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Design of evacuation strategies with crowd density feedback

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Abstract A second-order stochastic model describing a large scale crowd is formulated, and an efficient evacuation strategy for agents in complex surroundings is proposed and solved numerically. The method consists in reshaping the crowd contour by making use of the crowd density feedback that is commonly available from geolocation technologies, and Kantorovich distance is used to transport the current shape into the desired one. The availability of the crowd density enables to solve the otherwise challenging forward-backward problem. Using this approach, we demonstrate via numerical results that the crowd migrates through the complex environment as designed.

Keywords crowd dynamics, multi-agent system, stochastic differential equation, optimal control, congestion control

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1 Introduction

Crowd behaviors exist commonly in nature and human society: birds move in groups during migration, soldiers fight together in wars and robots can be designed to coordinate their activities for one special task. The mechanisms hidden behind these phenomena are various and have been attracting a considerable amount of researchers' attention from different areas such as biology or engineering [1,2]. As the number of agents in crowds grows, congestion problems happen more frequently. An example is the crowding phenomenon appearing in case of accidents or weak administration. When the congestion problem becomes serious, casualties can occur, followed by complex social issues. Thus, a relevant topic, the design of evacuation strategies, appears to be important. A good evacuation strategy can reduce the societal loss to the minimal level. Therefore, it is not surprising to find a plenty of literature in physics, mathematics and engineering, etc. discussing the congestion and evacuation problems in crowd dynamics [3–6].

Previous work in modeling and analyzing the congestion and evacuation problems can be classified into three main cases: (1) social psychology analysis, (2) multi-agent microscopic model and (3) mean field macroscopic model. All the models are able to deal with large scale crowds, but have different perspectives.

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The social psychology analysis pays attention to theories such as the planned behavior [7,8]. The multi-agent microscopic model, developed from the animal swarming behavior [9], refers to the individually identified entities, in which position and velocity for each individual are specified. In the past decades, the multi-agent system theory has been widely employed in the crowd analysis and has been developed flourishingly. The well known social force model and opinion model are perfect examples suitable for such setting [10–12]. The mean-field macroscopic description is expressed via the averaged gross quantities of the system state. It can be derived from the microscopic model together with proper assumptions of necessary empirical functions. Such a model is a dynamical system involving two coupled forward-backward partial differential equations (PDE). Mean field model is widely applied in finance, engineering and chemistry areas [13–15], and has been adopted to study congestion problems recently [16].

Although specific models have been proposed to simulate the congestion and evacuation, the crowd shape has been slightly neglected in previous efforts. It has been recognized that well organized crowd encounters less congestion consequences. This is the reason why passengers are supposed to queue up for buses. Another good example is the evacuation of "Battle of Dunkirk" where the troops were well organized during the retreat. In [17], a method was proposed to shape crowds by tracking the desired crowd moments in a deterministic leader-follower model. This method was proposed based on the fact that shape can be decided by the moments [18]. However, it was found that the results were not accurate by controlling the crowd moments. And, the computing cost is high when higher-order moments are desired.

With the development of geolocation technologies, it is now very convenient to obtain the crowd density information. In this article, we design a strategy to evacuate crowds by controlling their shapes. The strategy is proposed in a novel viewpoint by utilizing the crowd density feedback. In the multi-agent game model studied in this paper, each agent is assumed to be rational and cooperative. The control law aiming at reshaping the crowd contour is derived from the optimal control method and applied on each agent's acceleration. The control strategy is accomplished by tracking the approximate destinations derived from the current and desired crowd densities through properly defined Kantorovich distance. The motion dynamic is assumed of the same format for all the agents for calculation simplification. In our work, stochastic version of the model is considered. The white noises are introduced to describe the complex psychological changes and the unpredictable interruptions, thus the model is realistic.

The outline of this paper is as follows: in Section 2, the model is established and transformed. This process leads to a new model where the crowd density feedback is included. In Section 3, the control law of the tracking problem with fixed destinations is derived from the optimal control method. In Section 4, Kantorovich distance is reformulated and solved numerically. In Section 5, numerical experiments are carried out with the proposed strategy and algorithm. Section 6 is conclusion and perspective of the presented work.

2 Mathematical model statement

2.1 Social force interaction

Interactions between agents are unavoidable in networks [19,20]. Among diverse interaction patterns, social force is essential because every agent needs a balance position for itself in the network. For a large scale crowd in the n-dimensional Euclidean space whose agents are with positions x^i $(i = 1, 2, \ldots, N)$, the interplay between individuals i and j could be assumed as a general "repulsive-attractive" social force with the following expression [11],

$$S\left(x^i, x^j\right) = S(\widetilde{d}) = -\widetilde{d}\left[a - b\exp\left(-\frac{\|\widetilde{d}\|^2}{c}\right)\right], \tag{1}$$

