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A junction-by-junction feedback-based strategy with convergence analysis for dynamic traffic assignment

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Abstract By considering the traffic assignment problem as a control problem, this paper develops a new real-time route guidance strategy for accurate convergence of the traffic flows to user equilibrium (UE) or system optimum (SO), in the presence of drivers' response uncertainties. With the new guidance strategy, the drivers make routing decisions based on the route guidance information from junction to junction. Specifically, instead of total travel cost of every route from origin to destination, the travel cost of every alternative link plus the average cost to destination from the next junction corresponding to the alternative link is sent to the drivers at each specific junction. The drivers' response to the route guidance information is directly modeled by the splitting rates at the junctions, which are simply negatively correlated with the comparison of related cost information and are able to take into account the drivers' response uncertainties. With the proposed route guidance strategy, in the case of fixed travel demands, the accurate convergence of the traffic flows to a UE is guaranteed in the presence of drivers' response uncertainties by using LaSalle's invariance principle. When marginal travel cost information, instead of travel cost information, is sent to the drivers, a system optimum can be achieved under a mild condition on the marginal cost function.

Keywords dynamic traffic assignment, user equilibrium (UE), system optimum (SO), convergence, LaSalle's invariance principle

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1 Introduction

Due to the progress of urbanization and population growth, high travel costs caused by congestions become more severe [1]. Researchers and practicing engineers have been seeking innovative solutions based on advanced information technologies to make the existing transportation infrastructure more efficient, reliable and effective. Although more and more advanced software and devices become available to collect real-time traffic information, there are still significant problems in making use of the collected information to effectively improve the efficiency of the transportation system by appropriately guiding

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the users (drivers). The purpose of this paper is to develop a new route guidance strategy based on a relaxed driver response model involving uncertainties.

The study of traffic assignment can be traced back to the 1950's, when the two currently well-known traffic assignment objectives, user equilibrium (UE) and system optimum (SO) were introduced in [2]. The UE objective is achieved if no driver can reduce his travel cost by unilaterally switching his route, while the SO objective is such that the total system cost is minimized. Originally, static traffic assignment models were mainly studied for long-term traffic network planning. In the static models, the traffic environment is assumed to be time-invariant, and the traffic flows and travel costs on network links are studied at/around equilibria [3]. See, e.g., [4] and the references therein for the recent developments of the literature.

With an increasing desire for the analysis and control of the day-to-day and within-day evolution of the traffic flows, there has been a trend to study dynamic traffic assignment models by taking into account the time-varying traffic flows and travel demands [5–7]. The traffic flows in the networks are often used to represent the states of the system, and dynamic models using differential or difference equations are widely used to describe the evolution of the traffic flows. Basically, day-to-day models aim to capture day-to-day fluctuations [8–10] while within-day models are used to analyze the short-term fluctuations of the traffic flows influenced by the real-time traffic information [11, 12]. There have also been quite a few models developed to study the common behaviors of both within-day and day-to-day traffic assignment; see, e.g., [13–15]. The traffic assignment problem is closely related to the traffic signal control problem. The combination of the two problems has also been considered in some of the recent results [14, 16, 17]. Also note that there are two major kinds of models defined based on different kinds of flows: path-based model [8,12,13,15] and link-based model [9,10]. In [6,14], splitting rate models were employed to overcome some drawbacks of the path-based model.

Given a specific traffic network, by sending appropriate real-time traffic information to the drivers, it is expected that the traffic flows are controlled to ultimately converge to UE or SO. There have been two major approaches for the design of dynamic traffic assignment strategies: optimization methods (by using, for example, game theory and nonlinear programming [18]) and feedback control theory. Along the second line of research, convergence problems of the traffic flows can often be considered as Lyapunov stability problems of nonlinear dynamic systems. Intuitively, if the system is asymptotically stable (in the sense of Lyapunov) at UE, then the traffic flows ultimately converge to UE. Related results can be found in [8, 12, 15, 19] and recent papers [9, 20]. For a dynamic traffic assignment model, the Lyapunov stability property can be guaranteed if there exists an appropriate Lyapunov function. Moreover, even if a Lyapunov function cannot be found, the convergence may still be guaranteed by using LaSalle's invariance principle [13, 21]. See, e.g., [22] for Lyapunov stability theory and LaSalle's invariance principle. In addition, the advanced control theory provides powerful tools for the active control of the traffic flows. See, e.g., [23] for an optimal control design, [24] for a decentralized control design, [25, 26] for an H_∞ control design, [27] for a model predictive control design, [28] for a control design with feedback linearization, and [29] with agent-based control structure. More applications of feedback control theory to traffic control can be found in [16, 17].

