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L-quantum spaces

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Abstract In this paper, based on a complete residuated lattice L, we introduce the definitions of L-quantum spaces and continuous mappings. Then we establish an adjunction between the category of stratified L-quantum spaces and the opposite category of two-sided L-quantales. We also prove that the category of sober L-quantum spaces is dually equivalent to the category of spatial two-sided L-quantales.

Keywords L-quantum space, L-quantale, adjunction, stratified L-quantum space, spatial two-sided L-quantale

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1 Introduction

A topological duality is a correspondence between two mathematical structures involving points and predicates such that isomorphic structures can be identified. Stone [1] originally found such a duality between topology and logic. Abramsky related the important application of Stone duality in theoretical computer science, particularly in denotational semantics of computer programming languages [2]. It provides the proper framework for understanding the relationship between denotational semantics and program logic. In [3], Isbell constructed an adjunction between the category of topological spaces and the category of locales (the opposite category of frames), which induces a Stone-type duality between the category of sober spaces and that of spatial frames. The Stone-type duality is exploited in theoretical computer science for the study of formal semantics. In order to study the fuzzy counterpart of the Isbell adjunction between topological spaces and locales, Yao defined L-frames via fuzzy posets, and then he established an adjunction between the category of stratified L-topological spaces and the category of L-locales. He also showed the equivalence between sobriety of stratified L-topological spaces and spatiality of L-locales [4], which can be considered as a generalization of the Stone-type duality.

Quantum logic was introduced by Birkhoff and von Neumann [5] in the 1930s as the logic of quantum mechanics. The theory of quantum logic has attracted widespread attention from theoretical computer scientists. It has been successfully applied in the study of automata theory [6–8]. Quantales were first introduced by Mulvey [9] in order to provide a new mathematical model for quantum mechanics. The term quantale was coined much more recently as a combination of "quantum logic" and "locale" by Mulvey; he proposed using quantales for studying the foundations of quantum logic. Quantale theory,

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which is connected with the operational semantics and symbolic semantics of computer language and is used to characterize the equivalences of various observations of the process semantics [10–12], is one of the mathematical foundations of theoretical computer science. In 1987, Girard proposed a new logic system (linear logic system) to provide logic support system for parallel computation and parallel processing in theoretical computer science [13]. In [14], Yetter applied quantale theory to the study of linear logic semantics and constructed the relationship between phase semantics of linear logic and quantales.

Program semantics of computer can be done using topological concepts [15]. In 1989, Borceux and Bossche proposed a more general model of non-commutative topology strongly based on the notion of the quantale. They called this non-commutative topology space a quantum space [16], which could be regarded as a topological analog for a quantale. In [17], He and Luo modified the definition of quantum space, and showed the duality between the full subcategory of quantales and the category of quantum spaces. Wang applied fuzzy orders to the theory of quantales and introduced the notions of L-quantales [18]. Inspired by papers [4,17,18], we introduce the notion of L-quantum spaces, and discuss the relationship between the category of stratified L-quantum spaces and the category of L-quantales.

The outline of this article is as follows: In Section 2, we review some basic definitions and results used in the rest of the paper. In Section 3, we introduce the notions of L-quantum spaces and continuous mappings and study their properties. In Section 4, we establish an adjunction between the category of stratified L-quantum spaces and the opposite category of two-sided L-quantales. In Section 5, we obtain the duality between sobriety of stratified L-quantum spaces and spatiality of two-sided L-quantales.

2 Preliminaries

For category theory, we refer to [19].

Definition 2.1 (See [20,21]). A residuated lattice is an algebraic structure $\mathcal{L} = (L;\ \wedge,\ \vee,\ *,\ \rightarrow,\ 0,\ 1)$ of type (2, 2, 2, 2, 0, 0) such that:

(RL1) $(L;\ \wedge,\ \vee,\ 0,\ 1)$ is a bounded lattice with the least element 0 and the greatest element 1;

(RL2) $(L;\ *,\ 1)$ is a commutative monoid with the identity 1;

(RL3) $\mathcal{L}$ satisfies the adjointness property, i.e., $\forall\ x,\ y,\ z \in L,\ x * y \leqslant z$ iff $x \leqslant y \rightarrow z$.

A residuated lattice is called complete if the underlying lattice is complete.

Theorem 2.2 (See [22,23]). Let L be a residuated lattice. Then

(R1) $a * b \leqslant a \wedge b$;

(R2) $a = 1 \rightarrow a$;

(R3) $a \leqslant b \Leftrightarrow a \rightarrow b = 1$;

(R4) $a \rightarrow (b \rightarrow c) = (a * b) \rightarrow c$;

(R5) $a \rightarrow b \geqslant b$.

When L is complete,

(R6) $a * (\bigvee_i b_i) = \bigvee_i (a * b_i)$;

(R7) $a \rightarrow (\bigwedge_i b_i) = \bigwedge_i (a \rightarrow b_i)$;

(R8) $(\bigvee_i b_i) \rightarrow a = \bigwedge_i (b_i \rightarrow a)$.

In this paper, L always denotes a complete residuated lattice unless stated otherwise.

Definition 2.3 (See [20,24]). A fuzzy poset is a pair (X, e) such that X is a set and $e : X \times X \longrightarrow L$ is a mapping (called a fuzzy partial order over X), satisfying the following conditions for all $x, y, z \in X$:

(E1) $e(x, x) = 1$ (reflexivity);

(E2) $e(x, y) * e(y, z) \leqslant e(x, z)$ (transitivity);

(E3) $e(x, y) = e(y, x) = 1$ implies $x = y$ (antisymmetry).

