} = -\phi_{k_2}\gamma$. That $\phi_{k_1}, \phi_{k_2} > 0$ infers that $(\Psi^{\mathrm{T}}\gamma)_{k_1}$ and $(\Psi^{\mathrm{T}}\gamma)_{k_2}$ have opposite sign, which rejects the case that $\Psi^{\mathrm{T}}\gamma \geqslant 0$.

Remark 7. If the model dimension n is large, the effective property of data is not easy to verify through Definition 2. Here we construct a new criterion, which only needs the existence of one point at N-dimensional space.

Criterion 1. If there exists $\rho \in \mathbb{R}^N > 0$, s.t. $\Psi\rho = 0$, then data $(\mathscr{I}_N, \mathscr{O}_N)$ is effective.

Proof. Suppose there exists $\rho \in \mathbb{R}^N > 0$, s.t. $\Psi\rho = 0$. Then, if the data $(\mathscr{I}_N, \mathscr{O}_N)$ is ineffective, there is a non-zero vector $\gamma \in \mathbb{R}^n$ such that $\Psi^{\mathrm{T}}\gamma \geqslant 0$. Coupled with $\rho > 0$, we can conclude that $\rho^{\mathrm{T}}(\Psi^{\mathrm{T}}\gamma) > 0$. While because $\Psi\rho = 0$, $\rho^{\mathrm{T}}\Psi^{\mathrm{T}} = 0$ infers $\rho^{\mathrm{T}}(\Psi^{\mathrm{T}}\gamma) = 0$, which concludes the contradiction. Hence, data $(\mathscr{I}_N, \mathscr{O}_N)$ is effective.

Example 3. We give an example to illustrate the use of Criterion 1, which also shows the wide existence of effective data. Assume N input signals have m possible options $\phi_1, \phi_2, \ldots, \phi_m$. For any $i \leqslant m$, there are N_i input signals ϕ_i, including N_{i0} signals with observation 0 and N_{i1} signals with observation 1. Then, the integrate matrix Ψ given in Definition 1 is

$$\Psi = (\underbrace{\phi_1, \ldots, \phi_1}_{N_{10}}, \underbrace{-\phi_1, \ldots, -\phi_1}_{N_{11}}, \ldots, \underbrace{\phi_m, \ldots, \phi_m}_{N_{m0}}, \underbrace{-\phi_m, \ldots, -\phi_m}_{N_{m1}}).$$

Select an N-dimensional vector

$$\rho = (\underbrace{N_{11}, \ldots, N_{11}}_{N_{10}}, \underbrace{N_{10}, \ldots, N_{10}}_{N_{11}}, \ldots, \underbrace{N_{m1}, \ldots, N_{m1}}_{N_{m0}}, \underbrace{N_{m0}, \ldots, N_{m0}}_{N_{m1}})^{\mathrm{T}}.$$

It is easy to prove that

$$\Psi\rho = \sum_{i=1}^{m}(N_{i1}N_{i0}\phi_i - N_{i1}N_{i0}\phi_i) = 0.$$

If data length N is large enough, then there are great chances that $N_{i1} > 0, N_{i2} > 0$. Based on Criterion 1, the data is effective.

4 Convergence of the iterative estimate

The former two sections demonstrated that MLE is valid for the parameter estimation of a binary-valued system. In this section, we will prove the proposed iterative algorithm (6) converging to the ML estimate with an exponential rate by using a Lyapunov method.

Assumption 2. Data $(\mathscr{I}_N, \mathscr{O}_N)$ is effective.

Remark 8. As Theorem 2 shows, if data $(\mathscr{I}_N, \mathscr{O}_N)$ is ineffective, the log-likelihood function does not have any finite maximum point. Hence, Assumption 2 is necessary for us to analyze the convergence to the ML estimate.

The following main result infers that if the ML estimate exists, then the iterative estimate $\{\hat{\theta}_t\}$ constructed by (6) certainly converges to the ML estimate with exponential convergence rate.

Theorem 3. Under Assumptions 1 and 2, there exists $1 > \epsilon > 0$ such that the iteration $\{\hat{\theta}_t\}$ generated by (6) satisfies

$$\|\hat{\theta}_t - \hat{\theta}\| \leqslant \sqrt{\frac{Q_1}{\lambda_{\min}(A)}} \cdot \frac{\sqrt{(1-\epsilon)}^t}{1-\sqrt{(1-\epsilon)}},$$

where $\hat{\theta}$ is the ML estimate given in (3); $A = \sum_{k=1}^{N}\phi_k\phi_k^{\mathrm{T}}$, $\lambda_{\min}(A)$ is the minimal eigen value of A; $Q_1 = (1-\epsilon)^{-1}(\hat{\theta}_2 - \hat{\theta}_1)^{\mathrm{T}}A(\hat{\theta}_2 - \hat{\theta}_1)$; and $\|\cdot\|$ is the Euclidean norm.

Proof. Since EM algorithm increases the log-likelihood function value as the iteration goes on, $\{\hat{\theta}_t : t \geqslant 1\} \subset \{\theta : l(\theta) \geqslant l(\hat{\theta}_1)\}$ is bounded according to Lemma 3. Assuming there is an upper bound r such that for all $t \geqslant 1, \|\hat{\theta}_t\| \leqslant r$.

Eq. (6) can be transformed to

$$\hat{\theta}_{t+1} = \hat{\theta}_t + \left(\sum_{k=1}^{N}\phi_k\phi_k^{\mathrm{T}}\right)^{-1}\nabla l(\hat{\theta}_t) = \hat{\theta}_t + A^{-1}(\nabla l(\hat{\theta}_t)),$$

where $\nabla l(\hat{\theta})$ is the gradient vector of the log-likelihood function. Its accurate form can be seen in Lemma 2.

