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Controllability of Boolean control networks with state-dependent constraints

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Abstract This paper investigates the controllability of Boolean control networks (BCNs) with state-dependent constraints. A kind of input transformation is proposed to transfer a BCN with state-dependent input constraints into a BCN with free control input. Based on the proposed technique, a necessary and sufficient condition for controllability is obtained. It is shown that state-dependent constraints for the state can be equivalently expressed as input constraints. When a BCN has both input and state constraints, there is a possibility that the sets of admissible controls for some states are the empty set. To treat this kind of BCN, a variation of the input transformation is proposed and the problem of controllability is solved. An illustrative example is provided to explain the proposed method and results.

Keywords Boolean control network, state-dependent constraint, controllability, semi-tensor product, input transformation

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1 Introduction

A Boolean network (BN) is a special kind of finite-state dynamical system originally proposed by Kauffman in 1969 to model genetic regulatory networks [1,2] and they have been used to model the macroscopic behavior of a wide variety of complex systems. BNs are used in many fields including chemistry, biology, economics and computer science. In recent decades, BNs have been extensively studied. See, for instance [3–9], just to name a few.

Recently, Cheng and coauthors proposed a novel mathematical tool called the semi-tensor product (STP) of matrices, which has become a powerful tool for the analysis and design of finite-state dynamical systems [10]. Using STP, any logical expression can be expressed in a unified multi-linear form and thus a BN can be converted into a linear discrete-time dynamical system. Since the invention of STP, it has been successfully applied to many analysis and design problems of Boolean control networks (BCNs) and multi-valued logical systems. See, for instance [11–27]. In particular, this method has been applied to solve controllability of BCNs with free control inputs. See, for instance [28–30]. For recent developments of controllability of BCNs, refer to [28,31,32] and references therein. For a complete introduction to STP and its applications in different fields, refer to [33,34].

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Note that state and control constraints are very common in practical logical systems like gene regulatory networks. Recently, the control of BCNs with constraints has been attracting increasing attention. The first paper addressing this issue under the framework of STP is [35] where the controllability and stabilization of BCNs with state-independent constraints have been completely solved by a novel technique called the constrained incidence matrix method. Recently, reachability and controllability in avoiding undesirable states are discussed in [36].

Note, in many practical systems, the constraints for system state and control input are dependent on the state. For instance, we can model Chinese chess as a multi-valued logical system with two control inputs. At each step, the admissible strategies of each player and the allowable states at the next step as well are not free but depend on the current state. These restrictions are predefined by the rules of the game. If at a certain step, the set of admissible strategies for one of the players becomes the empty set, then the other player wins the game.

As far as we know, the control of BCNs with state-dependent constraints has not been addressed in the literature. In this paper, we propose an input transformation that is simple but effective in handling the state-dependent constraints of BCNs. Using the proposed input transformation, a BCN with input constraints is converted into a state-driven switched BCN and finally into a BCN with free control input. Based on this technique, the controllability problem of BCNs with state-dependent input constraints is completely solved. In addition, the largest controllable subset containing a given point is also discussed for BCNs with state-dependent input constraints.

This paper also shows that state-dependent state constraints can be equivalently expressed as state-dependent input constraints. To treat the case where the set of admissible control inputs becomes empty, a variation of an input transformation is proposed. This input transformation transfers a BCN with both input and state constraints into a discrete-time system with finite states. The system obtained is not completely equivalent to the original BCN because the input transformation is irreversible. However, the reachability between nonzero states for the original BCN and the obtained system is equivalent. Based on this, the controllability problem for this kind of BCN is solved.

This paper is arranged as follows. In Section 2, preliminaries for the STP of matrices, the algebraic form of BCNs and the properties of Boolean matrices are introduced. In Section 3, formulations and definitions are provided. In Section 4, the controllability of BCNs with input constraints is discussed. Section 5 discusses the controllability of BCNs with both input and state constraints. Section 6 gives an intuitive explanation for the input transformation through state transfer graphs. An illustrative example is given in Section 7 and some concluding remarks are drawn in Section 8.

2 Preliminaries

In this section, we briefly review some preliminaries including the basic concept of the STP of matrices, the algebraic form of BCNs and the properties of Boolean matrices. Some lemmas used in this paper are also introduced. For details, refer to [33].

2.1 STP of matrices and algebraic form of BCNs

Definition 1 (see [33]). Let A and B be $m \times n$ and $p \times q$ matrices, respectively. The semi-tensor product of A and B is

$$A \ltimes B = (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p}), \tag{1}$$

where $\otimes$ represents the Kronecker product, α is the least common multiple of n and p, and I_k denotes the $k \times k$ identity matrix.

The logical domain $\mathscr{D} := \{T := 1, F := 0\}$, from which any logical variable takes its values. A logical function with n arguments is a mapping $f : \mathscr{D}^n \to \mathscr{D}$. For any positive integer p, Δ_q represents the set of columns of the identity matrix I_q and δ_q^j represents the jth column of I_q. Also the vectors $T := 1 \sim \delta_2^1$ and $F := 0 \sim \delta_2^2$. Here, $\sim$ represents equivalence. With this notation, the logical domain $\mathscr{D}$ can be equivalently regarded as Δ_2 and any logical function with n arguments can be viewed as a mapping $f : \Delta_2^n \to \Delta_2$. Using the STP of matrices, Δ_2^n can be equivalently regarded as Δ_{2^n} by the one-to-one correspondence $\phi : \Delta_2^n \to \Delta_{2^n}$, $(x_1, \dots, x_n) \mapsto x_1 \ltimes \cdots \ltimes x_n$. Any logical function $f(x_1, \dots, x_n)$ with logical arguments $x_1, \dots, x_n \in \Delta_2$ can be expressed in a multi-linear form $f : \Delta_{2^n} \to \Delta_2$ as [33]

$$f(x_1, \dots, x_n) = M_f \ltimes x_1 \ltimes x_2 \ltimes \cdots \ltimes x_n, \tag{2}$$

where $M_f \in \mathscr{L}_{2\times 2^n}$, which is uniquely determined by f, is called the structural matrix of f. Here, $\mathscr{L}_{m\times n}$ represents the set of all $m \times n$ logical matrices, i.e., all of the $m \times n$ matrices A with $\mathrm{Col}(A) \subseteq \Delta_m$, where $\mathrm{Col}(A)$ represents the set of the columns of A. STP degenerates to the traditional product of matrices if all the matrices involved have matching sizes, thus the symbol $\ltimes$ can be omitted without causing any confusion. To determine the structural matrix of a given logical function, refer to [33].

A BCN with n state nodes and m input nodes is defined as

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ \quad\vdots \\ x_n(t+1) = f_n(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \end{cases}$$

where $f_i : \mathscr{D}^{n+m} \to \mathscr{D}$, $i = 1, 2, \dots, n$, are logical functions. Using the STP of matrices, a BCN can be alternatively expressed as the algebraic form [33]

$$x(t+1) = Lu(t)x(t), \tag{3}$$

where $x := x_1 \ltimes \cdots \ltimes x_n \in \Delta_{2^n}$ and $u := u_1 \ltimes \cdots \ltimes u_m \in \Delta_{2^m}$. $L \in \mathscr{L}_{2^n\times 2^{n+m}}$ is the structural matrix that is uniquely determined by the logical functions f_i. In this paper, we directly use the algebraic form (3) to present our results.