For a set X, L^X denotes the set of all L-fuzzy subsets of X, that is, the set of all mappings from X to L. For $\lambda \in L$, the constant mapping with the value λ is denoted by λ_X.

Example 2.4. (1) (The canonical fuzzy partial order on L) Define $e_L : L \times L \rightarrow L$ by $e_L(x, y) = x \rightarrow y$ for all $x, y \in L$. Then (L, e_L) is a fuzzy poset.

(2) Let X be a set. For all $A, B \in L^X$, the subsethood degree [25] $\mathrm{sub}_X(A, B)$ of A in B is defined as $\mathrm{sub}_X(A, B) = \bigwedge_{x\in X}(A(x) \to B(x))$. Then (L^X, sub_X) is a fuzzy poset [20].

(3) Let (X, e) be a fuzzy poset. Then $\leqslant_e = \{(x, y) \in X \times X \mid e(x, y) = 1\}$ is a partial order over X.

Definition 2.5 (See [26,27]). Let (X, e) be a fuzzy poset and $A \in L^X$. An element $x_0 \in X$ is called a join of A, expressed symbolically as $x_0 = \sqcup A$, if for all $x \in X$,

(J1) $A(x) \leqslant e(x, x_0)$;

(J2) $\bigwedge_{y\in X}(A(y) \to e(y, x)) \leqslant e(x_0, x)$.

Proposition 2.6 (See [26,27]). Let (X, e) be a fuzzy poset and $A \in L^X$. Then $x_0 = \sqcup A$ iff for all $x \in X,\ e(x_0, x) = \bigwedge_{y\in Y}(A(y) \to e(y, x))$.

Definition 2.7 (See [26,27]). A fuzzy poset (X, e) is called a fuzzy complete lattice if for all $A \in L^X$, $\sqcup A$ and $\sqcap A$ exist.

Proposition 2.8 (See [26,27]). Let (X, e) be a fuzzy poset. Then the following statements are equivalent:

(1) (X, e) is a fuzzy complete lattice.

(2) For all $A \in L^X, \sqcup A$ exists.

(3) For all $A \in L^X, \sqcap A$ exists.

Example 2.9. (L, e_L) is a fuzzy complete lattice, and for all $\phi \in L^L$, $\sqcup\phi = \bigvee_{a\in L}(\phi(a) * a)$, $\sqcap\phi = \bigwedge_{a\in L}(\phi(a) \to a)$.

Definition 2.10 (See [28]). Let $f : X \to Y$ be a mapping. The Zadeh forward powerset operator $f_L^{\to} : L^X \to L^Y$ and the Zadeh backward powerset operator $f_L^{\leftarrow} : L^Y \to L^X$ are, respectively, defined by $f_L^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$ for all $A \in L^X, y \in Y$ and by $f_L^{\leftarrow}(B) = B \circ f$ for all $B \in L^Y$.

Proposition 2.11 (See [4]). Let X be a set and $f : X \to L$ be a mapping. Then for all $A \in L^X,\ \sqcup f_L^{\to}(A) = \bigvee_{x\in X}(f(x) * A(x))$.

Definition 2.12 (See [18]). An L-quantale is a fuzzy complete lattice (X, e) with an associative binary operation & satisfying: $\forall\, a \in X,\ a\&\_ : X \to X$ and $\_\&a : X \to X$ are L-fuzzy join-preserving mappings, that is, $a\& (\sqcup A) = \sqcup(a\&\_)_L^{\to}(A)$ and $(\sqcup A)\&\, a = \sqcup(\_\&a)_L^{\to}(A)$ for all $A \in L^X$. If $\forall\, a \in X,\ a\&\, \top_X = a$ and $\top_X\&a = a$ ($\top_X$ being the top element of $(X, \leqslant_{e_X})$), we call $(X,\&,e)$ a two-sided L-quantale.

Example 2.13. The complete residuated lattice L itself is an L-quantale with respect to e_L.

Definition 2.14 (See [18]). Let $(X,\&,e)$ and $(Y,\&,e)$ be L-quantales. A mapping $f : X \to Y$ is called an L-quantale homomorphism if it satisfies $f(a\&b) = f(a)\&f(b)$ for all $a, b \in X$ and $f(\sqcup A) = \sqcup f_L^{\to}(A)$ for all $A \in L^X$. If f is also a bijection, we call f an L-quantale isomorphism. An L-quantale homomorphism is said to be strong if it satisfies $f(\top_X) = \top_Y$ ($\top_X$ and $\top_Y$ being, respectively, the top elements of $(X, \leqslant_{e_X})$ and $(Y, \leqslant_{e_Y})$).

Let L-Quant denote the category of two-sided L-quantales and strong L-quantale homomorphisms.

Proposition 2.15 (See [29,30]). Let A and B be fuzzy complete lattices. If $f : A \to B$ is an L-fuzzy join-preserving mapping, then $f : (A, \leqslant_{e_A}) \to (B, \leqslant_{e_B})$ preserves arbitrary sups.