Furthermore, based on mean value theorem, we can see that

$$\hat{\theta}_{t+1} - \hat{\theta}_t = \hat{\theta}_t - \hat{\theta}_{t-1} + A^{-1}\Big(\nabla l(\hat{\theta}_t) - \nabla l(\hat{\theta}_{t-1})\Big)$$

$$\begin{aligned}&= \hat{\theta}_t - \hat{\theta}_{t-1} + A^{-1}\Big(\nabla^2 l(\check{\theta}_{t,t-1})(\hat{\theta}_t - \hat{\theta}_{t-1})\Big)\\ &= \Big(I + A^{-1}\big(\nabla^2 l(\check{\theta}_{t,t-1})\big)\Big)(\hat{\theta}_t - \hat{\theta}_{t-1}),\end{aligned} \tag{7}$$

where $\check{\theta}_{t,t-1}$ is between $\hat{\theta}_t$ and $\hat{\theta}_{t-1}$, to be more accurate, there exists $\lambda_{t,t-1} \in (0,1)$ such that $\check{\theta}_{t,t-1} = \lambda_{t,t-1}\hat{\theta}_t + (1-\lambda_{t,t-1})\hat{\theta}_{t-1}$.

The boundedness of $\{\theta_t, t \geqslant 1\}$ indicates that $\|\check{\theta}_{t,t-1}\| \leqslant r$. Furthermore, based on Lemma 2, there exists $\epsilon > 0$ such that for all t,

$$-A < \nabla^2 l(\check{\theta}_{t,t-1}) \leqslant -\epsilon \textstyle\sum_{k=1}^{N} \phi_k \phi_k^{\mathrm{T}} = -\epsilon A.$$

For the simplicity, for all $t \geqslant 1$, we define

$$B_t = -\nabla^2 l(\check{\theta}_{t,t-1}), \qquad x_t = \hat{\theta}_{t+1} - \hat{\theta}_t.$$

Hence, $\forall t \geqslant 1, A > B_t > \epsilon A > 0$. Eq. (7) can be translated to the following form:

$$x_t = (I_n - A^{-1}B_t)x_{t-1}. \tag{8}$$

We prove the convergence of $\{\hat{\theta}_t\}$ by analyzing the property of a Lyapunov function $Q_t = (1-\epsilon)^{-t}x_t^{\mathrm{T}}Ax_t$,

$$\begin{aligned}Q_t &= (1-\epsilon)^{-t}x_t^{\mathrm{T}}Ax_t\\ &= (1-\epsilon)^{-t}x_{t-1}^{\mathrm{T}}(I_n - A^{-1}B_t)^{\mathrm{T}}A(I_n - A^{-1}B_t)x_{t-1}\\ &= Q_{t-1} + (1-\epsilon)^{-t}x_{t-1}^{\mathrm{T}}\big[(I_n - A^{-1}B_t)^{\mathrm{T}}A(I_n - A^{-1}B_t) - (1-\epsilon)A\big]x_{t-1}\\ &= Q_{t-1} + (1-\epsilon)^{-t}x_{t-1}^{\mathrm{T}}\big[A - 2B_t + B_tA^{-1}B_t - (1-\epsilon)A\big]x_{t-1}\\ &= Q_{t-1} + (1-\epsilon)^{-t}x_{t-1}^{\mathrm{T}}\big[B_tA^{-1}B_t - B_t + \epsilon A - B_t\big]x_{t-1}.\end{aligned}$$

For all $t \geqslant 1, A > B_t > 0$, so that $A^{-1} < B_t^{-1}$, and furthermore, $B_tA^{-1}B_t < B_tB_t^{-1}B_t = B_t$. Coupled with $\epsilon A < B_t$, we can see that

$$B_tA^{-1}B_t - B_t + \epsilon A - B_t < 0 \Rightarrow Q_t \leqslant Q_{t-1}.$$

For all $t > 1$,

$$Q_t \leqslant Q_1 \Rightarrow (1-\epsilon)^{-t}x_t^{\mathrm{T}}Ax_t \leqslant Q_1 \Rightarrow \|x_t\|^2 \leqslant \frac{Q_1}{\lambda_{\min}(A)}(1-\epsilon)^t,$$

where $\lambda_{\min}(A)$ is the minimal eigen value of matrix A. Hence,

$$\begin{aligned}\|\hat{\theta}_{t+r} - \hat{\theta}_t\| &= \|x_t + x_{t+1} + \cdots + x_{t+r-1}\|\\ &\leqslant \|x_t\| + \|x_{t+1}\| + \cdots + \|x_{t+r-1}\|\\ &\leqslant \sqrt{\frac{Q_1}{\lambda_{\min}(A)}}\Big[\sqrt{(1-\epsilon)}^{t} + \sqrt{(1-\epsilon)}^{t+1} + \cdots + \sqrt{(1-\epsilon)}^{t+r-1}\Big]\\ &= \sqrt{\frac{Q_1}{\lambda_{\min}(A)}} \cdot \frac{\sqrt{(1-\epsilon)}^{t}(1 - \sqrt{(1-\epsilon)}^{r})}{1 - \sqrt{(1-\epsilon)}}\\ &\to 0 \quad \text{as} \quad t \to \infty, r \to \infty.\end{aligned}$$

Based on Cauchy criterion, $\{\hat{\theta}_t\}$ convergent to some point $\hat{\theta}_0$. Additionally, fix t and let $r \to \infty$,

$$\|\hat{\theta}_0 - \hat{\theta}_t\| \leqslant \sqrt{\frac{Q_1}{\lambda_{\min}(A)}} \cdot \frac{\sqrt{(1-\epsilon)}^{t}}{1 - \sqrt{(1-\epsilon)}}.$$

We can see that convergence rate is of the same order with the exponential rate $\sqrt{(1-\epsilon)}^t$.