where a, b and c are positive constants satisfying $a < b$. Parameter a domains the attraction character, $b\exp(-\|\widetilde{d}\|^2/c)$ represents the repulsion effect and c is the tolerability parameter representing the level at which the social force switches the sign. $\widetilde{d} = x^i - x^j$, and $\|\widetilde{d}\|^2 = \widetilde{d}^{\mathrm{T}}\widetilde{d}$ is the square distance between agents

i and j. S appears to be of attraction character for large distances while repulsion for short distances. This individual based attraction/repulsion model is consistent with the usual crowd behaviors, that is, the agents look for suitable positions related to other agents' positions. Let $a - b\exp(-\|\widetilde{d}\|^2/c) = 0$, one can obtain the critical positive distance $\widetilde{d}_0 = \sqrt{c\ln(b/a)}$ at which the attraction and repulsion influences are balanced. To specify the social impact coming from x^j ($j = 1, 2, \ldots, N$, $j \neq i$) and acting on x^i, we introduce a new symbol $S^i\left(x^i, x^j\right)$, whose value is equal to $S\left(x^i, x^j\right)$, i.e., $S^i\left(x^i, x^j\right) = S\left(x^i, x^j\right)$. The total social force acting on agent i is $\sum_{j=1,j\neq i}^{N} S^i\left(x^i, x^j\right)/N$, with which each agent is connected with others.

2.2 Stochastic multi-agent model

Consider a crowd of N autonomous agents labeled as $1, 2, \ldots, N$ in two-dimensional Euclidean space $\mathbb{R}^2$. Denote the assemblage of all agents as a finite set $\boldsymbol{K}$, and the equation of motion for each $i \in \boldsymbol{K}$ is as follows:

$$\begin{aligned} &\dot{X}^i(t) = V^i(t), \\ &\dot{V}^i(t) = U^i(t) + \frac{k^i}{N} \sum_{j=1,j\neq i}^{N} S^i(X^i, X^j, t) - f^i V^i(t) + g^i(X^i, V^i, t)\xi^i(t), \end{aligned} \tag{2}$$

where, $X^i = [X_x^i, X_y^i]^{\mathrm{T}}$ and $V^i = [V_x^i, V_y^i]^{\mathrm{T}}$ represent the position and velocity respectively. Capital letters mean they are random processes. $U^i = [U_x^i, U_y^i]^{\mathrm{T}}$ is the control law applied on the acceleration. $k^i \sum_{j=1,j\neq i}^{N} S^i(X^i, X^j, t)/N$ means the mass influence applying on agent i. k^i is a positive parameter representing the cooperative willingness of agent i (a larger k^i means a stronger willingness). $f^i V^i(t)$ is a damping term caused by mechanical resistances such as force of friction. $\xi^i(t)$ is a Gaussian white noise of Stratonovich sense with the autocorrelation function $E[\xi^i(t)\xi^i(t+\tau)] = 2D^i\delta(\tau)$ [21]. All disturbances are assumed to be independent to present the different agent mental activities. The excitation term could be either external if $g^i(X^i, V^i, t)$ is a constant or parametric otherwise. The existence of the inter-individual interactions implies that model (2) is a large scale complex network who shows incredible complexity as N increases.

2.3 Model transformation

The distance between agents is an important key of model (2). Research work regarding the "distance information" is widely studied in areas such as robotics. Generally, the sensors capture the distances to other agents inside a certain neighborhood and use these information to make decisions. However, failures can occur because situations in which the distances between agents are difficult to detect happen frequently. Furthermore, as the number of agents grows, the network becomes more complex and it is more difficult to solve control laws even when only local information is adopted. Here, we intend to assess this problem on a statistic level by introducing the crowd density $p(X, t)$, which allows us to uncouple the interaction between agent i and the other agents and reinterpret the model in terms of social force field. Since all agents in the crowd are assumed to be independent identically distributed (i.i.d), according to the strong law of large numbers, we derive the following approximate social influence for each agent i:

$$\frac{1}{N} \sum_{j=1,j\neq i}^{N} S^i(X^i, X^j, t) \xrightarrow{a.s.} \int_X S^i(X^i, X, t)p(X, t) = \int_X S(X^i, X, t)p(X, t). \tag{3}$$

Thus, model (2) is rewritten together with (3) as follows,

$$\begin{aligned} &\dot{X}^i(t) = V^i(t), \\ &\dot{V}^i(t) = U^i(t) + k^i \int_X S(X^i, X, t)p(X, t) - f^i V^i(t) + g^i(X^i, V^i, t)\xi^i(t). \end{aligned} \tag{4}$$

With model (4), agents are decoupled, and they interact with a social force field described by $p(X, t)$. The unknown quantities in model (4) are agent i own position X^i, velocity V^i and the current crowd density $p(X,t)$. We must point out that $p(X,t)$ is assumed as an extra input available from geolocation techniques. Availability of $p(X,t)$ generates a new model including statistical information. Another advantage of model (4) is that it allows to avoid dealing with the high dimensional Fokker-Planck (F-P) equation including both $X(t)$ and $V(t)$. For further derivation, the following stochastic differential equations of both Stratonovich and Itô types are expressed.