Definition 2.16 (See [17]). A quantum space is a set X endowed with a family $\Omega(X) \subseteq \mathcal{P}(X)$ of open subsets and a binary operation,

$$\& : \Omega(X) \times \Omega(X) \to \Omega(X)$$

satisfying the following axioms:

(QS1) $\emptyset,\ X \in \Omega(X)$;

(QS2) $\forall\, \{U_i \mid i \in I\} \subseteq \Omega(X),\ \cup_{i\in I}U_i \in \Omega(X)$;

(QS3) $\forall\, U,\ V,\ W \in \Omega(X),\ (U\&V)\&W = U\&(V\&W)$;

(QS4) $\forall\, U,\ V \in \Omega(X),\ U \cap V \subseteq U\&V$;

(QS5) $\forall\, U \in \Omega(X), \{U_i \mid i \in I\} \subseteq \Omega(X),\ U\&(\cup_{i\in I}U_i) = \cup_{i\in I}(U\&U_i)$;

(QS6) $\forall\, U \in \Omega(X), \{U_i \mid i \in I\} \subseteq \Omega(X),\ (\cup_{i\in I}U_i)\&U = \cup_{i\in I}(U_i\&U)$.

Definition 2.17 (See [17]). Let $(X, \Omega(X))$ and $(Y, \Omega(Y))$ be quantum spaces. A mapping $f : X \to Y$ is called a continuous mapping if it satisfies the conditions that $f^{-1}(U) \in \Omega(X)$ for all $U \in \Omega(Y)$ and $f^{-1}(U)\&f^{-1}(V) = f^{-1}(U\&V)$ for all $U,\ V \in \Omega(Y)$.

3 *L*-quantum spaces

In this section, we shall introduce the definition of L-quantum spaces and study some properties of continuous mappings and open mappings.

Definition 3.1. Let X be a set, $\delta(X) \subseteq L^X$, and $\& : \delta(X) \times \delta(X) \to \delta(X)$. Consider the following conditions:

(1) $0_X, 1_X \in \delta(X)$;

(2) $\forall \{A_i \mid i \in I\} \subseteq \delta(X), \bigvee_{i\in I} A_i \in \delta(X)$;

(3) $\forall A, B, C \in \delta(X), (A\&B)\&C = A\&(B\&C)$;

(4) $\forall A \in \delta(X), B_i \subseteq \delta(X), A\&(\bigvee_{i\in I} B_i) = \bigvee_{i\in I}(A\&B_i)$;

(5) $\forall A \in \delta(X), B_i \subseteq \delta(X), (\bigvee_{i\in I} B_i)\&A = \bigvee_{i\in I}(B_i\&A)$;

(6) $\forall \lambda \in L, \lambda_X \in \delta(X)$;

(7) $\forall A \in \delta(X), x \in X, (A\&\lambda_X)(x) = (\lambda_X\&A)(x) = A(x) * \lambda$.

(i) $(X, \delta(X), \&)$ is called an L-quantum space if it satisfies (1)–(5).

(ii) $(X, \delta(X), \&)$ is called a stratified L-quantum space, if it satisfies (1)–(7).

Example 3.2. Let X be a set with more than two points, $\delta(X) \subseteq L^X$ be closed under pointwise union and $0_X,\ 1_X \in \delta(X)$. For all $A,\ B \in \delta(X)$, we define

$$A\&B = \begin{cases} 0_X, & B = 0_X, \\ A, & B \neq 0_X. \end{cases}$$

Then $(X, \delta(X), \&)$ is an L-quantum space but not a stratified L-quantum space.

Example 3.3. Let X be a set and $\delta(X) \subseteq L^X$ be all the constant mappings of X. For all $\lambda_X,\ \mu_X \in \delta(X),\ x \in X$, we define $(\lambda_X\&\ \mu_X)(x) = \lambda * \mu$. Then $(X, \delta(X), \&)$ is a stratified L-quantum space.

Example 3.4. Letting $X=\{x, y\}$ and $L=\{0, 1\}$, we put $\delta(X) = \{0_X, A, 1_X\}$, where $A(x) = 0, A(y)$=1. For all $A,\ B \in \delta(X), x \in X$, we define

$$(A\&B)(x) = \begin{cases} \lambda \wedge B(x), & A = \lambda_X, \\ A(x) \wedge \lambda, & B = \lambda_X, \\ 0, & \text{otherwise}. \end{cases}$$

Then $(X, \delta(X), \&)$ is a stratified L-quantum space.

Definition 3.5. Let $(X, \delta(X), \&)$ and $(Y, \delta(Y), \&)$ be L-quantum spaces. A mapping $f : X \to Y$ is called a continuous mapping if it satisfies $f_L^{\leftarrow}(A) \in \delta(X)$ for all $A \in \delta(Y)$ and $f_L^{\leftarrow}(A\&B) = f_L^{\leftarrow}(A)\&f_L^{\leftarrow}(B)$ for all $A,\ B \in \delta(Y)$.

Remark 3.6. Let $(X, \delta(X), \&)$ be an L-quantum space. Then $id_X : X \to X$ is a continuous mapping, where id_X is an identity mapping on the underlying set.

Let SL-QSp denote the category of stratified L-quantum spaces and continuous mappings.

Proposition 3.7. Let $(X, \delta(X), \&)$, $(Y, \delta(Y), \&)$, and $(Z, \delta(Z), \&)$ be L-quantum spaces. If $f : X \to Y$and $g : Y \to Z$ are continuous mappings, then $g \circ f$ is a continuous mapping.

Definition 3.8. Let $(X, \delta(X), \&)$ and $(Y, \delta(Y), \&)$ be L-quantum spaces. A mapping $f : X \to Y$ is called an open mapping if it satisfies $f_L^{\rightarrow}(A) \in \delta(X)$ for all $A \in \delta(X)$ and $f_L^{\rightarrow}(A\&B) = f_L^{\rightarrow}(A)\&f_L^{\rightarrow}(B)$ for all $A,\ B \in \delta(X)$.