Recall that $\hat{\theta}_{t+1} = \hat{\theta}_t + A^{-1}\nabla l(\hat{\theta}_t)$. As $t \to \infty$,

$$\hat{\theta}_0 = \hat{\theta}_0 + A^{-1}\nabla l(\hat{\theta}_0) \Rightarrow \nabla l(\hat{\theta}_0) = 0.$$

This concludes that $\hat{\theta}_0$ is an ML estimate $\hat{\theta}$. In addition, by Theorem 2, $\hat{\theta}$ is the unique maximum value of log-likelihood function.

Remark 9. Theorem 3 proves the convergence property of iterative algorithm and obtains the convergence rate at the same time. The introduction of Lyapunov function simplifies the analysis and the whole proof process is very concise. We hope the analysis and method could bring inspiration into related researches.

5 Discussion about the threshold and error variance

Recall the binary-valued model (1),

$$\begin{cases} y_k = \phi_k^{\mathrm{T}}\theta + e_k, \\ s_k = I_{[y_k \leqslant C]}, \quad 1 \leqslant k \leqslant N. \end{cases}$$

Here, we assume the threshold C known and variance of error $\{e_k\}$ fixed to be 1. These assumptions seem a little debatable. In this section, we give a brief description to show that these assumptions are not the nature of the problem.

5.1 Threshold detect

Even if we assume that the threshold is unknown, we can still detect it by a simple transformation process. Denote $y_k^* = y_k - C$, model (1) can be transformed as

$$\begin{cases} y_k^* = \phi_k^{\mathrm{T}}\theta - C + e_k, \\ s_k = I_{[y_k^* \leqslant 0]}, \quad 1 \leqslant k \leqslant N, \end{cases}$$

where the new threshold is fixed at 0, and C is converted to a new parameter to be estimated. The corresponding "input data" related to "parameter" C are all -1.

5.2 Influence of the error variance

The assumption that error $\{e_k\}$ is independent and identically distributed is very general and acceptable. But it is a little questionable whether the error variance should be fixed at 1. Relax the restriction and assume the error $\{e_k\}$ obeys normal distribution with mean 0 and variance σ^2. By transformation

$$y_k^\star = y_k/\sigma, \quad \theta^\star = \theta/\sigma, \quad e_k^\star = e_k/\sigma, \quad C^\star = C/\sigma,$$

The model (1) actually can be converted to the following equivalent model:

$$\begin{cases} y_k^\star = \phi_k^{\mathrm{T}}\theta^\star + e_k^\star, \\ s_k = I_{[y_k^\star \leqslant C^\star]}, \quad 1 \leqslant k \leqslant N, \end{cases}$$

where $e_k^\star$ follows normal distribution with mean 0 and variance 1. Hence, the algorithm (with assumption that error variance is 1) obtains a group of estimated parameters and thresholds which are $\theta^\star, C^\star$ rather than the actual ones θ, C. But is it not enough to obtain a group of parameters that share a constant multiple? To answer the question, we briefly give an example where the model describes the whole healthy condition of individual. For the detail of the example, see [23].

Example 4. Suppose input data ϕ_k represents the major different aspects for individual k such as smoking frequency, sleeping time per day, genotype of some important loci and so on; y_k represents the latent healthy index, which is unobservable; if the healthy index is high enough to exceed the threshold C, then the individual is healthy, denoted as $s_k = 0$, otherwise $s_k = 1$ indicates a poor condition. We are interested in two questions:

(1) Which aspect influences the healthy condition?

(2) Can we deduce whether the individual is healthy with the knowledge of every aspect information?

The answer to (1) can be roughly concluded by the abstract value of corresponding parameter. The parameter more close to zero is less sensitive for y_k with change in corresponding input, which indicates less effect. And obviously, the relative relationship among parameters is unaffected by the common multiple.

To resolve the problem (2), we can make use of the estimated parameter and known input to generate an estimated $\hat{y}_k$. Furthermore, by comparing the relative relationship between the $\hat{y}_k$ and threshold C to deduce the $\hat{s}_k$. If the threshold and parameters vary proportionally, the deduction results will not be influenced totally.

The example shows that sometimes only the ratios of estimated parameters are necessary for the practical problem. Thus, we can fix the variance of error as 1 and ignore the proportional change of estimated parameters.

6 Numerical simulations

In this section, we illustrate the main results through extensive simulations.

6.1 Log-likelihood function curve

To illustrate the log-likelihood function intuitively, we limit the model dimension to $n = 1$. In this case, Assumption 1 degenerates $\sum_{k=1}^{N} \phi_k^2 > 0$. If the assumption is not satisfied, then $\forall k, \phi_k = 0$, means we cannot obtain any useful input information.