2.2 Boolean matrix

A Boolean matrix $X = (x_{ij})$ is an $m \times n$ matrix with $x_{ij} \in \mathscr{D}$. The set of all $m \times n$ Boolean matrices is denoted by $\mathscr{B}_{m\times n}$. When a logical operator acts on Boolean matrices, it is assumed to act on them element by element. For instance, for any two Boolean matrices $X = (x_{ij})$, $Y = (y_{ij}) \in \mathscr{B}_{m\times n}$, $X \wedge Y := (x_{ij} \wedge y_{ij})$ and $X \vee Y := (x_{ij} \vee y_{ij})$. The symbols $\vee$ and $\wedge$ represent logical OR and AND, respectively. The addition of Boolean matrices is defined as

$$\begin{cases} \alpha +_{\mathscr{B}} \beta := \alpha \vee \beta, \quad \forall \alpha, \beta \in \mathscr{D}, \\ (\mathscr{B}) \sum\limits_{i=1}^{n} \alpha_i := \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n, \quad \forall \alpha_i \in \mathscr{D}, \\ X +_{\mathscr{B}} Y = (x_{ij} +_{\mathscr{B}} y_{ij}) \in \mathscr{B}_{m\times n}, \quad \forall X, Y \in \mathscr{B}_{m\times n}. \end{cases}$$

For any $X \in \mathscr{B}_{m\times n}$ and $Y \in \mathscr{B}_{n\times p}$, the Boolean product of X and Y is defined as

$$X \ltimes_{\mathscr{B}} Y := Z = (z_{ij})_{m\times p} \in \mathscr{B}_{m\times p},$$

$$z_{ij} = (\mathscr{B}) \sum_{k=1}^{n} x_{ik} \wedge y_{kj}.$$

For $X \in \mathscr{B}_{n\times n}$, Boolean powers are defined as

$$X^{(k)} := \underbrace{X \ltimes_{\mathscr{B}} X \ltimes_{\mathscr{B}} \cdots \ltimes_{\mathscr{B}} X}_{k}, \quad \forall k \in \mathbb{Z}_+.$$

See, for instance [33] for the detail.

In this paper, we regard the set of logical matrices $\mathscr{L}_{m\times n}$ as a subset of $\mathscr{B}_{m\times n}$, i.e., $\mathscr{L}_{m\times n} \subset \mathscr{B}_{m\times n}$. In particular, $\Delta_m \subset \mathscr{B}_{m\times 1}$. Thus the operations defined above for Boolean matrices also apply to logical matrices. For any nonzero Boolean vector $x \in \mathscr{B}_{m\times 1}$, a logical vector $z \in \Delta_m$ is called a logical component of x if $z \wedge x = z$. $\mathscr{S}(x)$ is the set of all logical components of x, i.e.,

$$\mathscr{S}(x) := \{z \in \Delta_m \mid z \wedge x = z\}. \tag{4}$$

If $x = 0_{m\times 1}$, then $\mathscr{S}(x) = \emptyset$. For any $x, y \in \mathscr{B}_{m\times 1}$,

$$\mathscr{S}(x) \cap \mathscr{S}(y) = \mathscr{S}(x \wedge y),$$
$$\mathscr{S}(x) \cup \mathscr{S}(y) = \mathscr{S}(x \vee y).$$

2.3 Some lemmas

Lemma 1 (see [33]). Let $X \in \mathbb{R}^m$ and $Y = \mathbb{R}^n$. Then $YX = W_{[m,n]}XY$, where $W_{[m,n]}$ is the swap matrix with index $[m, n]$ defined by

$$W_{[m,n]} := [I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \ldots, I_n \otimes \delta_m^m].$$

Lemma 2 (see [33]). Let $X \in \Delta_k$. Then $X^2 = M_{r,k}X$, where $M_{r,k}$ is the base-k power reducing matrix defined by

$$M_{r,k} := [\delta_k^1 \otimes \delta_k^1, \delta_k^2 \otimes \delta_k^2, \ldots, \delta_k^k \otimes \delta_k^k].$$

Lemma 3. Let $Z \in \mathscr{B}_{p\times 1}$ be any nonzero Boolean vector and $L \in \mathscr{B}_{p\times q}$ be any nonzero Boolean matrix with $\mathrm{Col}(L) \subseteq \Delta_p \cup \{0_{p\times 1}\}$. Then

(1) If a logical vector $v \in \Delta_q$ satisfies

$$v \wedge (L^{\mathrm{T}} \ltimes_{\mathscr{B}} Z) = v, \tag{5}$$

then

$$(Lv) \wedge Z = Lv. \tag{6}$$

(2) If v is a logical vector such that $Lv \neq 0_{p\times 1}$, then Eq. (6) also implies (5).

Proof. Since Z is nonzero, it can be decomposed as the summation of its logical components as $Z = \sum_{s=1}^{k} \delta_p^{i_s}$. Thus

$$L^{\mathrm{T}} \ltimes_{\mathscr{B}} Z = (\mathscr{B}) \sum_{s=1}^{k} L^{\mathrm{T}} \delta_p^{i_s} = (\mathscr{B}) \sum_{s=1}^{k} \mathrm{Col}_{i_s}(L^{\mathrm{T}}).$$

The condition (5) holds with $v = \delta_q^j$ if and only if there exists an s such that $\delta_q^j \wedge \mathrm{Col}_{i_s}(L^{\mathrm{T}}) = \delta_q^j$, which is in turn equivalent to the existence of s such that $(L^{\mathrm{T}})_{ji_s} = 1$, i.e., $(L)_{i_s j} = 1$. This implies that $\mathrm{Col}_j(L) = \delta_p^{i_s}$, thus

$$\delta_p^{i_s} \wedge \mathrm{Col}_j(L) = \mathrm{Col}_j(L).$$

Or equivalently,

$$\delta_p^{i_s} \wedge (Lv) = Lv. \tag{7}$$

Thus eq. (6) holds. The claim 2 can be proved by noting that the analysis above is reversible if $Lv \neq 0_{p\times 1}$.

3 Problem setting

A BCN with state-dependent input constraints as considered in this paper is described as

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ u(t) \in \mathscr{U}_{\sigma(x(t))}, \end{cases} \tag{8}$$

with $x(t) \in \Delta_{2^n}$, $u(t) \in \Delta_{2^m}$ and $L \in \mathscr{L}_{2^n \times 2^{n+m}}$. $\sigma : \Delta_{2^n} \to \{1, 2, \ldots, 2^n\}$ is a mapping defined as

$$\sigma(\delta_{2^n}^i) := i, \quad 1 \leqslant i \leqslant 2^n, \tag{9}$$

and for each $1 \leqslant i \leqslant 2^n$, there is an associated subset $\mathscr{U}_i \subseteq \Delta_{2^m}$. The collection of subsets $\{\mathscr{U}_i\}_{1\leqslant i\leqslant 2^n}$ characterizes the state-dependent input constraint. Whenever $x(t) = \delta_{2^n}^i$, $\sigma(x(t)) = i$ and $\mathscr{U}_{\sigma(x(t))} = \mathscr{U}_i$, which means that the input is only allowed to take values from $\mathscr{U}_i$.

Definition 2. For BCN (8), a state $X_d \in \Delta_{2^n}$ is said to be k-step reachable from the initial state $X_0 \in \Delta_{2^n}$ if there exists a control sequence $\{u(i)\}_{0 \leqslant i \leqslant k-1}$ such that

(1) $x(0) = X_0$ and $x(k) = X_d$;

(2) $u(t) \in \mathscr{U}_{\sigma(x(t))}$, $\forall 0 \leqslant t \leqslant k-1$.

X_d is said to be reachable from X_0 if it is k-step reachable from X_0 for some $k \in \mathbb{Z}_+$.

The sets of all points that are k-step reachable and reachable from X_0 are denoted by $\mathscr{R}^{(k)}(X_0)$ and $\mathscr{R}(X_0)$, respectively. Since Δ_{2^n} is a finite set, it is not difficult to show that

$$\mathscr{R}(X_0) = \bigcup_{s=1}^{2^n} \mathscr{R}^{(s)}(X_0).$$

For any $X_d \in \Delta_{2^n}$, define

$$\mathscr{R}_{-1}^{(k)}(X_d) := \{x \in \Delta_{2^n} \mid X_d \in \mathscr{R}^{(k)}(x)\},$$
$$\mathscr{R}_{-1}(X_d) := \{x \in \Delta_{2^n} \mid X_d \in \mathscr{R}(x)\}.$$

Definition 3. BCN (8) is said to be controllable if $\mathscr{R}(X) = \Delta_{2^n}$, $\forall X \in \Delta_{2^n}$.