Proposition 3.9. Let $(X, \delta(X), \&)$, $(Y, \delta(Y), \&)$, and $(Z, \delta(Z), \&)$ be L-quantum spaces. Then

(1) $id_X : X \to X$ is an open mapping.

(2) If $f : X \to Y$ and $g : Y \to Z$ are open mappings, then $g \circ f$ is an open mapping.

Definition 3.10. Let $(X, \delta(X), \&)$ and $(Y, \delta(Y), \&)$ be L-quantum spaces. A mapping $f : X \to Y$ is called a homeomorphism if f is a bijection and both f and f^{-1} are continuous mappings.

Proposition 3.11. Let $(X, \delta(X), \&)$ and $(Y, \delta(Y), \&)$ be L-quantum spaces and $f : X \to Y$ be a mapping. Then f is a homeomorphism iff f is a bijection and f is both continuous and open.

4 An adjunction between SL-QSp and L-Quant^{op}

Generally speaking, if $\mathcal{A}$ is a category, let $\mathcal{O}b(\mathcal{A})$ denote the class of $\mathcal{A}$-objects and $\mathcal{M}or(\mathcal{A})$ denote the class of $\mathcal{A}$-morphisms. In this section, we establish an adjunction between the category of stratified L-quantum spaces and the opposite category of two-sided L-quantales.

Proposition 4.1. Let $(X, \delta(X), \&)$ be a stratified L-quantum space. Then $(\delta(X), \&, \mathrm{sub}_X)$ is a two-sided L-quantale.

Proof. For all $\mathcal{A} \in L^{\delta(X)}$, since $(X, \delta(X), \&)$ is a stratified L-quantum space, we have $\bigvee_{U\in\delta(X)} U\&(\mathcal{A}(U))_X \in \delta(X)$. Now we shall prove that $(\delta(X), \mathrm{sub}_X)$ is a fuzzy complete lattice. For all $B \in \delta(X)$, one can easily see that

$$\begin{aligned}\bigwedge_{U\in\delta(X)} (\mathcal{A}(U) \to \mathrm{sub}_X(U, B)) &= \bigwedge_{U\in\delta(X)} \left(\mathcal{A}(U) \to \left(\bigwedge_{x\in X} U(x) \to B(x)\right)\right) \\ &= \bigwedge_{x\in X} \bigwedge_{U\in\delta(X)} (\mathcal{A}(U) * U(x) \to B(x)) \\ &= \bigwedge_{x\in X} \left(\left(\bigvee_{U\in\delta(X)} U\&(\mathcal{A}(U))_X\right)(x) \to B(x)\right) \\ &= \mathrm{sub}_X\left(\bigvee_{U\in\delta(X)} U\&(\mathcal{A}(U))_X, B\right).\end{aligned}$$

By Theorem 2.6, we conclude that $\sqcup\mathcal{A} = \bigvee_{U\in\delta(X)} U\&(\mathcal{A}(U))_X$. Therefore,

$$\begin{aligned}A\&(\sqcup\mathcal{A}) &= A\&\left(\bigvee_{U\in\delta(X)} (\mathcal{A}(U))_X\&U\right) \\ &= \bigvee_{U\in\delta(X)} A\&((\mathcal{A}(U))_X\&U) \\ &= \bigvee_{V\in\delta(X)} \left(\bigvee_{U\in\delta(X), A\&U=V} (\mathcal{A}(U))_X\&V\right) \\ &= \bigvee_{V\in\delta(X)} ((A\&\_)^{\to}_L(\mathcal{A})(V))_X\&V \\ &= \sqcup(A\&\_)^{\to}_L(\mathcal{A}).\end{aligned}$$

Similarly, $(\sqcup\mathcal{A})\&A = \sqcup(\_\&A)^{\to}_L(\mathcal{A})$. It is easy to prove that $\top_{\delta(X)} = 1_X$, where $\top_{\delta(X)}$ is the top element of $(\delta(X), \leqslant_{\mathrm{sub}_X})$. Then for all $A \in \delta(X)$, $A\&\top_{\delta(X)} = A\&1_X = 1_X\&A = A = \top_{\delta(X)}\&A$.

Thus $(\delta(X), \&, \mathrm{sub}_X)$ is a two-sided L-quantale.

Proposition 4.2. Let $(X, \delta(X), \&)$ and $(Y, \delta(Y), \&)$ be stratified L-quantum spaces and $f : X \to Y$ be a continuous mapping. Then $f^{\leftarrow}_L : (\delta(Y), \&, \mathrm{sub}_Y) \to (\delta(X), \&, \mathrm{sub}_X)$ is a strong L-quantale homomorphism.

Proof. (1) Since $f : X \to Y$ is continuous between stratified L-quantum spaces, we have $\forall\ A,\ B \in \delta(Y)$, $f^{\leftarrow}_L(A\&B) = f^{\leftarrow}_L(A)\&f^{\leftarrow}_L(B)$.