$$\begin{aligned} &\mathrm{d}X^i(t) = V^i(t)\mathrm{d}t, \\ &\mathrm{d}V^i(t) = \left[U^i(t) + k^i \int_X S(X^i, X, t)p(X,t)\right]\mathrm{d}t - f^i V^i(t)\mathrm{d}t + \sigma^i(X^i, V^i, t) \circ \mathrm{d}B^i(t). \end{aligned} \tag{5}$$

Eq. (5) is the Stratonovich stochastic differential equations derived straightforward from (4). $B^i(t) = [B_x^i, B_y^i]^{\mathrm{T}}$ is the standard Wiener process. $\sigma_j^i = [\boldsymbol{\sigma^i}]_j^{\mathrm{T}}$, $j = 1,2$ with $\boldsymbol{\sigma^i} = [\sigma_x^i, \sigma_y^i]^{\mathrm{T}} = \boldsymbol{L^i g^i}$. $\boldsymbol{g^i} = [g_x^i, g_y^i]^{\mathrm{T}}$, and $\boldsymbol{L^i L^{iT}} = 2\boldsymbol{D^i}$ with $\boldsymbol{D^i} = \mathrm{diag}(D_x^i, D_y^i)$. The equivalent Itô stochastic differential equations are

$$\begin{aligned} &\mathrm{d}X^i(t) = V^i(t)\mathrm{d}t, \\ &\mathrm{d}V^i(t) = \left[U^i(t) + k^i \int_X S(X^i, X, t)p(X,t)\right]\mathrm{d}t - f^i V^i(t)\mathrm{d}t + \frac{1}{2}\Gamma^i\mathrm{d}t + \sigma^i(X^i, V^i, t)\mathrm{d}B^i(t), \end{aligned} \tag{6}$$

where $\Gamma^i = [\sigma_1^i \partial\sigma_1^i/\partial v_1^i, \sigma_2^i\partial\sigma_2^i/\partial v_2^i]^{\mathrm{T}}$, and $v_j^i = [\boldsymbol{v^i}]_j$, $j = 1,2$ with $\boldsymbol{v^i} = [V_x^i, V_y^i]^{\mathrm{T}}$. $1/2\Gamma^i$ is the Wong-Zakai correction terms. If only external excitations exist, the Wong-Zakai terms become zero.

3 Optimal crowd control

3.1 Introduction of the optimal control framework

Assume agent i minimizes its own performance objective $J_i(U^i)$ in a finite time horizon $[t_0, t_f]$, then the general optimal control problem for agent i is written together with (6) as follows:

$$\begin{aligned} &\mathrm{d}X^i(t) = V^i(t)\mathrm{d}t, \\ &\mathrm{d}V^i(t) = \left[U^i(t) + k^i \int_X S(X^i, X, t)p(X,t)\right]\mathrm{d}t - f^i V^i(t)\mathrm{d}t + \frac{1}{2}\Gamma^i\mathrm{d}t + \sigma^i(X^i, V^i, t)\mathrm{d}B^i(t), \\ &|U^i| \leqslant U_{\max}^i, \quad X^i(0) = x^i(0), \quad V^i(0) = v^i(0), \end{aligned} \tag{7}$$

where $x^i(0)$ and $v^i(0)$ are the given initial position and velocity. $U^i(t)$ is assumed to be with a maximal boundary $U_{\max}^i > 0$ for the reality consideration. $J_i(U^i) = E[\int_{t_0}^{t_f} \phi_i(X^i, U^i, t)\mathrm{d}t + \psi_i(X_{t_f}^i)]$, where $\phi_i(X^i, U^i, t)$ and $\psi_i(X_{t_f}^i)$ are the carrying cost and final cost, respectively. Obviously, proper cost functions should be designed for different applications.

3.2 Optimal control for tracking precise destinations

In a tracking problem, the performance objective of agent i with destination X_e^i can be written as follows:

$$J_i(U^i) = E\left[\int_{t_0}^{t_f} \|X^i(t) - X_e^i\|\mathrm{d}t\right]. \tag{8}$$

And, the corresponding Hamilton-Jacobi-Bellman (HJB) equation is derived from (7) and (8) [22,23],

$$\begin{aligned} -\frac{\partial Z_i\left(X^i, V^i, t\right)}{\partial t} = &\min_{|U^i| \leqslant U_{\max}^i} \left\{ \|X^i(t) - X_e^i\| + \frac{1}{2}\sum_{j=1}^{2} \sigma_j^{i2} \frac{\partial^2 Z_i}{\partial v_j^{i2}} \right. \\ &\left. + \left(\frac{\partial Z_i}{\partial V^i}\right)^{\mathrm{T}} \left[k^i \int_X S(X^i, X, t)p(X,t) - f^i V^i(t)\right] + \left(\frac{\partial Z_i}{\partial V^i}\right)^{\mathrm{T}} \left(U^i + \frac{1}{2}\Gamma^i\right) + \left(\frac{\partial Z_i}{\partial X^i}\right)^{\mathrm{T}} V^i(t) \right\}. \end{aligned} \tag{9}$$