Definition 4. The k-step controllability matrix is defined as $\boldsymbol{T}_{k,C} := (\gamma_{ij})_{2^n \times 2^n}$, where

$$\gamma_{ij} := \begin{cases} 1, & \delta_{2^n}^i \in \mathscr{R}^{(k)}(\delta_{2^n}^j), \\ 0, & \text{otherwise}. \end{cases}$$

The controllability matrix is defined as $\boldsymbol{T}_C := (\gamma_{ij})_{2^n \times 2^n}$, where

$$\gamma_{ij} := \begin{cases} 1, & \delta_{2^n}^i \in \mathscr{R}(\delta_{2^n}^j), \\ 0, & \text{otherwise}. \end{cases}$$

From the definition above, $\delta_{2^n}^i$ is reachable from $\delta_{2^n}^j$ if and only if $(\boldsymbol{T}_C)_{ij} = 1$. Thus BCN (8) is controllable if and only if all of the entries of $\boldsymbol{T}_C$ are 1's.

In this paper, we also consider BCNs with both input and state constraints described by

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ u(t) \in \mathscr{U}_{\sigma(x(t))}, \\ x(t+1) \in \mathscr{X}_{\sigma(x(t))}, \quad x(0) \in \mathscr{X}_0, \end{cases} \tag{10}$$

with $x(t) \in \Delta_{2^n}$, $u(t) \in \Delta_{2^m}$ and $L \in \mathscr{L}_{2^n \times 2^{n+m}}$. $\sigma : \Delta_{2^n} \to \{1, 2, \ldots, 2^n\}$ is defined in (9). $\{\mathscr{U}_i\}_{1 \leqslant i \leqslant 2^n}$ is a collection of subsets of Δ_{2^m} that characterizes the input constraint. Similarly, the state constraint is characterized by the collection of subsets $\{\mathscr{X}_i\}_{0 \leqslant i \leqslant 2^n}$ of Δ_{2^n}. Whenever $x(t) = \delta_{2^n}^i$, then the state at the next time step is limited to $x(t+1) \in \mathscr{X}_i \subseteq \Delta_{2^n}$. In particular, the initial state is restricted with $x(0) \in \mathscr{X}_0$.

Definition 5. For BCN (10), a state $X_d \in \Delta_{2^n}$ is said to be k-step reachable from $X_0 \in \Delta_{2^n}$ if there exists a control sequence $\{u(i)\}_{0 \leqslant i \leqslant k-1}$ such that

(1) $x(0) = X_0$ and $x(k) = X_d$,

(2) $x(t+1) \in \mathscr{X}_{\sigma(x(t))}$, $\forall 0 \leqslant t \leqslant k-1$,

(3) $u(t) \in \mathscr{U}_{\sigma(x(t))}$, $\forall 0 \leqslant t \leqslant k-1$,

X_d is said to be reachable from $X_0 \in \Delta_{2^n}$ if it is k-step reachable from X_0 for some $k \in \mathbb{Z}_+$.

The subsets $\mathscr{R}^{(k)}(X_0)$, $\mathscr{R}_{-1}^{(k)}(X_0)$, $\mathscr{R}(X_0)$ and $\mathscr{R}_{-1}(X_0)$ are defined in the same way as before.

Remark 1. Note that Definition 5 does not require $X_0 \in \mathscr{X}_0$. The reason is that, for any $X_d \in \mathscr{R}(X_0)$ with $X_0 \in \mathscr{X}_0$, the destination state X_d and the intermediate states in the path from X_0 to X_d do not necessarily belong to $\mathscr{X}_0$. Thus, there is a necessity to investigate reachability between states that do not belong to $\mathscr{X}_0$.

For BCN (10), the concepts of controllable subsets and controllability matrices are literally the same as Definition 7 and Definition 4, respectively, but the underlying reachability is replaced by Definition 5.

Definition 6. BCN (10) is said to be controllable if $\mathscr{R}(X) = \Delta_{2^n}$, $\forall X \in \mathscr{X}_0$.

4 Controllability of BCN with input constraints

4.1 Input transformation and controllability

Given a collection of $p \times q$ matrices U_i, $1 \leqslant i \leqslant 2^n$, and the mapping $\sigma : \Delta_{2^n} \to \{1, 2, \dots, 2^n\}$ defined in (9),

$$U_{\sigma(x)} = \boldsymbol{U}x, \quad \forall x \in \Delta_{2^n},$$

where

$$\boldsymbol{U} := [U_1\ U_2\ \cdots\ U_{2^n}]. \tag{11}$$

Note that the above was also used to solve the output controllability of state-driven switched BCNs in [32]. Now we are ready to state one of the main results.

Proposition 1. Suppose that $U_i \in \mathscr{L}_{2^m \times 2^m}$, $1 \leqslant i \leqslant 2^n$, is a collection of logical matrices satisfying

$$\mathrm{Col}(U_i) = \mathscr{U}_i, \tag{12}$$

and define $\boldsymbol{U}$ as in (11). Then the k-step controllability matrix and the controllability matrix of BCN (8) are, respectively

$$\boldsymbol{T}_{k,C} = (L_U \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})^{(k)}, \tag{13}$$

$$\boldsymbol{T}_C = (\mathscr{B}) \sum_{s=1}^{2^n} (L_U \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})^{(s)}, \tag{14}$$

where

$$L_U = L\boldsymbol{U}W_{[2^n,2^{n+m}]}M_{r,2^n}W_{[2^m,2^n]}. \tag{15}$$

Proof. First of all, note that the matrices U_i satisfying (12) always exist. For instance, if $m = 2$ and $\mathscr{U}_i = \{\delta_4^1, \delta_4^3\}$, then we can choose $U_i = \delta_4[1\ 1\ 3\ 3]$. Construct an input transformation

$$u(t) = U_{\sigma(x(t))}v(t), \tag{16}$$

for BCN (8), where $v \in \Delta_{2^m}$ is a new control input. Under this transformation, BCN (8) is changed to the switched BCN

$$x(t+1) = LU_{\sigma(x(t))}v(t)x(t). \tag{17}$$

By Lemmas 1 and 2, we have

$$\begin{aligned} x(t+1) &= LU_{\sigma(x(t))}v(t)x(t) \\ &= L\boldsymbol{U}x(t)v(t)x(t) \\ &= L\boldsymbol{U}W_{[2^n,2^{n+m}]}x^2(t)v(t) \\ &= L\boldsymbol{U}W_{[2^n,2^{n+m}]}M_{r,2^n}x(t)v(t) \\ &= L\boldsymbol{U}W_{[2^n,2^{n+m}]}M_{r,2^n}W_{[2^m,2^n]}v(t)x(t). \end{aligned}$$

Thus we finally change the switched BCN (17) to the form

$$x(t+1) = L_U v(t)x(t), \tag{18}$$

where L_U is defined in (15). According to the construction of U_i, for any control sequence $\{u(t)\}$ satisfying the constraint $u(t) \in \mathscr{U}_{\sigma(x(t))}$, we can always choose a control sequence $\{v(t)\}$ such that Eq. (16) holds. In addition, if Eq. (16) is satisfied, then $u(t) \in \mathscr{U}_{\sigma(x(t))}$ and the switched BCN (18) and BCN (8) will produce the same solution provided that the initial states are the same. Based on this observation, the controllability matrices of these two systems coincide. Split L_U into 2^m blocks with equal sizes as

$$L_U = [\mathrm{Blk}_1(L_U)\ \mathrm{Blk}_2(L_U)\ \cdots\ \mathrm{Blk}_{2^m}(L_U)]. \tag{19}$$

Define

$$\mathcal{M}_U := (\mathscr{B})\sum_{i=1}^{2^m} \mathrm{Blk}_i(L_U) = L_U \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}.$$

Using the results in [20], the k-step controllability matrix and the controllability matrix of BCN (18) are respectively

$$\boldsymbol{T}_{k,C} = \mathcal{M}_U^{(k)}, \tag{20}$$

$$\boldsymbol{T}_C = (\mathscr{B})\sum_{s=1}^{2^n} \mathcal{M}_U^{(s)}. \tag{21}$$

Thus the claims follow.