(2) $\forall\ \mathcal{A}\in L^{\delta(Y)},\ x \in X$,

$$\begin{aligned}f^{\leftarrow}_L(\sqcup\mathcal{A})(x) &= f^{\leftarrow}_L\left(\bigvee_{V\in\delta(Y)} V\&(\mathcal{A}(V))_X\right)(x) \\ &= \left(\bigvee_{V\in\delta(Y)} V\&(\mathcal{A}(V))_X\right)(f(x)) \\ &= \bigvee_{V\in\delta(Y)} f^{\leftarrow}_L(V)(x) * \mathcal{A}(V) \\ &= \bigvee_{U\in\delta(X)} \left(U(x) * \bigvee_{f^{\leftarrow}_L(V)=U} \mathcal{A}(V)\right) \\ &= \bigvee_{U\in\delta(X)} (U\&(f^{\leftarrow}_L)^{\to}_L(\mathcal{A})(U))(x) \\ &= \sqcup(f^{\leftarrow}_L)^{\to}_L(\mathcal{A})(x).\end{aligned}$$

Then $f^{\leftarrow}_L(\sqcup\mathcal{A}) = \sqcup(f^{\leftarrow}_L)^{\to}_L(\mathcal{A})$.

(3) It is easy to show that $f^{\leftarrow}_L(\top_{\delta(Y)}) = \top_{\delta(X)}$, where $\top_{\delta(X)}$ and $\top_{\delta(Y)}$ are, respectively, the top elements of $(\delta(X), \leqslant_{\mathrm{sub}_X})$ and $(\delta(Y), \leqslant_{\mathrm{sub}_Y})$. Therefore, $f^{\leftarrow}_L$ is a strong L-quantale homomorphism.

By Propositions 4.1 and 4.2, we can obtain the following proposition.

Proposition 4.3. Define $\Omega_L : SL\text{-QSp}\to L\text{-Quant}^{op}$ by $\Omega_L(f) = f_L^{\leftarrow} : (\delta(Y), \&, \mathrm{sub}_Y) \to (\delta(X), \&, \mathrm{sub}_X)$ for all $f : (X, \delta(X), \&) \to (Y, \delta(Y), \&) \in \mathcal{M}or(SL\text{-QSp})$. Then Ω_L is a functor.

Suppose $(A, \&, e)$ is a two-sided L-quantale, $pt_L(A) = \{p \mid p : A \to L$ is a strong L-fuzzy join-preserving mapping$\}$. $\forall\ a \in A$, define $\Phi_L(a) : pt_L(A) \to L$ by $\Phi_L(a)(p) = p(a)$ for all $p \in pt_L(A)$.

Proposition 4.4. Let $(A, \&, e)$ be a two-sided L-quantale, and let $\Phi_L(A) = \{\Phi_L(a) \mid a \in A\}$. Define $\& : \Phi_L(A) \times \Phi_L(A) \to \Phi_L(A)$ by $(\Phi_L(a)\&\Phi_L(b))(p) \triangleq \Phi_L(a\&b)(p)$ for all $a, b \in A,\ p \in pt_L(A)$. Then $Pt_L(A) = (pt_L(A), \Phi_L(A), \&)$ is a stratified L-quantum space.

Proof. Step 1: $\forall\ p \in pt_L(A),\ \Phi_L(\top_A)(p) = p(\top_A) = 1 = 1_{pt_L(A)}(p),\ \Phi_L(\bot_A)(p) = p(\bot_A) = 0 = 0_{pt_L(A)}(p)$. Hence, $1_{pt_L(A)} = \Phi_L(\top_A) \in \Phi_L(A),\ 0_{pt_L(A)} = \Phi_L(\bot_A) \in \Phi_L(A)$.

Step 2: $\forall\ \{\Phi_L(a_i) \mid i \in I\} \subseteq \Phi_L(A),\ p \in pt_L(A)$, by Proposition 2.14, we have

$$\bigvee_{i\in I} \Phi_L(a_i)(p) = \bigvee_{i\in I} p(a_i) = p\bigg(\bigvee_{i\in I} a_i\bigg) = \Phi_L\bigg(\bigvee_{i\in I} a_i\bigg)(p).$$

Then $\bigvee_{i\in I} \Phi_L(a_i) = \Phi_L(\bigvee_{i\in I} a_i) \in \Phi_L(A)$.

Step 3: $\forall\ \Phi_L(a),\ \Phi_L(b),\ \Phi_L(c) \in \Phi_L(A)$,

$$\begin{aligned}(\Phi_L(a)\&\Phi_L(b))\&\Phi_L(c) &= \Phi_L(a\&b)\&\Phi_L(c) = \Phi_L((a\&b)\&c)\\ &= \Phi_L(a\&(b\&c)) = \Phi_L(a)\&(\Phi_L(b)\&\Phi_L(c)).\end{aligned}$$

Then $(\Phi_L(a)\&\Phi_L(b))\&\Phi_L(c) = \Phi_L(a)\&(\Phi_L(b)\&\Phi_L(c))$.

Step 4: $\forall\ \Phi_L(a) \in \Phi_L(A),\ \{\Phi_L(b_i) \mid i \in I\} \subseteq \Phi_L(A)$,

$$\begin{aligned}\Phi_L(a)\&\bigg(\bigvee_{i\in I} \Phi_L(b_i)\bigg) &= \Phi_L(a)\&\bigg(\Phi_L(\bigvee_{i\in I} b_i)\bigg) = \Phi_L\bigg(a\&\bigg(\bigvee_{i\in I} b_i\bigg)\bigg)\\ &= \Phi_L\bigg(\bigvee_{i\in I}(a\&b_i)\bigg) = \bigvee_{i\in I}(\Phi_L(a)\&\Phi_L(b_i)).\end{aligned}$$

Similarly, we have $\bigvee_{i\in I} \Phi_L(b_i)\&\Phi_L(a) = \bigvee_{i\in I}(\Phi_L(b_i)\&\Phi_L(a))$.