Examples 1 and 2 give two examples of ineffective data case and effective case respectively. Actually, that data $\{\mathscr{I}_N, \mathscr{O}_N\}$ is ineffective is equivalent one of (A1) and (A2) is true,

$$\begin{aligned}
&\text{(A1).}\quad \text{For all } k \text{ that input } \phi_k \geqslant 0,\ \text{observation } s_k = 1;\\
&\qquad\quad\ \text{For all } k \text{ that input } \phi_k \leqslant 0,\ \text{observation } s_k = 0.\\
&\text{(A2).}\quad \text{For all } k \text{ that input } \phi_k \leqslant 0,\ \text{observation } s_k = 1;\\
&\qquad\quad\ \text{For all } k \text{ that input } \phi_k \geqslant 0,\ \text{observation } s_k = 0.
\end{aligned}$$

If we generate the data based on the model (1), cases $(A1)$ and $(A2)$ hardly emerge. To illustrate the necessity of "effective property", we adopt a kind of data generation process, which has nothing to do with the model.

Fix data length $N = 10$, generate the observations $\mathscr{O}_N$ with the following Matlab codes:

$$\begin{aligned}
&\text{(B1).}\quad s = [\text{ones}(N/2, 1); \text{zeros}(N/2, 1)];\\
&\text{(B2).}\quad s = [\text{ones}(N, 1)].
\end{aligned}$$

Command ones$(a, 1)$ generates an a-dimensional vector with elements all 1, and command zeros$(a, 1)$ generates an a-dimensional vector with elements all 0. The procedure can generate (B1) and (B2) as Table 1.

As for input data $\mathscr{I}_N$, two cases are considered. The matlab codes are below.

$$\begin{aligned}
&\text{(C1).}\quad \text{input} = [\text{rand}(N/2, 1); -\text{rand}(N/2, 1)];\\
&\text{(C2).}\quad \text{input} = [-\text{rand}(N, 1)].
\end{aligned}$$

Table 1 Observations generated by (B1) and (B2)

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
B1	1	1	1	1	1	0	0	0	0	0
B2	1	1	1	1	1	1	1	1	1	1

Table 2 Signs of input generated by (C1) and (C2)

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}
C1	+	+	+	+	+	−	−	−	−	−
C2	−	−	−	−	−	−	−	−	−	−

Command rand$(a, 1)$ can generate an a-dimensional vector with elements uniformly distributed at random numbers, which are positive. The signs of input in two cases are shown in Table 2.

Hence, (B1)+(C1) corresponds to the ineffective case (A1), and (B2)+(C2) corresponds to the ineffective case (A2). The other two combinations (B1)+(C2) and (B2)+(C1) correspond to the effective cases.

Log-likelihood function $l(\theta)$, where $\theta \in (-10, 10)$ is shown in Figure 1. We can see that "effective property" indeed provides the existence and uniqueness of finite ML estimate.

6.2 Convergence of the proposed iterative algorithm

In this section, the convergence of the proposed iterative algorithm (6) is illustrated by numerical simulations. The brief simulation process is as follows:

Step 1. Data generate. Fix the data length $N = 500$, the model dimension $n = 7$, the sensor threshold $C = 0$, and the model parameter $\theta = (-3, -2, -1, 0, 1, 2, 3)^{\mathrm{T}}$. Error $\mathscr{E}_N$ and input data $\mathscr{I}_N$ are generated based on standard normal distribution. Matlab codes are below:

$$\text{error} = [\text{randn}(N, 1)]; \qquad \text{input} = [\text{randn}(N, 7)].$$

The binary-valued observations $\mathscr{O}_N$ is generated according to model (1).

Step 2. Initial vector select. To prove that EM algorithm can converge to the unique ML estimate, under the same effective data $\{\mathscr{I}_N, \mathscr{O}_N\}$, we adopt a random vector as the iterative initial vector $\hat{\theta}_1$. All components of $\hat{\theta}_1$ are generated by normal distribution with mean 0 and standard deviation 3.

$$\text{initial} = \text{randn}(7, 1) \times 3.$$

Step 3. Parameter estimate. Based on the initial value $\hat{\theta}_1$ and iteration process (6), we can generate the iteration estimates $\{\hat{\theta}_t, t \geqslant 1\}$.

The simulation results of iteration estimates are shown in Figure 2. Under various initial vectors, all components of estimates $\{\hat{\theta}_t\}$ converge to the unique ML estimate, which is quite close to the true parameter. In addition, the curves of Figure 2 indicate the exponential convergence rate.

6.3 Consistency of the ML estimate

In this section, we illustrate the consistency of the ML estimate by numerical simulations. In other words, we identify whether the ML estimate converges to the actual parameter with the increase of data length.

Select data length $N = 500, 100, 2000, 10000$, respectively. In each case, we repeat the data generation and parameter estimate to obtain the estimated parameter for 100 times.

Figure 3 shows the distribution of each component of the ML estimates. We can see that with the increase of data length, the ML estimate converges to the actual parameter.

6.4 Comparisons with other methods

In this section we compare the proposed EM-typed algorithm with the traditional Newton-Raphson (NR) method by numerical simulations. The theoretical results show that these two methods have their own