Remark 2. In general, the choice of U_i is not unique, but the results obtained do not depend on the choice provided that Eq. (12) holds.

Proposition 2. The following claims hold

(1) For any $X_0 \in \Delta_{2^n}$,

$$\mathscr{R}^{(k)}(X_0) = \mathscr{S}(\boldsymbol{T}_{k,C}X_0),$$
$$\mathscr{R}(X_0) = \mathscr{S}(\boldsymbol{T}_C X_0).$$

(2) For any $X_d \in \Delta_{2^n}$,

$$\mathscr{R}_{-1}^{(k)}(X_d) = \mathscr{S}(\boldsymbol{T}_{k,C}^{\mathrm{T}}X_d),$$
$$\mathscr{R}_{-1}(X_d) = \mathscr{S}(\boldsymbol{T}_C^{\mathrm{T}} X_d).$$

(3) BCN (8) is controllable if and only if

$$(\boldsymbol{T}_C)_{ij} = 1, \quad \forall 1 \leqslant i, j \leqslant 2^n.$$

Proof. Claim 3 above is obviously true by the definition of the controllability matrix $\boldsymbol{T}_C$. In the following, we only prove the first formula of claim 1. The other formula and claim 2 can be proved similarly.

Suppose that $X_0 = \delta_{2^n}^j$. According to the definition of $\boldsymbol{T}_{k,C}$,

$$\begin{aligned}\mathscr{R}^{(k)}(X_0) &= \{\delta_{2^n}^i \mid (\boldsymbol{T}_{k,C})_{ij} = 1\}\\ &= \{\delta_{2^n}^i \mid \delta_{2^n}^i \wedge (\boldsymbol{T}_{k,C}X_0) = \delta_{2^n}^i\}\\ &= \mathscr{S}(\boldsymbol{T}_{k,C}X_0).\end{aligned}$$

4.2 Shortest control sequence design

For any pair of states $\delta_{2^n}^{i_0}$ and $\delta_{2^n}^{i_d}$ satisfying $\delta_{2^n}^{i_d} \in \mathscr{R}(\delta_{2^n}^{i_0})$, a shortest control sequence $\{v(k)\}$ for (18) that drives the system from $\delta_{2^n}^{i_0}$ to $\delta_{2^n}^{i}$ can be designed using the algorithm proposed in [20]. As soon as $\{v(k)\}$ is obtained, it can then be transferred to a control sequence $\{u(k)\}$ for the original BCN (8) using (16). The design procedure is as follows:

(1) Determine the shortest transition period k by

$$k := \min_{s\in\mathbb{Z}_+}\{s \mid (\boldsymbol{T}_{s,C})_{i_d i_0} = 1\}.$$

(2) Choose an admissible path $\delta_{2^n}^{i_0} \to \delta_{2^n}^{i_1} \to \cdots \to \delta_{2^n}^{i_k} = \delta_{2^n}^{i_d}$ as follows: First of all, $\delta_{2^n}^{i_1}$ should be chosen from the intersection of $\mathscr{R}^{(1)}(\delta_{2^n}^{i_0})$ and $\mathscr{R}_{-1}^{(k-1)}(\delta_{2^n}^{i_d})$. By Proposition 2,

$$\begin{aligned}\delta_{2^n}^{i_1} &\in \mathscr{R}^{(1)}(\delta_{2^n}^{i_0}) \cap \mathscr{R}_{-1}^{(k-1)}(\delta_{2^n}^{i_d})\\ &= \mathscr{S}[\mathrm{Col}_{i_0}(\boldsymbol{T}_{1,C}) \wedge \mathrm{Col}_i(\boldsymbol{T}_{k-1,C}^{\mathrm{T}})].\end{aligned}$$

Similarly, $\delta_{2^n}^{i_s}$, $1 \leqslant s \leqslant k-1$, can be chosen recursively as

$$\delta_{2^n}^{i_s} \in \mathscr{S}[\mathrm{Col}_{i_{s-1}}(\boldsymbol{T}_{1,C}) \wedge \mathrm{Col}_i(\boldsymbol{T}_{k-s,C}^{\mathrm{T}})], \quad 1 \leqslant s \leqslant k-1.$$

(3) Design a control sequence $\{v(s) = \delta_{2^m}^{j_s}\}_{0 \leqslant s \leqslant k-1}$ for (18) where j_s is determined through

$$j_s \in \{j \mid \mathrm{Col}_{i_s}(\mathrm{Blk}_j(L_U)) = \delta_{2^n}^{i_{s+1}}\}.$$

(4) Transfer $\{v(s)\}$ to the control sequence $\{u(s)\}_{0 \leqslant s \leqslant k-1}$ for the original BCN by

$$u(s) = U_{i_s} v(s) = \mathrm{Col}_{j_s}(U_{i_s}).$$

4.3 Controllable subset

Definition 7. A subset $\mathscr{C} \subseteq \Delta_{2^n}$ is called a controllable subset if $X_1 \in \mathscr{R}(X_2)$, $\forall X_1, X_2 \in \mathscr{C}$.

Suppose that $\mathscr{C}_1$, $\mathscr{C}_2 \subseteq \Delta_{2^n}$ are two controllable subsets. If $\mathscr{C}_1 \cap \mathscr{C}_2 \neq \emptyset$, then the union $\mathscr{C}_1 \cup \mathscr{C}_2$ is also a controllable subset. If there is a controllable subset that contains $X \in \Delta_{2^n}$, then the union of all the controllable subsets containing $X \in \Delta_{2^n}$ is still a controllable subset. This subset is called the largest controllable subset containing X and it is denoted by $\mathscr{C}(X)$. If a state X does not belong to any controllable subset, then $\mathscr{C}(X) = \emptyset$.

Proposition 3. For any $X \in \Delta_{2^n}$,

$$\mathscr{C}(X) = \mathscr{R}(X) \cap \mathscr{R}_{-1}(X). \tag{22}$$

Proof. According to the definitions of $\mathscr{R}(X)$ and $\mathscr{R}_{-1}(X)$, we always have $\mathscr{C}(X) \subseteq \mathscr{R}(X) \cap \mathscr{R}_{-1}(X)$. We prove the claim in two cases:

(1) If $\mathscr{R}(X) \cap \mathscr{R}_{-1}(X) = \emptyset$, then $\mathscr{C}(X) = \emptyset$ and Eq. (22) is obviously true.

(2) If $\mathscr{R}(X) \cap \mathscr{R}_{-1}(X) \neq \emptyset$, then $X \in \mathscr{R}(X) \cap \mathscr{R}_{-1}(X)$. In this case, for any X_1, $X_2 \in \mathscr{R}(X) \cap \mathscr{R}_{-1}(X)$, then $X_1 \in \mathscr{R}(X)$ and $X \in \mathscr{R}(X_2)$. Thus $\mathscr{R}(X) \cap \mathscr{R}_{-1}(X)$ is a controllable subset containing X. According to the definition of $\mathscr{C}(X)$, $\mathscr{R}(X) \cap \mathscr{R}_{-1}(X) \subseteq \mathscr{C}(X)$.

By Propositions 2 and 3, we have the following.

Corollary 1. For any $\delta_{2^n}^j$, $\mathscr{C}(\delta_{2^n}^j) = \mathscr{S}\left[\mathrm{Col}_j(\boldsymbol{T}_C) \wedge \mathrm{Col}_j(\boldsymbol{T}_C^{\mathrm{T}})\right]$.

Corollary 2. Suppose that j^* is an integer arbitrarily chosen from $\{1, 2, \ldots, 2^n\}$. Then BCN (8) is controllable if and only if $\mathrm{Col}_{j^*}(\boldsymbol{T}_C) \wedge \mathrm{Col}_{j^*}(\boldsymbol{T}_C^{\mathrm{T}}) = \mathbf{1}_{2^n}$.