Step 5: $\forall\ \lambda \in L,\ B = \lambda_A \in L^A,\ a_0 = \sqcup B \in A$, by Proposition 2.10, we have

$$\begin{aligned}\Phi_L(a_0)(p) &= p(a_0) = p(\sqcup B) = \sqcup p_L^{\rightarrow}(B) = \bigvee_{a\in A}(p(a) * B(a)) = \bigvee_{a\in A}(p(a) * \lambda)\\ &= \lambda * \bigvee_{a\in A} p(a) = \lambda * p(\bigvee_{a\in A} a) = \lambda * p(\top_A) = \lambda.\end{aligned}$$

Then $\Phi_L(a_0) = \lambda_{pt_L(A)} \in \Phi_L(A)$. Next, we shall prove $(\Phi_L(a)\&\lambda_{pt_L(A)})(p) = p(a) * \lambda$. In fact,

$$\begin{aligned}(\Phi_L(a)\&\lambda_{pt_L(A)})(p) &= (\Phi_L(a)\&\Phi_L(\sqcup\lambda_A))(p) = p(a\&(\sqcup\lambda_A))\\ &= p(\sqcup(a\&\_)_L^{\rightarrow}(\lambda_A)) = \sqcup(p \circ (a\&\_))_L^{\rightarrow}(\lambda_A).\end{aligned}$$

$\forall\ u \in L$,

$$\begin{aligned}\bigwedge_{c\in L}((p \circ (a\&\_))_L^{\rightarrow}(\lambda_A)(c) \to e_L(c, u)) &= \bigwedge_{c\in L}\bigg(\bigvee_{p(a\&b)=c} \lambda_A(b) \to e_L(c, u)\bigg) = \bigwedge_{b\in A}(\lambda \to e_L(p(a\&b), u))\\ &= \bigg(\bigvee_{b\in A} \lambda * p(a\&b)\bigg) \to u = \bigg(\lambda * \bigg(\bigvee_{b\in A} p(a\&b)\bigg)\bigg) \to u\\ &= \bigg(\lambda * \bigg(p\bigg(a\& \bigvee_{b\in A} b\bigg)\bigg)\bigg) \to u = (\lambda * p(a\&\top_A)) \to u\\ &= (\lambda * p(a)) \to u = e_L(\lambda * p(a), u).\end{aligned}$$

Then $(\Phi_L(a)\&\lambda_{pt_L(A)})(p) = \lambda * p(a)$. Similarly, we have $(\lambda_{pt_L(A)}\&\Phi_L(a))(p) = p(a) * \lambda$.

Therefore, $Pt_L(A) = (pt_L(A), \Phi_L(A), \&)$ is a stratified L-quantum space.

Proposition 4.5. Let $f : (A, \&, e_A) \to (B, \&, e_B)$ be a morphism in L-Quant^{op}. Define $Pt_L(f) : Pt_L(A) \to Pt_L(B)$ by $Pt_L(f)(p) = p \circ f$ for all $p \in pt_L(A)$. Then $Pt_L(f)$ is continuous.

Proof. (1) $\forall\ a \in B,\ p \in pt_L(A)$,

$$\begin{aligned}(Pt_L(f))_L^{\leftarrow}(\Phi_L(a))(p) &= \Phi_L(a)(Pt_L(f)(p)) = \Phi_L(a)(p \circ f)\\ &= (p \circ f)(a) = \Phi_L(f(a))(p).\end{aligned}$$

Then $(Pt_L(f))_L^{\leftarrow}(\Phi_L(a)) = \Phi_L(f(a)) \in \Phi_L(A)$.

(2) $\forall\ \Phi_L(a_1),\ \Phi_L(a_2) \in \Phi_L(B)$,

$$\begin{aligned}(Pt_L(f))_L^{\leftarrow}(\Phi_L(a_1)\&\Phi_L(a_2)) &= (Pt_L(f))_L^{\leftarrow}(\Phi_L(a_1\&a_2)) = \Phi_L(f(a_1\&a_2))\\ &= \Phi_L(f(a_1)\&f(a_2)) = \Phi_L(f(a_1))\&\Phi_L(f(a_2))\\ &= ((Pt_L(f))_L^{\leftarrow}(\Phi_L(a_1))\&(Pt_L(f))_L^{\leftarrow}(\Phi_L(a_2))).\end{aligned}$$

Proposition 4.6. Define $Pt_L : L$-Quant$^{op} \to SL$- QSp by $Pt_L(f)(p) = p \circ f$ for all $f : (A, \&,\ e_A) \to (B, \&, e_B) \in \mathcal{M}or(L$-Quant$^{op})$, $p \in pt_L(A)$. Then Pt_L is a functor.

Theorem 4.7. $\Omega_L \dashv Pt_L : SL$-QSp$\rightharpoonup L$-Quant^{op}.

Proof. For all $(B, \&, e_B) \in \mathcal{O}b(L$-Quant$^{op})$, we shall show that there exists a co-universal mapping μ_B.