Here, we present our opinions on the mean-field model that can be derived by sequentially applying averaging procedures. In the problem proposed in this paper, $p(X,t)$ is adopted in solving the control

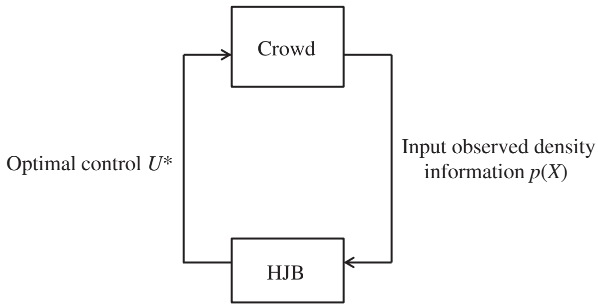

Figure 1 The flow diagram of the control process on each observation time step.

laws. Therefore, HJB is coupled with both the $p(X,t)$ and FP describing the density $p(X,V,t)$. Thus, our setting leads to a very difficult if not unsolvable mean-field system. However, in the current paper, we assume $p(X,t)$ to be available from geolocation technologies, so that the complex forward-backward system can be avoid. And our assumption potentially implies a strategy to solve the mean field model. Our conception is explained in the flow diagram as shown in Figure 1.

Thus, a simpler optimal problem with performance objective (10) can be solved on each observation time step.

$$J_i(U^i) = E\left[\int_t^{t+\triangle t} \|X^i(t) - X_e^i(t)\|\mathrm{d}t\right]. \tag{10}$$

In the calculation, the time horizon is discretized into small steps in which the crowd density and the social force are assumed to be constants. This simplification could also be termed as model predictive control (MPC). Although we could not get a global optimal control law with such a simpler formulation, this simplification allows us to subsequently design agents' motion trajectories. Hence, the agents are able to avoid known threats and obstacles. Analogously, the HJB equation with performance objective (10) is:

$$-\frac{\partial Z_i\left(X^i,V^i,t\right)}{\partial t} = \min_{|U^i|\leqslant U^i_{\max}}\left\{\|X^i(t) - X_e^i(t)\| + \frac{1}{2}\sum_{j=1}^{2}\sigma_j^{i2}\frac{\partial^2 Z_i}{\partial v_j^{i2}}\right.$$
$$\left. + \left(\frac{\partial Z_i}{\partial V^i}\right)^{\mathrm{T}}\left[k^i\int_X S(X^i,X)p(X) - f^iV^i(t)\right] + \left(\frac{\partial Z_i}{\partial V^i}\right)^{\mathrm{T}}\left(U^i + \frac{1}{2}\Gamma^i\right) + \left(\frac{\partial Z_i}{\partial X^i}\right)^{\mathrm{T}} V^i(t)\right\}. \tag{11}$$

In case of the bounded input, one can easily deduce the optimal control law from (11) as follows,

$$U^{i*} = -U^i_{\max}\frac{\partial Z_i}{\partial V^i}\left\|\frac{\partial Z_i}{\partial V^i}\right\|^{-1}. \tag{12}$$

The direction of the control vector is opposite to that of $\partial Z_i/\partial V^i$. When $\|\partial Z_i/\partial V^i\| = 0$, an arbitrary control law can be applied. The existence of noises allows the agents to keep moving. By substituting (12) into (11), the final HJB equation is rewritten as follows,

$$-\frac{\partial Z_i\left(X^i,V^i,t\right)}{\partial t} = \|X^i(t) - X_e^i(t)\| + \left(\frac{\partial Z_i}{\partial X^i}\right)^{\mathrm{T}} V^i(t) + \frac{1}{2}\sum_{j=1}^{2}\sigma_j^{i^2}\frac{\partial^2 Z_i}{\partial v_j^{i2}}$$
$$+ \left(\frac{\partial Z_i}{\partial V^i}\right)^{\mathrm{T}}\left[k^i\int_X S(X^i,X)p(X) - f^iV^i(t)\right] + \left(\frac{\partial Z_i}{\partial V^i}\right)^{\mathrm{T}}\left(-U^i_{\max}\frac{\partial Z_i}{\partial V^i}\left\|\frac{\partial Z_i}{\partial V^i}\right\|^{-1} + \frac{1}{2}\Gamma^i\right). \tag{13}$$

3.3 Computation techniques

The calculation is performed on each agent with its own information of X^i and V^i, and the crowd density $p(X)$. The calculation is carried out between (4) and (13). Eq. (4) is solved by Euler algorithm with small time step $\triangle t$. The noise term $\xi^i(t)$ at each discrete time instance t_k is generated from the following formula [24],

$$\xi^i(t_k) = \sqrt{\frac{2D^i}{\triangle t}} W_k, \tag{14}$$

where W_k are independent samples of Gaussian random variables with zero means and unit variance. Eq. (13) is a PDE with five variables: X_x^i, X_y^i, V_x^i, V_y^i and t, which is a challenge to solve. We use the finite-difference method of the following central step scheme,

$$\begin{aligned} \frac{\partial y\left(x_1, x_2\right)}{\partial x_1} &\approx \frac{y(x_1+\widetilde{h}, x_2) - y(x_1-\widetilde{h}, x_2)}{2\widetilde{h}}, \\ \frac{\partial^2 y\left(x_1, x_2\right)}{\partial x_1^2} &\approx \frac{y(x_1+\widetilde{h}, x_2) - 2y(x_1, x_2) + y(x_1-\widetilde{h}, x_2)}{\widetilde{h}^2}. \end{aligned} \tag{15}$$