Remark 3. Corollary 1 can be used to find all of the largest controllable subsets of a BCN.

5 Controllability of BCNs with both input and state constraints

In this section, we consider controllability of BCN (10) with both input and state constraints. First of all, we show that the state constraint can be equivalently expressed as an input constraint.

Lemma 4. For BCN (10), the state constraint

$$x(t+1) \in \mathscr{X}_{\sigma(x(t))} \tag{23}$$

is satisfied if and only if

$$u(t) \in \hat{\mathscr{U}}_{\sigma(x(t))},$$

where

$$\hat{\mathscr{U}}_i := \mathscr{S}\left[(LW_{[2^n,2^m]}\delta_{2^n}^i)^{\mathrm{T}} \ltimes_{\mathscr{B}} \chi_i\right],$$
$$\chi_i := \sum_{x \in \mathscr{X}_i} x.$$

Proof. Note that $x(t+1) = Lu(t)x(t) = LW_{[2^n,2^m]}x(t)u(t)$. Suppose that $x(t) = \delta_{2^n}^i$. Note that $LW_{[2^n,2^m]}\delta_{2^n}^i$ is a logical matrix, thus for any $u(t) \in \Delta_{2^m}$, $LW_{[2^n,2^m]}\delta_{2^n}^i u(t) \neq 0$. Thus the state constraint (23) is equivalent to

$$\chi_i \wedge [LW_{[2^n,2^m]}\delta_{2^n}^i u(t)] = [LW_{[2^n,2^m]}\delta_{2^n}^i u(t)]. \tag{24}$$

By Lemma 3, Eq. (24) is equivalent to

$$u(t) \wedge \left[(LW_{[2^n,2^m]}\delta_{2^n}^i)^{\mathrm{T}} \ltimes_{\mathscr{B}} \chi_i\right] = u(t).$$

That is,

$$u(t) \in \mathscr{S}\left[(LW_{[2^n,2^m]}\delta_{2^n}^i)^{\mathrm{T}} \ltimes_{\mathscr{B}} \chi_i\right].$$

By Lemma 4, BCN (10) is equivalent to

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ u(t) \in \mathscr{U}_{\sigma(x(t))} \cap \hat{\mathscr{U}}_{\sigma(x(t))}, \\ x(0) \in \mathscr{X}_0. \end{cases} \tag{25}$$

Except for the initial state constraint $x(0) \in \mathscr{X}_0$, another major difference between (25) and (8) is that, for BCN (25), there is a possibility that $\mathscr{U}_{\sigma(x(t))} \cap \hat{\mathscr{U}}_{\sigma(x(t))} = \emptyset$, which means that no admissible control input exists such that $x(t+1) \in \mathscr{X}_{\sigma(x(t))}$. In this case, the solution cannot be extended over time any more and $\mathscr{R}(x(t)) = \emptyset$.

Proposition 4. Suppose that $\hat{U}_i \in \mathscr{B}_{2^m \times 2^m}$, $1 \leqslant i \leqslant 2^n$, is any collection of Boolean matrices satisfying

$$\mathrm{Col}(\hat{U}_i) = \begin{cases} \mathscr{U}_i \cap \hat{\mathscr{U}}_i, & \mathscr{U}_i \cap \hat{\mathscr{U}}_i \neq \emptyset, \\ \{0_{2^m \times 1}\}, & \mathscr{U}_i \cap \hat{\mathscr{U}}_i = \emptyset. \end{cases} \tag{26}$$

Then the k-step controllability matrix and the controllability matrix of BCN (10) are respectively

$$\boldsymbol{T}_{k,C} = (L_{\hat{U}} \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})^{(k)}, \tag{27}$$

$$\boldsymbol{T}_C = (\mathscr{B}) \sum_{s=1}^{2^n} \boldsymbol{T}_{s,C}, \tag{28}$$

where $L_{\hat{U}} \in \mathscr{B}_{2^n \times 2^{n+m}}$ is defined as

$$L_{\hat{U}} := L\hat{\boldsymbol{U}}W_{[2^n,2^{n+m}]}M_{r,2^n}W_{[2^m,2^n]}. \tag{29}$$

Proof. By Lemma 4, the original BCN (10) is equivalent to BCN (25). Construct an input transformation for BCN (25) as

$$u(t) = \hat{U}_{\sigma(x(t))}v(t), \tag{30}$$

where $\hat{U}_i$, $1 \leqslant i \leqslant 2^n$, are given in (26). Obviously, $\hat{U}_{\sigma(x)} = \hat{\boldsymbol{U}}x$, $\forall x \in \Delta_{2^n}$, where $\hat{\boldsymbol{U}} := [\hat{U}_1 \ \hat{U}_2 \ \cdots \ \hat{U}_{2^n}] \in \mathscr{B}_{2^m \times 2^{n+m}}$. Thus the input transformation can be equivalently expressed as

$$u(t) = \hat{\boldsymbol{U}}x(t)v(t). \tag{31}$$

Under this transformation, BCN (25) is transferred into the discrete-time system

$$\begin{cases} x(t+1) = L\hat{\boldsymbol{U}}x(t)v(t)x(t), \\ x(0) \in \mathscr{X}_0. \end{cases} \tag{32}$$

By Lemmas 1 and 2, and following the same argument as in Subsection 4.1, Eq. (32) can be rewritten as

$$\begin{cases} x(t+1) = L_{\hat{U}}v(t)x(t), \\ x(0) \in \mathscr{X}_0, \end{cases} \tag{33}$$

where $L_{\hat{U}}$ is given in (29).

Note that system (33) is no longer a BCN since some of the columns of $L_{\hat{U}}$ might be zero. Actually, $\mathrm{Col}(L_{\hat{U}}) \subseteq \{0_{2^n \times 1}\} \cup \Delta_{2^n} := \Delta_{2^n}^o$. Thus, the state space of system (33) is $\Delta_{2^n}^o$ instead of Δ_{2^n}. In addition, for any control sequence $\{v(k)\}$, the corresponding sequence $\{u(k)\}$ defined by the input transformation (30) is not necessarily an admissible control sequence for the original BCN since there is possibility that $u(k) = 0_{2^m\times 1} \notin \Delta_{2^m}$. Based on this observation, system (33) and the original BCN (25) are not equivalent. However, from the construction of the input transformation, we have the following relationships between (33) and the original BCN (10):

(1) If $\{x(k)\}$ is any solution to (33) under control sequence $\{v(k)\}$ satisfying $x(k) \in \Delta_{2^n}$, $\forall k \in \mathbb{Z}_+$, then it is also a solution to (25) under control sequence $\{u(k)\}$, which is defined in (30).

(2) If $\{x(k)\}$ is a solution to the original BCN (25) under control sequence $\{u(k)\}$, then there is a control sequence $\{v(k)\}$ satisfying (30) such that $\{x(k)\}$ is also a solution to (33) under $\{v(k)\}$.

Based on the observation above and the construction of the input transformation, one easily sees that the reachability between nonzero states are equivalent for these two systems. If we define the controllability matrices between nonzero states for system (33) as in Definition 4, i.e., ignore the information about reachability from and to $\delta_{2^n}^0 := 0_{2^n\times 1}$, then the controllability matrices of BCN (33) coincide with the controllability matrices between nonzero states for system (33). Thus, we only need to prove that the controllability matrices between nonzero states for system (33) can be expressed as the forms given in (27) and (28).