(1) $\forall\ (B, \&, e_B) \in \mathcal{O}b(L$-Quant$^{op})$, suppose $\mu_B = \Phi_L : \Omega_L \circ Pt_L(B) \to B$. We shall prove that $\mu_B : \Omega_L \circ Pt_L(B) \to B \in \mathcal{M}or(L$-Quant$^{op})$, that is, $\mu_B : B \to \Omega_L \circ Pt_L(B)$ is a strong L-quantale homomorphism. Firstly, $\forall\ a,\ b \in B$, $\Phi_L(a\&b) = \Phi_L(a)\&\Phi_L(b)$. Secondly, $\forall\ p \in pt_L(B)$, $\Phi_L(\top_B)(p) = p(\top_B) = 1 = 1_{pt_L(B)}(p) = \top_{\Phi_L(B)}(p)$, so $\Phi_L(\top_B) = \top_{\Phi_L(B)}$. Thirdly, $\forall\ A \in L^B,\ p \in pt_L(B)$, on the one hand,

$$\begin{aligned}\Phi_L(\sqcup A)(p) &= p(\sqcup A) = \sqcup p_L^{\rightarrow}(A) = \bigvee_{b\in B}(A(b) * p(b))\\ &= \bigvee_{b\in B}(A(b) * (\Phi_L(b)(p)))\\ &= \left(\bigvee_{b\in B} A(b)\&\Phi_L(b)\right)(p).\end{aligned}$$

Then $\Phi_L(\sqcup A) = \bigvee_{b\in B}(A(b)\&\Phi_L(b))$. On the other hand,

$$\begin{aligned}\sqcup(\Phi_L)_L^{\rightarrow}(A) &= \bigvee_{U\in\Phi_L(B)}((\Phi_L)_L^{\rightarrow}(A)(U)\&U) = \bigvee_{U\in\Phi_L(B)}\ \bigvee_{\Phi_L(b)=U}(A(b)\&U)\\ &= \bigvee_{b\in B}(A(b)\&\Phi_L(b)).\end{aligned}$$

Thus $\Phi_L(\sqcup A) = \sqcup(\Phi_L)_L^{\rightarrow}(A)$.

(2) $\forall\ X \in \mathcal{O}b(SL$-QSp$)$, $g : \Omega_L(X, \delta(X), \&) \to (B, \&, e_B) \in \mathcal{M}or(L$-Quant$^{op})$. Define $f : X \to Pt_L(B)$ by $f(x)(b) = g(b)(x)$ for all $x \in X,\ b \in B$. We shall prove that f is continuous. Clearly, f is well-defined. $\forall\ x \in X,\ b,\ b_1,\ b_2 \in B$, $f_L^{\leftarrow}(\Phi_L(b))(x) = (\Phi_L(b))(f(x)) = f(x)(b) = g(b)(x)$. Thus, $f_L^{\leftarrow}(\Phi_L(b)) = g(b) \in \delta(X)$. $f_L^{\leftarrow}(\Phi_L(b_1)\&\Phi_L(b_2)) = f_L^{\leftarrow}(\Phi_L(b_1\&b_2)) = g(b_1\&b_2) = g(b_1)\&g(b_2) = f_L^{\leftarrow}(\Phi_L(b_1))\&f_L^{\leftarrow}(\Phi_L(b_2))$.

(3) By (2), $f_L^{\leftarrow} \circ \Phi_L = g$, that is, $g = \Omega_L(f) \circ \mu_B$.

(4) In the following, we shall prove the uniqueness of f. Assume that $h : X \to Pt_L(B)$ satisfies $g = \Omega_L(h) \circ \mu_B$. Then $\forall\ x \in X,\ b \in B$,

$$\begin{aligned}f(x)(b) &= (\Phi_L(b))(f(x)) = (f_L)^{\leftarrow}(\Phi_L(b))(x)\\ &= (h_L)^{\leftarrow}(\Phi_L(b))(x) = \Phi_L(b)(h(x)) = h(x)(b).\end{aligned}$$

5 The duality between *Sob*-*SL*-QSp and *SL*-Quant

In this section, we introduce the notion of sober L-quantum space and prove that the category of sober L-quantum spaces is dually equivalent to the category of spatial two-sided L-quantales.

Definition 5.1. Let $(X,\delta(X),\&)$ be a stratified L-quantum space. $\forall\ x\in X$, define $\Psi(x):\delta(X)\to L$ by $\Psi(x)(U)=U(x)$ for all $U\in\delta(X)$. If $\Psi:X\to pt_L(\delta(X))$ is a bijection, then $(X,\delta(X),\&)$ is called a sober L-quantum space.

Let *Sob-SL-QSp* denote the category of sober L-quantum spaces and continuous mappings.

Proposition 5.2. Let $(X,\delta(X),\&)$ be a stratified L-quantum space. Then the following statements are equivalent:

(1) $(X,\delta(X),\&)$ is a sober L-quantum space;

(2) $\Psi:(X,\delta(X),\&)\to(pt_L(\delta(X)),\Phi_L(\delta(X)),\&)$ is a homeomorphism.