Thus far, the control law is obtained when destinations are known in advance for agents. In reality, a desired density $p_g(X)$ is often required for a crowd instead of precise destinations. In this purpose, a strategy is developed in the next section to estimate destinations from the crowd densities.

4 Destination estimation

In this section a procedure to track the desired crowd density $p_g(X)$ is extensively described based on Section 3 and the Kantorovich distance d_K for the crowds.

4.1 Kantorovich distance for crowd

Kantorovich distance originally proposed by Kantorovich in 1940s is a measure between different distributions [25]. Denote $(K,\ d)$ as a separable metric space. For any two Borel probability measures K_1 and K_2 on K, Kantorovich distance between K_1 and K_2 is defined as follows:

$$d_K = \sup\left\{\left|\int f\mathrm{d}K_1 - \int f\mathrm{d}K_2\right|,\ \|f\| \leqslant 1\right\}, \tag{16}$$

where $\|\cdot\|$ is the following Lipschitz semi-norm:

$$\|f\| = \sup_{x\neq y} \frac{|f\left(x\right) - f\left(x\right)|}{d\left(x,\ y\right)} \quad \left(f : K \to \mathbb{R}\right). \tag{17}$$

Based on the above mathematical definition, Kantorovich distance has been reformulated for different applications. For example, Werman et al. [26] proposed the Kantorovich distance for two-dimensional gray-valued images and was followed by Kaijser [27]. Here, a proper Kantorvich distance for crowds is proposed.

Consider our problem in a discrete Euclidean space Ω. Denote K as a finite set of all the sub areas in Ω. A crowd K_1 with support K is defined as an integer-valued non negative function K_1 defined on K. K_1 is valued with the number of agents in the crowd in each sub area, which can be estimated from $p(X)$ by integral. Label all sub areas included in the current and the desired densities as $A = \{A_1, A_2, \dots, A_m\}$ and $B = \{B_1, B_2, \dots, B_n\}$, respectively. They satisfy $\sum_{i=1}^m A_i = \sum_{j=1}^n B_j = N$, $A_i \geqslant 0$, $B_j \geqslant 0$. Define a transmitting crowd $K_t = \{K_t\left(A_i\right), A_i \in A\}$ and a receiving crowd $K_r = \{K_r\left(B_j\right), B_j \in B\}$. In Figure 2, an example of K_t and K_r with above definitions are illustrated. We clarify that A and B do not need to be disjointed, they can be the same or overlapped as well.

T is a transportation strategy from K_t to K_r if T has K_t as its transmitting crowd and K_r as its receiving crowd. Obviously, T is not unique. Denote $\Theta\left(K_t, K_r\right)$ as the set of T from K_t to K_r. The transfer cost $C\left(T\right)$ for an arbitrary $T \in \Theta\left(K_t, K_r\right)$ is defined based on the specified underlying distance $d_{ij}\left(A_i, B_j\right)$ as follows:

$$C\left(T\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}\left(A_i, B_j\right) n\left(i, j\right), \tag{18}$$

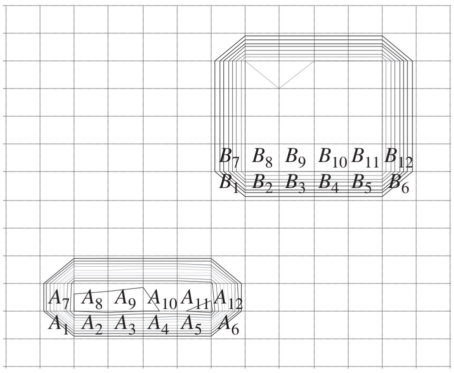

Figure 2 An example of transmitting crowd K_t and receiving crowd K_r.

Table 1 The format of matrix $M(m,n)$

	B_1	B_2	$\dots$	B_n
A_1				
A_2				
$\cdots$	$M(m,n)$		$d_{ij}(A_i,B_j)$	
A_m				

where $n\,(i,j)$ is the number of agents moving from A_i to B_j. Kantorovich distance d_K between the transmitting crowd K_t and the receiving crowd K_r with respect to the distance function $d_{ij}\,(A_i,B_j)$ is defined as follows:

$$d_K = \inf\left\{C\,(T)\,,\ T\in\Theta\,(K_t,K_r)\right\}. \tag{19}$$

Eq. (19) is equivalent to the following linear programming formulation:

$$\begin{aligned}&\min\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}\,(A_i,B_j)\,n\,(i,j)\,,\\ &\sum_{j=1}^{n} n\,(i,j)=A_i,\quad 1\leqslant i\leqslant m,\\ &\sum_{i=1}^{m} n\,(i,j)=B_j,\quad 1\leqslant j\leqslant n.\end{aligned} \tag{20}$$

When (19) or (20) is satisfied, transportation strategy T is optimal. Under such conceptions, the reshaping problem is reformulated as a large scale transportation problem where the key of Kantorovich distance lies in. All information needed to solve this problem, $d_{ij}\,(A_i,B_j)\,, n\,(i,j)\,, A_i$ and B_j, can be derived from $p(X)$ and $p_g(X)$.