The proof of this claim is similar to the argument in [20], the difference being that the system (33) considered here is not a BCN. We provide a detailed proof in the following. Split $L_{\hat{U}}$ into 2^m blocks with equal sizes as

$$L_{\hat{U}} = [\mathrm{Blk}_1(L_{\hat{U}})\ \mathrm{Blk}_2(L_{\hat{U}})\ \cdots\ \mathrm{Blk}_{2^m}(L_{\hat{U}})], \tag{34}$$

then obviously,

$$(\mathscr{B})\sum_{i=1}^{2^m} \mathrm{Blk}_i(L_{\hat{U}}) = L_{\hat{U}} \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}.$$

Thus the matrix $\boldsymbol{T}_{k,C}$ defined in (27) can be alternatively expressed as

$$\begin{aligned} \boldsymbol{T}_{k,C} &= \left[(\mathscr{B})\sum_{i=1}^{2^m} \mathrm{Blk}_i(L_{\hat{U}})\right]^{(k)} \\ &= (\mathscr{B})\sum_{1\leqslant j_1,\ldots,j_k\leqslant 2^m} \mathrm{Blk}_{j_1}(L_{\hat{U}})\,\mathrm{Blk}_{j_2}(L_{\hat{U}})\cdots\mathrm{Blk}_{j_k}(L_{\hat{U}}). \end{aligned} \tag{35}$$

Suppose that for system (33), $X_d = \delta_{2^n}^{i_d} \neq \delta_{2^n}^0$ is k-step reachable from $X_0 = \delta_{2^n}^{i_0} \neq \delta_{2^n}^0$, then there exists a control sequence $v(i) = \delta_{2^m}^{j_i}$, $0 \leqslant i \leqslant k-1$, such that

$$\begin{aligned} \delta_{2^n}^{i_d} &= (L_{\hat{U}}\delta_{2^m}^{j_{k-1}})(L_{\hat{U}}\delta_{2^m}^{j_{k-2}})\cdots(L_{\hat{U}}\delta_{2^m}^{j_0})\delta_{2^n}^{i_0} \\ &= \mathrm{Blk}_{j_{k-1}}(L_{\hat{U}})\,\mathrm{Blk}_{j_{k-2}}(L_{\hat{U}})\cdots\mathrm{Blk}_{j_0}(L_{\hat{U}})\delta_{2^n}^{i_0}. \end{aligned}$$

This implies that the (i_d, i_0)-entry of the matrix $\mathrm{Blk}_{j_{k-1}}(L_{\hat{U}})\ \mathrm{Blk}_{j_{k-2}}(L_{\hat{U}})\ \cdots\ \mathrm{Blk}_{j_0}(L_{\hat{U}})$ is 1. By the expression (35), $(\boldsymbol{T}_{k,C})_{i_d i_0} = 1$.

In contrast, if $(\boldsymbol{T}_{k,C})_{i_d i_0} = 1$, then by (35), there exist j_i, $0 \leqslant i \leqslant k-1$, such that the (i_d, i_0)-entry of the matrix $\mathrm{Blk}_{j_{k-1}}(L_{\hat{U}})\ \mathrm{Blk}_{j_{k-2}}(L_{\hat{U}})\ \cdots\ \mathrm{Blk}_{j_0}(L_{\hat{U}})$ is 1. Then we can construct a control sequence $v(i) = \delta_{2^m}^{j_i}$, $0 \leqslant i \leqslant k-1$, such that $x(0) = \delta_{2^n}^{i_0}$ and $x(k) = \delta_{2^n}^{i_d}$. This proves that the matrix $\boldsymbol{T}_{k,C}$ defined in (27) is the k-step controllability matrices between nonzero states for system (33).

The proof of the claim that the matrix $\boldsymbol{T}_C$ defined in (28) equals the controllability matrices between nonzero states for system (33) is trivial by noting that $(\boldsymbol{T}_C)_{ij} = 1$ if and only if there exists a k such that $(\boldsymbol{T}_{k,C})_{i_d i_0} = 1$, i.e., $\delta_{2^n}^j$ is k-step reachable from $\delta_{2^n}^i$ for some k.

Remark 4. If $\{x(k)\}$ is a solution to (33) under control sequence $\{v(k)\}$ satisfying $x(T) \in \Delta_{2^n}$ and $x(T+1) = 0$ for some $T \in \mathbb{Z}_+$, then $\hat{U}_{x(T)} = 0_{2^m\times 2^m}$. This means that, for BCN (25), $\mathscr{U}_{\sigma(x(T))} \cap \hat{\mathscr{U}}_{\sigma(x(T))} = \emptyset$. In other words, there are no admissible control inputs for the original BCN that can produce an admissible state at $T+1$.

Proposition 5. Suppose that $\mathscr{X}_0 = \{\delta_{2^n}^{i_1}, \delta_{2^n}^{i_2}, \ldots, \delta_{2^n}^{i_q}\}$ and let $\boldsymbol{X}_0 = \delta_{2^n}[i_1\ i_2\ \cdots\ i_q]$. Then the following claims hold:

(1) BCN (10) is controllable if and only if

$$(\boldsymbol{T}_C \boldsymbol{X}_0)_{ij} = 1, \quad \forall 1 \leqslant i \leqslant 2^n, \forall 1 \leqslant j \leqslant q. \tag{36}$$

(2) Suppose that j^* is an integer arbitrarily chosen from $\{1, 2, \ldots, q\}$. Then BCN (10) is controllable if and only if

$$\mathrm{Col}_{j^*}(\boldsymbol{T}_C \boldsymbol{X}_0) = \mathbf{1}_{2^n}, \quad \mathrm{Row}_{j^*}(\boldsymbol{T}_C \boldsymbol{X}_0) = \mathbf{1}_q^{\mathrm{T}}.$$

Proof.

(1) By the definition of $\boldsymbol{X}_0$, $\mathrm{Col}_j(\boldsymbol{T}_C \boldsymbol{X}_0) = \mathrm{Col}_{i_j}(\boldsymbol{T}_C)$. Thus Eq. (36) holds if and only if $\mathrm{Col}_{i_j}(\boldsymbol{T}_C) = \mathbf{1}_{2^n}$, which is equivalent to $\mathscr{R}(\delta_{2^n}^{i_j}) = \Delta_{2^n}$, $\forall j = 1, 2, \ldots, q$.

(2) On one hand, the condition $\mathrm{Col}_{j^*}(\boldsymbol{T}_C \boldsymbol{X}_0) = \mathbf{1}_{2^n}$ is equivalent to $\mathscr{R}(\delta_{2^n}^{i_{j^*}}) = \Delta_{2^n}$. On the other hand, note that $\mathrm{Row}_{j^*}(\boldsymbol{T}_C \boldsymbol{X}_0) = \mathrm{Row}_{j^*}(\boldsymbol{T}_C)\boldsymbol{X}_0$, thus the condition $\mathrm{Row}_{j^*}(\boldsymbol{T}_C \boldsymbol{X}_0) = \mathbf{1}_q^{\mathrm{T}}$ is equivalent to $(\boldsymbol{T}_C)_{j^* i_j} = 1$, $\forall j = 1, 2, \ldots, q$. This is in turn equivalent to $\delta_{2^n}^{j^*} \in \mathscr{R}(\delta_{2^n}^{i_j})$, $\forall j = 1, 2, \ldots, q$. Thus, for any $X_d \in \Delta_{2^n}$ and $X_0 \in \mathscr{X}_0$, $X_d \in \mathscr{R}(\delta_{2^n}^{i_{j^*}})$ and $\delta_{2^n}^{j^*} \in \mathscr{R}(X_0)$, which imply $X_d \in \mathscr{R}(X_0)$.

6 An intuitive explanation for the input transformation via a state transfer graph

This section gives the input transformation (30) an intuitive explanation via state transfer graphs. For BCN (10), there is a directed graph $\mathcal{G}_u$ whose nodes are the elements of Δ_{2^n}. The edges are defined as follows. There is a directed edge from $\delta_{2^n}^i$ to $\delta_{2^n}^j$ if and only if $\delta_{2^n}^j \in \mathscr{X}_i$ and there exists a $\delta_{2^m}^s \in \mathscr{U}_i$ such that $\delta_{2^n}^j = L\delta_{2^m}^s \delta_{2^n}^i$. The state transfer graph for the system (33), $\mathcal{G}_v$, is a directed graph whose nodes are the elements of $\Delta_{2^n} \cup \{\delta_{2^n}^0\}$ and the edges are defined as follows. There is a directed edge from $\delta_{2^n}^i$ to $\delta_{2^n}^j$ if and only if there exists a $\delta_{2^m}^s \in \Delta_{2^m}$ such that $\delta_{2^n}^j = L_{\hat{U}} \delta_{2^m}^s \delta_{2^n}^i$.