Proof. We only need to prove that (1) $\Rightarrow$ (2), that is, $\Psi:(X,\delta(X),\&)\to(pt_L(\delta(X)),\Phi_L(\delta(X)),\&)$ is continuous and open. $\forall\ U,\ V\in\delta(X),\ x\in X$,

$$\Psi_L^{\leftarrow}(\Phi_L(U))(x)=\Phi_L(U)\circ\Psi_L(x)=\Psi_L(x)(U)=U(x).$$

$$\Psi_L^{\leftarrow}(\Phi_L(U)\&\Phi_L(V))=\Psi_L^{\leftarrow}(\Phi_L(U\&V))=U\&V=\Psi_L^{\leftarrow}(\Phi_L(U))\&\Psi_L^{\leftarrow}(\Phi_L(V)).$$

Then Ψ is continuous. In addition, $\forall\ U,\ V\in\delta(X),\ p\in pt_L(\delta(X))$, since Ψ is a bijection, there exists $x\in X$ such that $p=\Psi(x)$. Hence,

$$\Psi_L^{\rightarrow}(U)(p)=\Psi_L^{\rightarrow}(U)(\Psi(x))=U(x)=\Psi(x)(U)=p(U)=\Phi_L(U)(p).$$

$$\Psi_L^{\rightarrow}(U\&V)=\Phi_L(U\&V)=\Phi_L(U)\&\Phi_L(V)=\Psi_L^{\rightarrow}(U)\&\Psi_L^{\rightarrow}(V).$$

Then $\Psi_L^{\rightarrow}(U)=\Phi_L(U)\in\Phi_L(\delta),\ \Psi_L^{\rightarrow}(U\&V)=\Psi_L^{\rightarrow}(U)\&\Psi_L^{\rightarrow}(V)$. Thus, Ψ is open.

Proposition 5.3. Let $(Q,\&,e)$ be a two-sided L-quantale. Then $Pt_L(Q)$ is a sober L-quantum space.

Proof. We only need to prove that $\Psi:pt_L(Q)\to pt_L(\Phi_L(Q))$ is a bijection. $\forall\ p,\ q\in pt_L(Q)$, if $p\neq q$, then there exists $a\in Q$ such that $p(a)\neq q(a)$. Since $\Phi_L(a)\in\Phi_L(Q)$, we have $\Psi(p)(\Phi_L(a))=(\Phi_L(a))(p)=p(a)\neq q(a)=(\Phi_L(a))(q)=\Psi(q)(\Phi_L(a))$. Thus, Ψ is injective. $\forall\ q\in pt_L(\Phi_L(Q))$, put $p=q\circ\Phi_L$. By the proof of Theorem 4.7(1), we obtain $p\in pt_L(Q)$. $\forall\ U=\Phi_L(b)\in\Phi_L(Q),\ b\in Q,\ \Psi(p)(U)=\Phi_L(b)(p)=p(b)=q(\Phi_L(b))=q(U)$. Thus, Ψ is surjective.

Proposition 5.4. Let $(Q,\&,e)$ be a two-sided L-quantale. Then the following statements are equivalent:

(1) $\Phi_L:Q\to\Phi_L(Q)$ is injective.

(2) $\Phi_L:(Q,\&,e)\to(\Phi_L(Q),\&,\mathrm{sub})$ is an isomorphism in L-Quant.

Proof. (1)$\Rightarrow$(2): It is easy to show that Φ_L is surjective. Now we only need to prove that Φ_L is a homomorphism. $\forall\ a,\ b\in Q,\ A\in L^Q,\ \Phi_L(a\&b)=\Phi_L(a)\&\Phi_L(b)$, In addition,

$$\begin{aligned}
&\bigwedge_{a\in Q}((\Phi_L)_L^{\rightarrow}(A)(\Phi_L(a))\to\mathrm{sub}(\Phi_L(a),\Phi_L(b)))\\
&=\bigwedge_{a\in Q}\left(\left(\bigvee_{\Phi_L(c)=\Phi_L(a)}A(c)\right)\to\mathrm{sub}(\Phi_L(a),\Phi_L(b))\right)\\
&=\bigwedge_{a\in Q}\left(A(a)\to\bigwedge_{p\in pt_L(Q)}(\Phi_L(a)(p)\to\Phi_L(b)(p))\right)\\
&=\bigwedge_{p\in pt_L(Q)}\left(\left(\bigvee_{a\in Q}A(a)*p(a)\right)\to\Phi_L(b)(p)\right)\\
&=\bigwedge_{p\in pt_L(Q)}(\sqcup p_L^{\rightarrow}(A)\to\Phi_L(b)(p))\\
&=\bigwedge_{p\in pt_L(Q)}(p(\sqcup A)\to\Phi_L(b)(p))\\
&=\bigwedge_{p\in pt_L(Q)}(\Phi_L(\sqcup A)(p)\to\Phi_L(b)(p))\\
&=\mathrm{sub}(\Phi_L(\sqcup A),\Phi_L(b)).
\end{aligned}$$

Then $\sqcup(\Phi_L)_L^{\rightarrow}(A)=\Phi_L(\sqcup A)$.

Therefore, $\Phi_L:(Q,\&,e)\to(\Phi_L(Q),\&,sub)$ is an isomorphism in L-Quant.

(2)$\Rightarrow$(1): It is straightforward.

A two-sided L-quantale will be called spatial if it satisfies the conditions in Proposition 5.4. Let SL-Quant denote the category of spatial two-sided L-quantales and strong L-quantale homomorphisms.

Proposition 5.5. Let $(X, \delta(X), \&)$ be a sober L-quantum space. Then $(\delta(X), \&, \mathrm{sub}_X)$ is a spatial two-sided L-quantale.

Proof. $\forall\, U,\ V \in \delta(X)$, if $U \neq V$, there exists $x \in X$ such that $U(x) \neq V(x)$. Putting $p = \Psi(x)$, we have $\Phi_L(U)(p) = p(U) = \Psi(x)(U) = U(x) \neq V(x) = \Psi(x)(V) = p(V) = \Phi_L(V)(p)$.

By Propositions 5.2–5.5, we have

Theorem 5.6. *Sob*-SL-QSp is dually equivalent to SL-Quant.

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