4.2 Computation of Kantorovich distance

In recent decades, the computation of Kantorovich distance has been widely discussed in different areas, such as in the image processing and computer science [28,29], and the primal-dual algorithm has been considered as an efficient method for solving such a problem. A detailed step-by-step description of this algorithm has been previously reported [30]. Here, we apply the Ford-Fulkerson primal-dual algorithm to solve our problem. Two main steps in our calculation are briefly presented in the following paragraphs.

Initialization Label two sets A and B derived from $p(X)$ and $p_g(X)$. Calculate the distance d_{ij} between arbitrary A_i and B_j, and create a zero matrix $M\,(m,n)$ indexed as shown in Table 1. This matrix is used to store the transportation plan.

Find minima $\alpha_i = \min\limits_{1\leqslant j\leqslant n} d_{ij}$ $(i=1,2,\dots,m)$ in each row of M, denote the column set including α_i as B_α, and mark the elements in M with indexes according to α_i. Then, calculate variable β_j $(j=1,2,\dots,n)$ for each column following (21), and mark the elements in M whose indexes satisfy $\beta_j>0$.

α_i $(i = 1, 2, \ldots, m)$ and β_j $(j = 1, 2, \ldots, n)$ are called the dual variables. The marked elements are stored according to the marking order. All marked elements formulate the potential optimal transportation plan.

$$\begin{aligned} &\beta_j = 0,\ j \in B_\alpha, \\ &\beta_j = \min_{1 \leqslant j \leqslant n} \left(d_{ij} - \alpha_i\right),\ j \notin B_\alpha. \end{aligned} \tag{21}$$

For each row i, the marked elements are valued following the marking order. Assume the first marked element is M_{ij}, then formula (22) is given as a rule to value the marked elements. Repeat (22) until all the marked elements are valued.

$$\begin{aligned} &M_{ij} = \min\left(A_i,\ B_j\right), \\ &A_i = A_i - M_{ij}, \\ &B_j = B_j - M_{ij}. \end{aligned} \tag{22}$$

In the first step, we obtain the initial valued transportation plan M and the remaining number of agents in A and B.

Update transportation plan matrix M After the first step, if $\sum_{i=1}^{m} A_i = \sum_{j=1}^{n} B_j = 0$, the optimal plan is obtained. Otherwise, M should be updated. Two aspects of updating are considered. The first one is updating the marked elements values, and the second is adding marked elements. Both aspects start with the marking process.

The marking process starts with marking all the rows i satisfying $A_i > 0$ in the order of $i = 1, 2, \ldots, m$, and denoting $L\alpha\left(k\right) = i$ and $L\alpha_{\text{mark}}\left(i\right) = \infty$, $k = 1, 2, \cdots \leqslant m$. Then, check the marked rows i in order, and mark the columns j satisfying the condition that M_{ij} was marked in the first step. Mark these columns as $L\beta_{\text{mark}}\left(j\right) = i$, and store column indexes as $L\beta\left(l\right) = j$, $l = 1, 2, \cdots \leqslant n$. Thirdly, check the marked columns j according to the marking order. If there is a new row i satisfying $i \notin L\alpha$ and $M_{ij} > 0$, mark this row as $L\alpha\left(k\right) = i$, $L\alpha_{\text{mark}}\left(i\right) = j$. Fourthly, check the marked rows i according to the marking order. If there is a new column j satisfying $j \notin L\beta$, and M_{ij} was marked in the first step, mark this column as $L\beta_{\text{mark}}\left(j\right) = i$, $L\beta\left(l\right) = j$. Repeat this marking process until either new rows or columns are marked. This marking process is quite fast in calculation, and the marks are renewed in every loop.