Moreover, the advanced traffic management and information systems are able to provide more traffic information to realize the advanced control strategies. In this paper, by introducing advanced control theory to the literature of transportation systems, we study the dynamic traffic assignment problem with route guidance. Note that this problem is closely related to the within-day dynamic traffic assignment problem. Figure 1 shows the general block diagram of a dynamic traffic assignment system with route guidance, which is a feedback (closed-loop) system.

Clearly, the dynamics of the close-loop system depend on the structure of the traffic network, the behavior of the drivers and the design of the route guidance system. From the viewpoint of feedback control, the drivers and the traffic network together are considered as the system being controlled, and the guidance system is the controller. Our objective is to design the controller such that the traffic flows ultimately converge to UE or SO after a dynamic evolution procedure.

A dynamic traffic assignment model depends heavily on the drivers' response to the route guidance

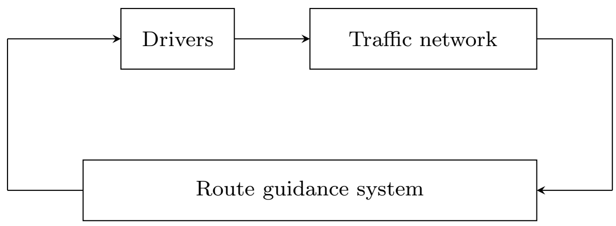

Figure 1 The block diagram of a dynamic traffic assignment system with route guidance.

information. However, many of the existing results (e.g., [13–15]) have not taken into account the uncertainties of the drivers' response, which limits the practical application of the results. In fact, it is shown in [30] that a converging perturbation may cause instability of a nonlinear system even if the perturbation-free system is asymptotically stable. On the other hand, the availability of advanced software and hardware for traffic information collection allows us to use traffic information other than the usually considered route travel costs [31], which makes it possible to handle the uncertainties of the drivers' response by designing improved route guidance strategies.

This paper takes a step forward toward the dynamic traffic assignment under uncertainties of drivers' response in a multi-origin multi-destination setting. By considering the traffic assignment problem as a control design problem, this paper answers the question: what real-time traffic information should be sent to the drivers, in the presence of drivers' response uncertainties, for accurate convergence? This problem is solved in this paper by developing a new junction-by-junction dynamic traffic assignment strategy. In the new model, the drivers' response to the route guidance information is directly represented by the splitting rates at junctions, which are simply negatively correlated with respect to the comparison of related cost information and significantly relaxes the requirement on the drivers' response; see Section 3 for detailed discussions. Instead of total travel cost of every route from origin to destination, the travel cost of every alternative link plus the average cost to destination from the next junction corresponding to the alternative link is sent to the drivers at specific junctions. With the proposed strategy, the drivers make routing decisions based on the route guidance information from junction to junction. The celebrated LaSalle's invariance principle [22] will be carefully used to guarantee the convergence of the traffic flows to a UE, in the presence of the drivers' response uncertainties. When marginal travel cost information, instead of travel cost information, is sent to the drivers, SO can be achieved under a convexity requirement on the SO objective function. This paper focuses on the problems caused by the uncertainties, and assumes fixed travel demands. Although this paper mainly focuses on the case of static travel demands, the result provides a solid basis to study the more practical case with time-varying travel demands. In this case, accurate convergence may not be expectable, and instead, practical convergence would be more meaningful. Due to the space limitation, this problem is left for future research. Also, the algorithm presented in this paper provides a solution to the calculation of the static traffic assignment solutions.