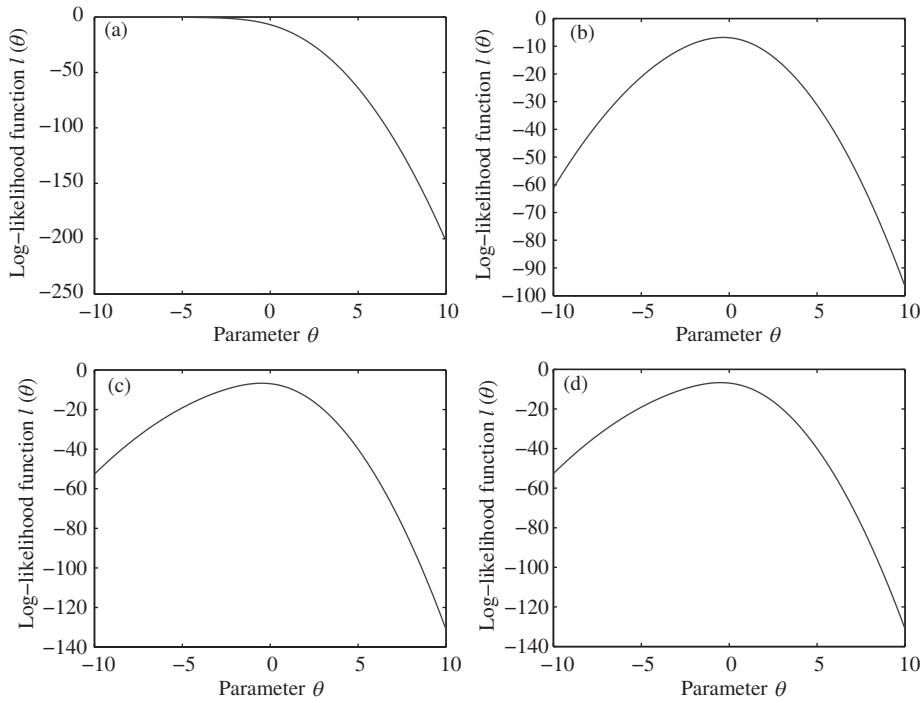

Figure 1 Curve of the log-likelihood function $l(\theta)$ under different cases with (a) (B1)+(C1), (b) (B1)+(C2), (c) (B2)+(C1), and (d) (B2)+(C2).

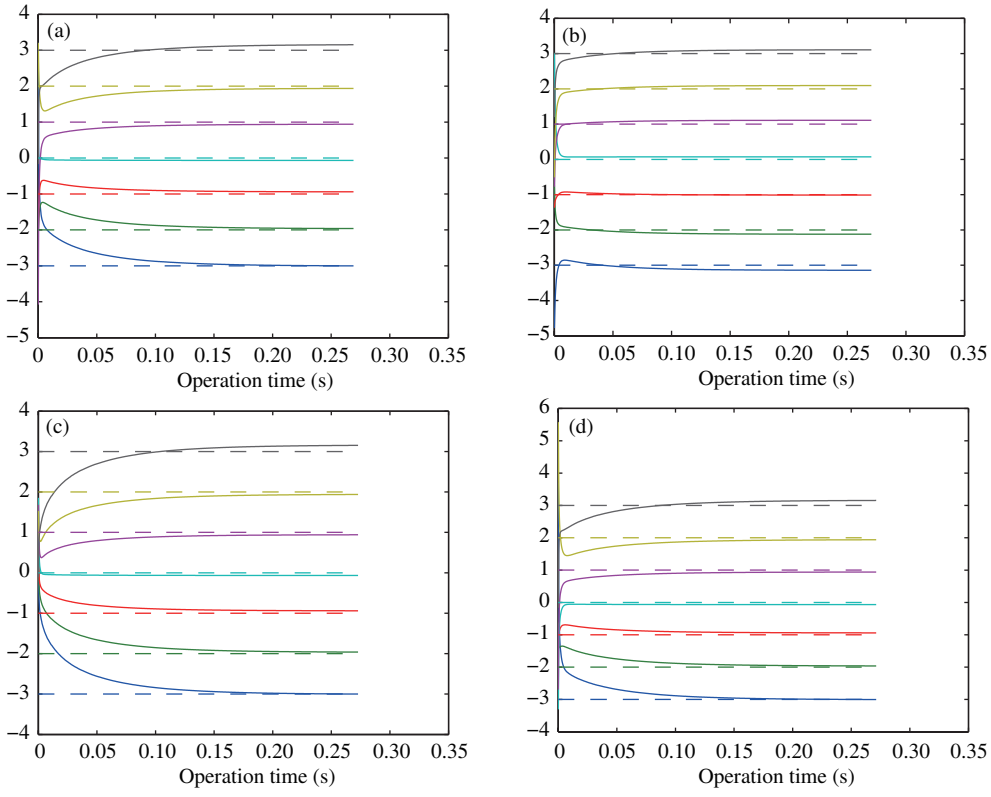

Figure 2 The solid lines are the curves of all 7 components of estimated parameter $\hat{\theta}_t$. The dotted lines correspond to the actual parameter. Estimates with (a) initial vector 1; (b) initial vector 2; (c) initial vector 3; (d) initial vector 4.