Note that BCN (10) is equivalent to BCN (25) and the system (33) is obtained from BCN (10) through input transformation (30). The relations between the state transfer graphs $\mathcal{G}_u$ and $\mathcal{G}_v$ of these two systems are as follows:

(1) If there is a directed edge from $\delta_{2^n}^i$ to $\delta_{2^n}^j$ in $\mathcal{G}_u$, then there exists a $\delta_{2^m}^s \in \mathscr{U}_i \cap \hat{\mathscr{U}_i}$ such that $\delta_{2^n}^j = L\delta_{2^m}^s \delta_{2^n}^i$. This implies that $\mathscr{U}_i \cap \hat{\mathscr{U}_i} \neq \emptyset$ and $\hat{U}_i \neq 0_{2^m \times 2^m}$, thus there is a $\delta_{2^m}^{s'}$ such that $\delta_{2^m}^s = \hat{U}_i \delta_{2^m}^{s'}$. This means that, in $\mathcal{G}_v$, there is also a directed edge from $\delta_{2^n}^i$ to $\delta_{2^n}^j$, but there is no edge from $\delta_{2^n}^i$ to $\delta_{2^n}^0$. In contrast, if there is a directed edge from $\delta_{2^n}^i$ to $\delta_{2^n}^j$ in $\mathcal{G}_v$ with $\delta_{2^n}^i$, $\delta_{2^n}^j \neq 0$, then there is also an edge from $\delta_{2^n}^i$ to $\delta_{2^n}^j$ in $\mathcal{G}_u$.

(2) If there is no edge from $\delta_{2^n}^i$ to any other state in $\mathcal{G}_u$, then $\mathscr{U}_i \cap \hat{\mathscr{U}_i} = \emptyset$ and $\hat{U}_i = 0_{2^m \times 2^m}$. Then for any $\delta_{2^m}^s \in \Delta_{2^m}$, $L_{\hat{U}} \delta_{2^m}^s \delta_{2^n}^i = \delta_{2^n}^0$. This means that, in $\mathcal{G}_v$, there is an edge from $\delta_{2^n}^i$ to $\delta_{2^n}^0$ and there are no edges from $\delta_{2^n}^i$ to any other nonzero states. In contrast, if there is an edge from $\delta_{2^n}^i$ to $\delta_{2^n}^0$ in $\mathcal{G}_v$, then $\hat{U}_i = 0_{2^m \times 2^m}$, i.e., $\mathscr{U}_i \cap \hat{\mathscr{U}_i} = \emptyset$ and thus there is no edge from $\delta_{2^n}^i$ to any other states in $\mathcal{G}_u$.

(3) In $\mathcal{G}_v$, there is a directed edge from $\delta_{2^n}^0$ to itself, but there is no edge from $\delta_{2^n}^0$ to any other nonzero state.

From the above analysis, the graph $\mathcal{G}_v$ can be obtained from $\mathcal{G}_u$ by adding an edge from $\delta_{2^n}^i$ to $\delta_{2^n}^0$ for any $\delta_{2^n}^i$ such that $\mathscr{U}_i \cap \hat{\mathscr{U}_i} = \emptyset$ and also one from $\delta_{2^n}^0$ to itself. From the relations between the state transfer graphs, though the input transformation is not reversible, the reachabilities between nonzero states are equivalent. See Example 1 in the following for an intuitive explanation.

Example 1. Consider a BCN of the form (10) with $L = \delta_4[2\ 3\ 4\ 1\ 3\ 1\ 3\ 2]$. $\mathscr{U}_i$ and $\mathscr{X}_i$ are listed in Table 1. The state transfer graph $\mathcal{G}_u$ is given in Figure 1(a). Choose $\hat{U}_i$ as in Table 1, then $L_{\hat{U}} = L\hat{\boldsymbol{U}} W_{[4,8]} M_{r,4} W_{[2,4]} = \delta_4[2\ 3\ 4\ 0\ 3\ 3\ 3\ 0]$. The state transfer graph $\mathcal{G}_v$ for system $x(t+1) = L_{\hat{U}} v(t) x(t)$ is given in Figure 1(b). One sees that $\mathcal{G}_u$ is a sub-graph of $\mathcal{G}_v$ and the connections between nonzero states in these two graphs are completely the same.

Table 1 The subsets and the choice of $\hat{U}_i$ in Example 1

i	$\mathscr{U}_i$	$\mathscr{X}_i$	$\hat{\mathscr{U}}_i$	$\mathscr{U}_i \cap \hat{\mathscr{U}}_i$	$\hat{U}_i$
1	Δ_2	Δ_4	Δ_2	Δ_2	$\delta_2[1\ 2]$
2	Δ_2	$\{\delta_4^3\}$	$\{\delta_2^1\}$	$\{\delta_2^1\}$	$\delta_2[1\ 1]$
3	Δ_2	Δ_4	Δ_2	Δ_2	$\delta_2[1\ 2]$
4	$\{\delta_2^1\}$	$\{\delta_4^2,\ \delta_4^3\}$	$\{\delta_2^2\}$	$\emptyset$	$\delta_2[0\ 0]$

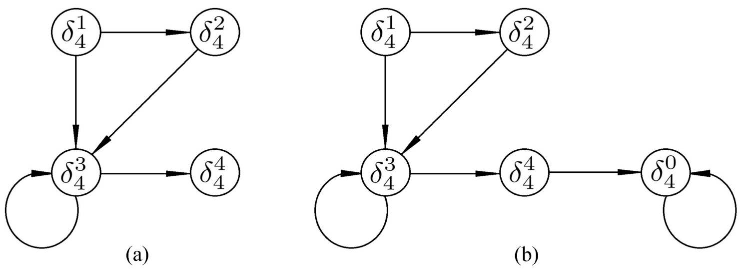

Figure 1 State transfer graphs (a) $\mathcal{G}_u$ and (b) $\mathcal{G}_v$ in Example 1.

Table 2 Determining the matrices $\hat{U}_i$

i	$\mathscr{U}_i$	$\hat{\mathscr{U}}_i$	$\mathscr{U}_i \cap \hat{\mathscr{U}}_i$	$\hat{U}_i$	i	$\mathscr{U}_i$	$\hat{\mathscr{U}}_i$	$\mathscr{U}_i \cap \hat{\mathscr{U}}_i$	$\hat{U}_i$
1	$\{\delta_4^1\}$	$\Delta_4 \setminus \{\delta_4^3\}$	$\{\delta_4^1\}$	$\delta_4[1\ 1\ 1\ 1]$	9	$\{\delta_4^2\}$	$\{\delta_4^3\}$	$\emptyset$	$\delta_4[0\ 0\ 0\ 0]$
2	$\{\delta_4^1\}$	Δ_4	$\{\delta_4^1\}$	$\delta_4[1\ 1\ 1\ 1]$	10	$\{\delta_4^2\}$	$\{\delta_4^2\}$	$\emptyset$	$\delta_4[0\ 0\ 0\ 0]$
3	Δ_4	Δ_4	Δ_4	$\delta_4[1\ 2\ 3\ 4]$	11	Δ_4	$\{\delta_4^2\}$	$\{\delta_4^2\}$	$\delta_4[2\ 2\ 2\ 2]$
4	Δ_4	$\{\delta_4^1,\ \delta_4^4\}$	$\{\delta_4^1,\ \delta_4^4\}$	$\delta_4[1\ 1\ 4\ 4]$	12	$\Delta_4 \setminus \{\delta_4^2\}$	$\{\delta_4^2\}$	$\emptyset$	$\delta_4[0\ 0\ 0\ 0]$
5	$\{\delta_4^2\}$	$\emptyset$	$\emptyset$	$\delta_4[0\ 0\ 0\ 0]$	13	Δ_4	$\Delta_4 \setminus \{\delta_4^4\}$	$\Delta_4 \setminus \{\delta_4^4\}$	$\delta_4[1\ 2\ 3\ 3]$
6	Δ_4	$\{\delta_4^4\}$	$\{\delta_4^4\}$	$\delta_4[4\ 4\ 4\ 4]$	14	$\{\delta_4^1\}$	Δ_4	$\{\delta_4^1\}$	$\delta_4[1\ 1\ 1\ 1]$
7	$\{\delta_4^2\}$	$\{\delta_4^4\}$	$\emptyset$	$\delta_4[0\ 0\ 0\ 0]$	15	Δ_4	$\Delta_4 \setminus \{\delta_4^4\}$	$\Delta_4 \setminus \{\delta_4^4\}$	$\delta_4[1\ 2\ 3\ 3]$
8	$\Delta_4 \setminus \{\delta_4^4\}$	$\{\delta_4^1,\ \delta_4^4\}$	$\{\delta_4^1\}$	$\delta_4[1\ 1\ 1\ 1]$	16	$\{\delta_4^1\}$	$\{\delta_4^1,\ \delta_4^4\}$	$\{\delta_4^1\}$	$\delta_4[1\ 1\ 1\ 1]$