The rest of this paper is organized as follows. Section 2 introduces our traffic network model based on graph representations. In Section 3, we propose a relaxed driver response model by directly using the splitting rates at junctions. The main results on the convergence to UE and SO are given in Section 4. Section 5 presents numerical simulations to validate our theoretical results. Section 6 contains some concluding remarks.

2 Preliminaries and problem formulation

In this section, we introduce our traffic network model by using graph-based representations.

For a road traffic network, we use $\mathcal{V} = \{v_1, \ldots, v_n\}$ to represent the set of the junctions and $\mathcal{E} = \{e_1, \ldots, e_m\}$ to represent the set of the directed links. For convenience of notations, we denote $\mathcal{N} = \{1, \ldots, n\}$ and $\mathcal{L} = \{1, \ldots, m\}$. For $l \in \mathcal{L}$, we use $h(l), t(l)$ to represent the indices of the head and the tail of link e_l, respectively. See, e.g., [32] for the related concepts of graph theory. It should be noted that there could be multiple directed links with the same direction between the same pair of junctions.

A route in the traffic network corresponds to a non-repeating sequence of vertices such that from each of its vertices there is a directed edge to the next vertex in the sequence. In a traffic network, there can be multiple origin-destination (OD) pairs. We use $\mathcal{OD} = \{od_1, \ldots, od_M\}$ to denote the set of all the OD pairs. Denote $\mathcal{K} = \{1, \ldots, M\}$. For $k \in \mathcal{K}$, we use $o(k), d(k)$ to represent the indices of the origin and destination of OD pair od_k, respectively. We also define $\mathcal{D} = \{d(k) : k \in \mathcal{K}\}$ as the index set of all the destination vertices.

For each $i \in \mathcal{N}$, $k \in \mathcal{K}$, we use $x_i^k \in \mathbb{R}_+$ to denote the traffic flow through junction v_i contributed by OD pair od_k. Recall that $\mathbb{R}_+$ represents the set of nonnegative real numbers. For each $l \in \mathcal{L}$, $k \in \mathcal{K}$, we employ $y_l^k \in \mathbb{R}_+$ to denote the traffic flow on link e_l contributed by OD pair od_k. Then, the total traffic flow on each link e_l, denoted by y_l, can be calculated by

$$y_l = \sum_{k \in \mathcal{K}} y_l^k. \tag{1}$$

In this paper, we assume that the traffic flows over the road network satisfy the flow conservation property. That is, for each $k \in \mathcal{K}$:

- If $i \neq o(k)$ and $i \neq d(k)$, then

$$\sum_{l:h(l)=i} y_l^k = x_i^k = \sum_{l':t(l')=i} y_{l'}^k. \tag{2}$$

- If $i = o(k)$, then only the second equality in (2) holds.
- If $i = d(k)$, then only the first equality in (2) is satisfied.

From the flow conservation property, it can be observed that

$$x_{o(k)}^k = x_{d(k)}^k := x_*^k, \tag{3}$$

for $k \in \mathcal{K}$. Here, x_*^k is known as the travel demand of OD pair od_k.

In this paper, we only consider the case of fixed travel demand and assume that x_*^k is constant, i.e.,

$$\dot{x}_*^k = 0, \tag{4}$$

for each $k \in \mathcal{K}$.

We can also represent the flow conservation property with a linear equation. According to the discussions above, for each $k \in \mathcal{K}$, we have

$$\sum_{l:h(l)=i} y_l^k - \sum_{l':t(l')=i} y_{l'}^k = 0, \quad \text{if } i \neq o(k) \text{ and } i \neq d(k); \tag{5}$$

$$\sum_{l':t(l')=i} y_{l'}^k = x_k^k, \quad \text{if } i = o(k); \tag{6}$$

$$\sum_{l:h(l)=i} y_l^k = x_k^k, \quad \text{if } i = d(k). \tag{7}$$

This leads to a linear equation $Ay = D$ where $y \in \mathbb{R}^{Mm}$ is the vector composed of all the y_l^k for $l \in \mathcal{L}$ and $k \in \mathcal{K}$, $D \in \mathbb{R}^{Mn}$ with the elements corresponding to the junctions, and $A \in \mathbb{R}^{Mn \times Mm}$ being a constant matrix. Moreover, given any specified network, the rows of A are linearly independent [3].