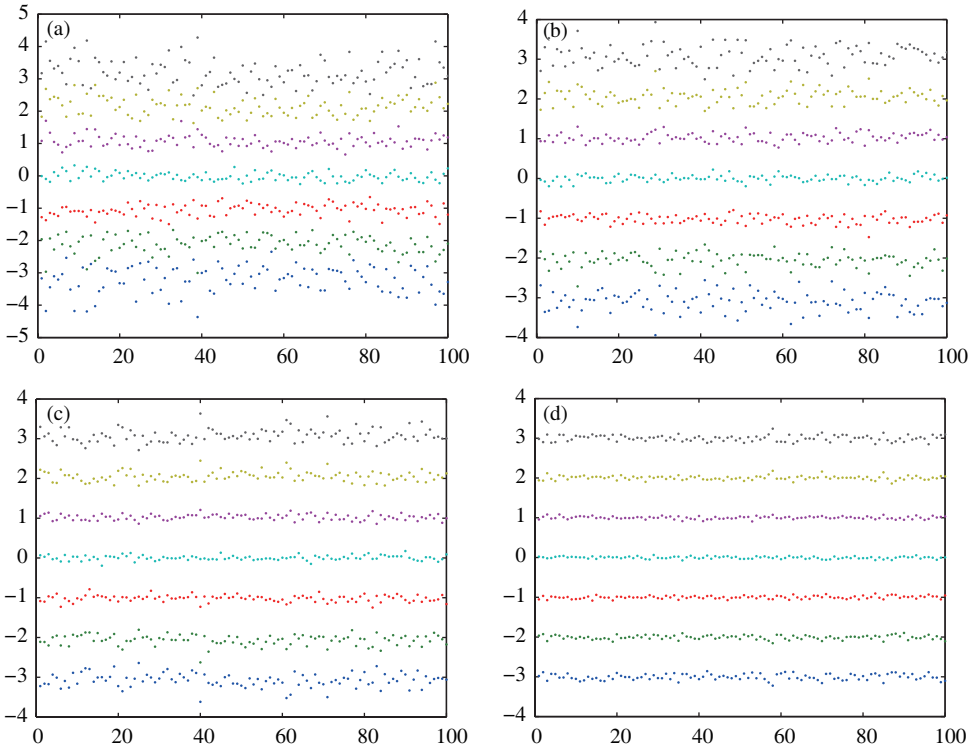

Figure 3 The corresponding component of the ML estimates of 100 times sampling. ML estimates with data length (a) $N = 500$; (b) $N = 1000$; (c) $N = 2000$; and (d) $N = 10000$.

Table 3 The computing time per iteration: the data length N is fixed as 1000 and the model dimension n ranges from 5 to 35

	n=5	n=10	n=15	n=20	n=25	n=30	n=35
EM	4.3×10^{-4}	4.9×10^{-4}	5.5×10^{-4}	5.2×10^{-4}	5.3×10^{-4}	5.3×10^{-4}	8.1×10^{-4}
NR	0.0087	0.0102	0.0104	0.0100	0.0113	0.0125	0.0136

Table 4 The comparison of computing time per iteration: the model dimension is fixed at $n = 10$ and data length n ranges from 5 to 35

	N=500	N=1000	N=1500	N=2000	N=2500	N=3000	N=3500
EM	3.6×10^{-4}	4.7×10^{-4}	7.5×10^{-4}	7.3×10^{-4}	0.0012	0.0011	0.0016
NR	0.0032	0.0108	0.0222	0.0372	0.0611	0.0811	0.1180

advantages. EM-typed algorithm is robust, easy to programme, fast within each iteration, and does not need much storage space. While the NR method has a much more fast convergence rate if the starting value is close to the true one. Here limited to the specific binary-valued model and proposed algorithm, the respective features are illustrated by simulations. The whole process is operated on an ordinary personal computer.

6.4.1 *Computing time per iteration*

First, we compare the computing time per iteration. Extensive simulations show that model dimension n and data length N are two major factors that influence. We first fix the data length $N = 1000$ and change the model dimension n from 5 to 35. The related result can be seen in Table 3. At the increase of n, the computing time per iteration increases slightly. The EM algorithm is always obviously faster than the NR method.

On the other hand, we fixed the model dimension $n = 10$ and altered the data length N from 500 to 3500, the related results can be seen in Table 4. As the increase of N, the computing time of both

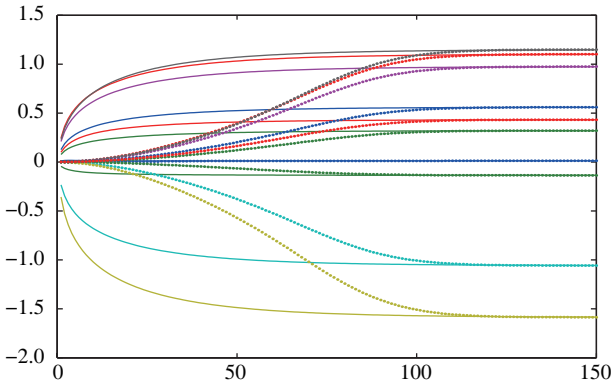

Figure 4 The iteration process of EM and NR methods with data length $N = 1000$ and model dimension $n = 10$. The solid line represents the EM algorithm and dotted line represents the NR method.

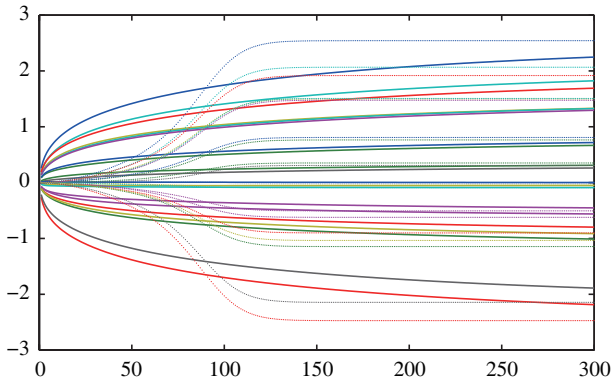

Figure 5 The iteration process of EM and NR methods with data length $N = 1000$ and model dimension $n = 20$. The solid line represents the EM algorithm and the dotted line represents the NR method.

methods increases significantly. The rate advantage of EM algorithm is still very obvious.

6.4.2 *Convergence rate comparison*

From the previous section, we can see that the proposed EM-typed algorithm needs much less time than NR method to carry out an iteration. But whether NR method needs much less iteration steps to complete the parameter estimation? Which one is faster comparing the whole computing time?