7 An illustrative example

Example 2. Consider a BCN of the form (8) with $n = 4$, $m = 2$ and $L = [L_1\ L_2\ L_3\ L_4]$, where

$$L_1 = \delta_{16}[15\ 3\ 14\ 5\ 12\ 7\ 8\ 6\ 8\ 9\ 10\ 11\ 16\ 13\ 2\ 4],$$
$$L_2 = \delta_{16}[4\ 15\ 15\ 12\ 11\ 10\ 9\ 12\ 8\ 7\ 6\ 1\ 4\ 15\ 2\ 9],$$
$$L_3 = \delta_{16}[9\ 15\ 6\ 8\ 12\ 11\ 12\ 9\ 5\ 9\ 10\ 12\ 14\ 15\ 1\ 7],$$
$$L_4 = \delta_{16}[16\ 15\ 4\ 13\ 9\ 5\ 6\ 1\ 9\ 5\ 8\ 9\ 11\ 3\ 7\ 5].$$

The control input constraints are as follows:
(1) When $x(t) = \delta_{16}^8$, $u(t) \neq \delta_4^4$;
(2) When $x(t) = \delta_{16}^{12}$, $u(t) \neq \delta_4^2$;
(3) When $x(t) \in \{\delta_{16}^1,\ \delta_{16}^2,\ \delta_{16}^{14},\ \delta_{16}^{16}\}$, only $u(t) = \delta_4^1$ is allowed;
(4) When $x(t) \in \{\delta_{16}^5,\ \delta_{16}^7,\ \delta_{16}^9,\ \delta_{16}^{10}\}$, only $u(t) = \delta_4^2$ is allowed.
In addition, we assume that the state is only allowed to evolve within

$$\mathscr{X} = \{\delta_{16}^1,\ \delta_{16}^2,\ \delta_{16}^3,\ \delta_{16}^4,\ \delta_{16}^5,\ \delta_{16}^6,\ \delta_{16}^{13},\ \delta_{16}^{14},\ \delta_{16}^{15},\ \delta_{16}^{16}\}.$$

The input and state constraints are transferred to the collections of subsets $\{\mathscr{U}_i\}$ and $\{\hat{\mathscr{U}}_i\}$, respectively. Then, based on $\{\mathscr{U}_i\} \cap \{\hat{\mathscr{U}}_i\}$, a collection of matrices $\{\hat{U}_i\}$ is selected; these are listed in Table 2. Define $\hat{\boldsymbol{U}} := [\hat{U}_1\ \hat{U}_2; \cdots\ \hat{U}_{16}]$, then

$$L_{\hat{U}} := L\hat{\boldsymbol{U}} W_{[8,32]} M_{r,8} W_{[4,8]}$$
$$= [\mathrm{Blk}_1(L_{\hat{U}}),\ \mathrm{Blk}_2(L_{\hat{U}}),\ \mathrm{Blk}_3(L_{\hat{U}}),\ \mathrm{Blk}_4(L_{\hat{U}})],$$

with

$$\begin{aligned}
\mathrm{Blk}_1(L_{\hat{U}}) &= \delta_{16}[15\ 3\ 14\ 5\ 0\ 5\ 0\ 6\ 0\ 0\ 6\ 0\ 16\ 13\ 2\ 4],\\
\mathrm{Blk}_2(L_{\hat{U}}) &= \delta_{16}[15\ 3\ 15\ 5\ 0\ 5\ 0\ 6\ 0\ 0\ 6\ 0\ 4\ 13\ 6\ 4],\\
\mathrm{Blk}_3(L_{\hat{U}}) &= \delta_{16}[15\ 3\ 6\ 13\ 0\ 5\ 0\ 6\ 0\ 0\ 6\ 0\ 14\ 13\ 1\ 4\ ],\\
\mathrm{Blk}_4(L_{\hat{U}}) &= \delta_{16}[15\ 3\ 4\ 13\ 0\ 5\ 0\ 6\ 0\ 0\ 6\ 0\ 14\ 13\ 1\ 4].
\end{aligned}$$

By Proposition 4, the controllability matrices $\boldsymbol{T}_{k,C}$ and $\boldsymbol{T}_C$ can be easily calculated; this calculation is omitted due to the length limitation. We can use $\boldsymbol{T}_C$ to determine the reachable set for any initial state and all of the largest controllable subsets. For instance, a simple calculation shows that

$$\begin{aligned}
[\mathrm{Col}_4(\boldsymbol{T}_C)]^{\mathrm{T}} &= [0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1],\\
\mathrm{Row}_4(\boldsymbol{T}_C) &= [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1].
\end{aligned}$$

Thus, by Proposition 2, we have

$$\mathscr{R}(\delta_{16}^4) = \mathscr{S}(\mathrm{Col}_4(\boldsymbol{T}_C)) = \left\{\delta_{16}^4, \delta_{16}^5, \delta_{16}^{13}, \delta_{16}^{14}, \delta_{16}^{16}\right\}$$

and by Corollary 1 the largest controllable subset containing δ_{16}^4 is

$$\begin{aligned}
\mathscr{C}(\delta_{16}^4) &= \mathscr{S}\left[\mathrm{Col}_4(\boldsymbol{T}_C) \wedge [\mathrm{Row}_4(\boldsymbol{T}_C)]^{\mathrm{T}}\right]\\
&= \left\{\delta_{16}^4, \delta_{16}^{13}, \delta_{16}^{14}, \delta_{16}^{16}\right\}.
\end{aligned}$$

In addition, by Proposition 5, this BCN is not controllable.

Choose $X_0 = \delta_{16}^4$ and $X_d = \delta_{16}^{14}$. It is easy to check that the shortest admissible path from X_0 to X_d is $\delta_{16}^4 \to \delta_{16}^{13} \to \delta_{16}^{14}$ and one of the admissible control sequences is $v(0) = \delta_4^3$ and $v(1) = \delta_4^4$. Thus the control sequence for the original BCN is $u(0) = \hat{U}_4 v(0) = \delta_4^3$ and $u(1) = \hat{U}_{13} v(1) = \delta_4^3$.

8 Concluding remarks

In this paper, we investigated controllability of BCNs with state-dependent constraints under the framework of STP of matrices and the algebraic form of BCNs. We proposed an input transformation that can transfer a BCN with input constraints into one with free input. Based on this, controllability matrices can be easily obtained. We showed that state constraints can be equivalently expressed as input constraints and a variation of the input transformation has been proposed for BCNs with both input and state constraints. An illustrative example has also been provided to explain the main idea and the results obtained in this paper.

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