After this marking process, the value of M_{ij} should be updated for the first $B_j > 0$, $j \in L\beta$ and $i = L\beta_{\text{mark}}\left(j\right)$. The updating strategy is performed through the admissible gain flow determined as follows:

$$L\alpha_{\text{mark}}\left(i\right) = \eta,\ L\beta_{\text{mark}}\left(\eta\right) = \gamma,\ L\alpha_{\text{mark}}\left(\gamma\right) = \mu, \cdots. \tag{23}$$

The procedure (23) stops when $L\alpha_{\text{mark}}\left(y\right) = \infty$. This terminal condition is reasonable since the agents should be transferred from the row y satisfying $A_y > 0$. The elements values are updated as follows:

$$\begin{aligned} &h_1 = \min\left\{B_j,\ M_{i\eta},\ M_{\gamma\mu}\ , \ldots,\ A_y\right\}, \\ &M_{ij} = M_{ij} + h_1,\ M_{i\eta} = M_{i\eta} - h_1, \\ &M_{\gamma\eta} = M_{\gamma\eta} + h_1,\ M_{\gamma\mu} = M_{\gamma\mu} - h_1, \\ &\qquad \cdots \\ &A_y = A_y - h_1,\ B_j = B_j - h_1. \end{aligned} \tag{24}$$

If there is no $B_j > 0$, $j \in L\beta$, M should be renewed by adding marked elements whose indexes satisfy rule (25),

$$\begin{aligned} &h_2 = \min_{i \in L\alpha, j \notin L\beta} \left(d_{ij} - \alpha_i - \beta_j\right), \\ &\alpha_i = \alpha_i + h_2,\ \beta_j = \beta_j - h_2. \end{aligned} \tag{25}$$

After a new M is obtained from (25), the calculation starts again from the beginning of the second step until all the agents are transferred, i.e., $\sum_{i=1}^{m} A_i = \sum_{j=1}^{n} B_j = 0$.

With the procedure presented above, we can obtain the optimal transportation strategy from which the destination for each agent is estimated. Together with the procedure described in Section 3, we are able to reshape the crowd with the feedback density $p(X)$ and the desired density $p_g(X)$.

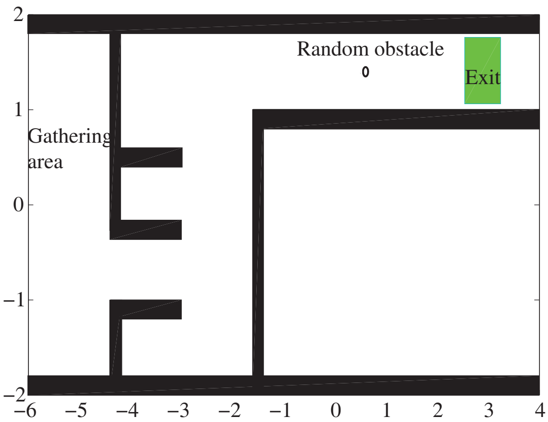

Figure 3 (Color online) The environment assumed for the simulation.

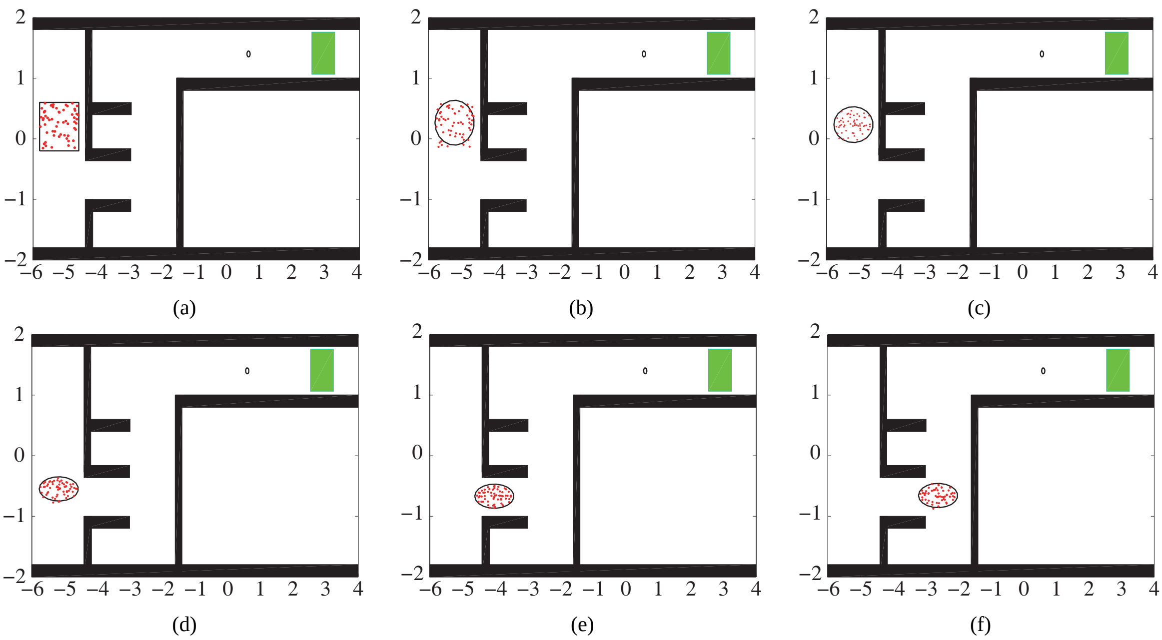

Figure 4 (Color online) Snapshots of the crowd behaviours in reshaping process and passing through the first bottleneck. (a) $t = 0$; (b) $t = 0.39$; (c) $t = 1.17$; (d) $t = 24.94$; (e) $t = 56.55$; (f) $t = 96.33$. The red points are the crowd agents positions. The black rectangle and ellipses are the desired shapes.