In this section, we compare the convergence rate, in short, we identify which one takes less iteration steps to complete the identification process. The actual parameter, input data and error terms are randomly selected based on standard normal distribution, the initial iterative parameter is fixed at zero point. The data length is fixed as 1000, Figure 4 shows the iteration process when model dimension is 10 and Figure 5 shows the case where model dimension is 20.

Figure 4 illustrates that if model dimension is relatively few, the EM algorithm needs less iteration steps to achieve the ML estimate than the NR method, which is very inspiring. We know that EM needs much less computing time per iteration than NR, hence, EM algorithm is much more competitive in the total computing time.

As the model dimension increases, the disadvantage of EM algorithm emerges. As is shown in Figure 5, when model dimension is 20, the EM algorithm needs more steps to complete the iteration (about twice steps). Considering the advantages in computing time per iteration, in this case, the EM algorithm is still faster than the NR method. But if the model dimension n increases to 50 or larger, the low convergence rate of EM-typed algorithm makes it almost unavailable.

6.4.3 *Robustness of the algorithm*

In our test procedure, we find that the EM-typed proposed algorithm indeed has extraordinary robustness. Whatever the data length N and model dimension n are, the procedure never messages error or warning.

For the NR method, if n is larger than 40, in some cases the procedure begins to fail, maybe because the condition number of Hessian matrix under specific parameter is excessively large. And if N is larger than 10000, the NR method always fails because the personal computer cannot provide enough storage space. Of course, many NR-updated algorithms are proposed to overcome these disadvantages. For example, the NR method or quasi-NR method with unfixed iteration step size is a good alternative but the step size calculation process is hard to programme and waste a lot of computing time.

7 Summary and discussion

In this paper, we considered system identification with batched binary data through the ML criterion. The local and global concave properties of log-likelihood function has been obtained by the negativeness of the Hessian matrix, which demonstrates that there is at most one finite ML estimate. Furthermore, a necessary and sufficient condition that there is a unique finite solution for the ML problem has been given. The ML estimate is proved to be a consistent estimate, which guarantees the rationality. An iterative algorithm is constructed to estimate the parameter and the convergence of the algorithm has been obtained, the limit of which is exactly the ML estimate. Surprisingly, the convergence has an exponential rate.

In this paper, we considered the binary-valued model, which is a common case in practical field. The single threshold brings much convenience for the analysis. If there are multiple thresholds, some techniques need to be updated, for example, the threshold detection problem will not be trivial any more. How to construct an algorithm to estimate the parameter and threshold simultaneously is an attractive question. The development from finite impulse response models to general linear and even nonlinear models is also promising.

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Conflict of interest The authors declare that they have no conflict of interest.

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Appendix A Proofs of Lemmas 1 and 3

Appendix A.1 Proof of Lemma 1

Proof. To prove the strictly decreasing property of the function $p(x)$, the derivation $p'(x)$ of $p(x)$ is needed:

$$p'(x) = \frac{-f(x)(F(x)+2f(x))}{F^3(x)}\left(F(x)x+f(x)\right).$$

Define $h(x) = F(x)x + f(x)$, then $h(x)$ has an opposite sign to $p'(x)$. In addition, $h'(x) = F(x) > 0$, and

$$\begin{aligned}\lim_{x\to-\infty} h(x) &= \lim_{x\to-\infty}\frac{F(x)}{1/x} = \lim_{x\to-\infty} -\frac{f(x)}{1/x^2} = \lim_{x\to-\infty}\left(-f(x)x^2\right)\\ &= \lim_{x\to-\infty}\left(-\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}x^2\right) = 0.\end{aligned}$$

We can see that $h(x) > 0$ holds for all $x \in \mathbb{R}$, which shows the strict decrease of $p(x)$.

To obtain the infimum value and supremum value of $p(x)$, we calculate the limit value of $p(x)$ at $+\infty$ and $-\infty$, which can be obtained by repeated use of L'Hôpital's rule, see, e.g. [1)],

$$\begin{aligned}\lim_{x\to+\infty} p(x) &= \lim_{x\to+\infty}\left(f(x)x+f^2(x)\right) = 0,\\ \lim_{x\to-\infty} p(x) &= \lim_{x\to-\infty}\frac{-f(x)x^2F(x)+f(x)F(x)-f^2(x)x}{2F(x)f(x)} = \frac{1}{2} - \lim_{x\to-\infty}\frac{x^2F(x)+f(x)x}{2F(x)}\\ &= \frac{1}{2} - \lim_{x\to-\infty}\frac{2xF(x)+x^2f(x)+f(x)-f(x)x^2}{2f(x)} = -\lim_{x\to-\infty}\frac{xF(x)}{f(x)}\\ &= -\lim_{x\to-\infty}\frac{F(x)+xf(x)}{-xf(x)} = 1+\lim_{x\to-\infty}\frac{F(x)}{xf(x)} = 1+\lim_{x\to-\infty}\frac{f(x)}{f(x)-f(x)x^2} = 1.\end{aligned}$$

Until now, the proof is completed.

As for example (2) of Remark 4, we give a graph to illustrate that $p_{t,d}(x) = -\frac{\mathrm{d}^2\log[F_{t,d}(x)]}{\mathrm{d}x^2} > 0$ does not always hold. It is trivial that $p_{t,d}(x) = -\frac{\mathrm{d}^2\log[F_{t,d}(x)]}{\mathrm{d}x^2} > 0$ is equivalent and that $\frac{\mathrm{d}\log[F_{t,d}(x)]}{\mathrm{d}x}$ strictly decreases. Figure A1 is the graph of function $\frac{\mathrm{d}\log[F_{t,d}(x)]}{\mathrm{d}x}$ under degree of freedom $d = 1$. We can see that function does not satisfy the strictly decreasing condition.

Appendix A.2 Proof of Lemma 3

Proof. The lemma is equivalent that for all $b < 0$, there exists an upper bound $r(b) > 0$, if $\|\theta\| \geqslant r(b), l(\theta) < b$. The below proof is to construct the upper bound $r(b)$.

For the simplicity, some definitions are given in advance,

$$\begin{aligned}f_{\alpha,k}(x) &= \log[F(C-\phi_k^{\mathrm{T}}\alpha x)]I_{[s_k=1]} + \log[F(\phi_k^{\mathrm{T}}\alpha x - C)]I_{[s_k=0]},\\ x_{b,k}(\alpha) &= \frac{C-F^{-1}(e^b)}{\phi_k^{\mathrm{T}}\alpha}I_{[s_k=1]} + \frac{C+F^{-1}(e^b)}{\phi_k^{\mathrm{T}}\alpha}I_{[s_k=0]},\end{aligned}$$

where $\alpha \in \mathbb{R}^n, b, x \in \mathbb{R}, k = 1, 2, \ldots, N$. Additionally, $f_{\alpha,k}(x) < 0$ because the CDF has an upper bound 1.

1) Apostol T M. Mathematical Analysis. 2nd ed. Boston: Addison Wesley Publishing Company, 1974.

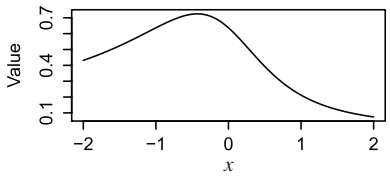

Figure A1 The function $\frac{\mathrm{dlog}[F_{t,d}(x)]}{\mathrm{d}x}$ under degree of freedom $d = 1$.

Arbitrarily select an unit vector α, Definition 2 tells that there exists a k_1 such that $\phi_{k_1}^{\mathrm{T}}\alpha(I_{[s_{k_1}=0]} - I_{[s_{k_1}=1]}) < 0$. Suppose $s_{k_1} = 1$ ($s_{k_1} = 0$ is a similar case), then $\phi_{k_1}^{\mathrm{T}}\alpha > 0$, furthermore, function $f_{\alpha,k_1}(x) = \log[F(C - \phi_k^{\mathrm{T}}\alpha x)]$ is a strictly decreasing function and tends to $-\infty$ as $x \to \infty$. Hence, for any given $b < 0$,

$$x \geqslant x_{b,k_1}(\alpha) \Rightarrow x \geqslant \frac{C - F^{-1}(e^b)}{\phi_k^{\mathrm{T}}\alpha} \Rightarrow C - \phi_k^{\mathrm{T}}\alpha x \leqslant F^{-1}(e^b)$$

$$\Rightarrow \log(F(C - \phi_k^{\mathrm{T}}\alpha x)) \leqslant b \Rightarrow f_{\alpha,k_1}(x) \leqslant b.$$

For all $x \geqslant x_{b,k_1}(\alpha), l(x\alpha) = \sum_{k=1}^{N} f_{\alpha,k}(x) < f_{\alpha,k_1}(x) \leqslant b$. This is equivalent that for any vector θ whose corresponding unit vector is α, if $\|\theta\| \geqslant x_{b,k_1}(\alpha), l(\theta) < b$. That is, the set $\{\theta, l(\theta) \geqslant b, \frac{\theta}{\|\theta\|} = \alpha\}$ is bounded by $x_{b,k_1}(\alpha)$.

Because $\phi_{k_1}^{\mathrm{T}}\alpha(I_{[s_{k_1}=0]} - I_{[s_{k_1}=1]}) < 0$, there exists an $\epsilon > 0$ and a set $A_\alpha(\epsilon) = \{\alpha_0 : \|\alpha_0\| = 1, \|\alpha_0 - \alpha\| \leqslant \epsilon\}$ such that for any $\beta \in A_\alpha(\epsilon), \phi_{k_1}^{\mathrm{T}}\beta(I_{[s_{k_1}=0]} - I_{[s_{k_1}=1]}) < 0$. This means for any $\beta \in A_\alpha(\epsilon)$, there exists $x_{b,k_1}(\beta)$ as the bound of $\{\theta, l(\theta) \geqslant b, \frac{\theta}{\|\theta\|} = \beta\}$. The continuity of function $x_{b,k_1}(\alpha_0)$ and compactness of $A_\alpha(\epsilon)$ infers that $\max_{\alpha_0 \in A_\alpha(\epsilon)} x_{b,k_1}(\alpha_0) < \infty$. Define an open subset $B_\alpha = \{\alpha_0 : \|\alpha_0\| = 1, \|\alpha_0 - \alpha\| < \epsilon/2\}$ and $r(\alpha) = \max_{\alpha_0 \in B_\alpha(\epsilon)} x_{b,k_1}(\alpha_0) < \infty$. We can see that for all θ whose corresponding unit vector is within an open set B_α, if $\|\theta\| \geqslant r(\alpha), l(\theta) < r$.

Let $S = \{\alpha : \|\alpha\| = 1\}$ denote N-dimensional sphere whose radius is 1. $\{B_\alpha, \alpha \in S\}$ is an open cover of S. Coupled with the compactness of S, there is a finite subcover $\{B_{\alpha_i}, i \leqslant m\}$ of S. Let r denote $\max_{1 \leqslant i \leqslant m} r(\alpha_i)$, we can see that for all θ, if $\|\theta\| \geqslant r, l(\theta) < b$.