5 Numerical results

In this section, numerical experiments are carried out to illustrate the proposed idea and procedure. All the agents are assumed to behave with the same following parameters for simplification:

$$\begin{aligned} &a = 0.1,\ b = 2,\ c = 0.05,\ N = 60,\ k^i = [1,\ 1]^{\mathrm{T}},\ D^i = [0.01,\ 0.01]^{\mathrm{T}}, \\ &U^i_{\max} = [0.7,\ 0.7]^{\mathrm{T}},\ f^i = [0.9,\ 0.9]^{\mathrm{T}},\ g^i(X^i, V^i, t) = [0.1,\ 0.1]^{\mathrm{T}}. \end{aligned} \tag{26}$$

The environment is assumed to be a room with some space separated by walls and a random circular obstacle as shown in Figure 3. The obstacle is assumed with center $X_{\rm ob}$ and radius $r_{\rm ob}$. In the simulation (also see in the supplementary video), we let the agents move with specified shape from the "gathering area" towards the "exit" without hitting against the barriers.

In Figure 4, the process of shape transformation in the first few seconds is represented. The crowd shape is modified in order to pass the first bottleneck. In Figure 5, we show the crowd behavior when it passes through the random obstacle. The agents active the obstacle avoidance force with the opposite direction of $\boldsymbol{XX}_{\rm ob}$ to avoid the obstacle [31]. This extra force does not affect our control law. And the agents form the desired shape after they pass the obstacle.

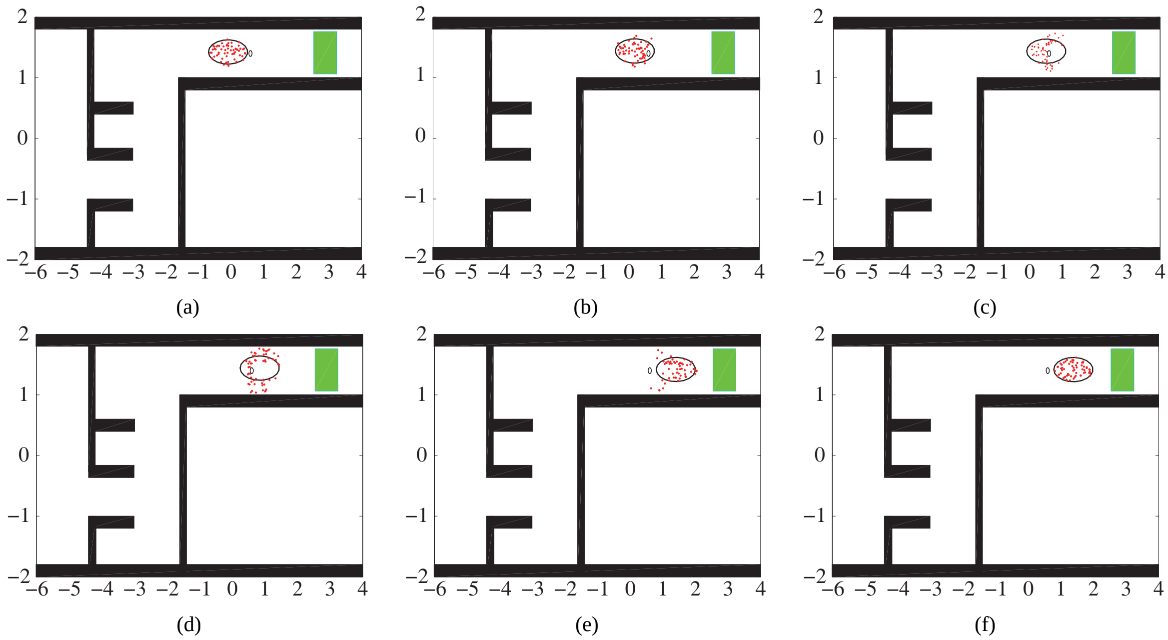

Figure 5 (Color online) Snapshots of the crowd behaviours passing through the obstacle. (a) $t = 238.29$; (b) $t = 244.14$; (c) $t = 245.31$; (d) $t = 246.09$; (e) $t = 248.43$; (f) $t = 249.6$. The red points are the crowd agents positions. The black ellipses are the desired shapes.

6 Conclusion

The crowd density directly reflects the crowd shape. Efficient control of the crowd shape can prevent congestion in a known complex environment. In this article, a novel optimal control strategy utilizing the density feedback is proposed to reshape the crowd. A new second-order crowd model concluding statistic information is generated. The control law is derived for each agent by using its own information, the current and the desired crowd densities. The results obtained from the numerical simulations illustrate the proposed idea and strategies.

The availability of density feedback successfully overcomes the complex coupled forward-backward system. We believe our work could also open an alternative door for analyzing the crowds models in different areas, such as traffic engineering, farm management and epidemic prevention.

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Conflict of interest The authors declare that they have no conflict of interest.

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