With the flow conservation property, for each $l \in \mathcal{L}$, $d \in \mathcal{D}$, we employ a variable β_l^d, which is called splitting rate, to represent the fraction of the total flow $\sum_{k \in \mathcal{K}:d(k)=d} x_{t(l)}^k$ through junction $v_{t(l)}$ that uses link e_l, i.e.,

$$\sum_{k \in \mathcal{K}:d(k)=d} y_l^k = \beta_l^d \sum_{k \in \mathcal{K}:d(k)=d} x_{t(l)}^k. \tag{8}$$

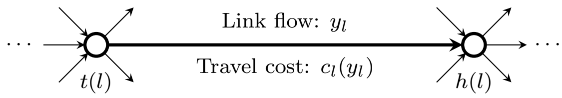

Figure 2 Definitions of $t(l)$, $h(l)$, y_l and $c_l(y_l)$ for link l.

Considering the physical meaning of the splitting rates and the flow conservation property of the traffic network, it can also be observed that, for each $d \in \mathcal{D}$,

$$\beta_l^d \geqslant 0 \text{ for all } l \in \mathcal{L}, \tag{9}$$

$$\sum_{l\in\mathcal{L}:t(l)=i} \beta_l^d = 1 \text{ for each } i \in \mathcal{N}\backslash\{d\}. \tag{10}$$

In particular, for each $d \in \mathcal{D}$, $i \in \mathcal{N}\backslash\{d\}$, property (10) implies

$$\sum_{l\in\mathcal{L}:t(l)=i} \dot{\beta}_l^d = 0. \tag{11}$$

For $l \in \mathcal{L}$, we assume that the travel cost along link e_l is solely determined by the traffic flow y_l on the link, and such relation is represented by a function $c_l : \mathbb{R}_+ \to \mathbb{R}_+$. That is, the travel cost along link e_l is $c_l(y_l)$. The travel cost along a route is the total cost of traveling along all the links of the route. For an $l \in \mathcal{L}$, Figure 2 shows the definitions of $t(l)$, $h(l)$, y_l and $c_l(y_l)$.

The dynamic traffic assignment problem with route guidance can generally be considered in a feedback framework as shown in Figure 1. The route guidance system measures/estimates the traffic information such as the traffic flows and the travel costs on the links, and provide route guidance to drivers by processing the collected information. The drivers make routing decision based on the guidance. The objective of this paper is to design a route guidance strategy such that the traffic flows in the transportation network converge to desired optimal points (e.g., UE and SO).

By considering the traffic assignment problem as a control problem, we will find what information should be sent to the drivers to achieve the convergence objective in the presence of drivers' response uncertainties. In this paper, the UE objective and the SO objective will both be considered. In Section 3, we will introduce a less restrictive model for the drivers' response to the guidance, and at the same time, propose an appropriate route guidance strategy such that the specific objectives can be achieved.

3 A relaxed driver response model

In the dynamic traffic assignment system, the drivers make decisions based on the guidance information sent from the route guidance system and the link traffic flows change accordingly. Intuitively, drivers switch to an alternative route if travel cost over the current route is higher. The uncertain reluctance of the drivers' response to information provided by the route guidance system significantly influence the convergence of the traffic flows to the optimal points. In this section, the splitting rates at the junctions are used to characterize the drivers' response.

The availability of advanced software and hardware for real-time traffic information collection allows us to use traffic information other than route travel costs for improved dynamic traffic assignment. It is vital to find out what information should be sent to the drivers and how.

In the section, we propose a guidance strategy, with which the guidance information sent to the drivers at one junction to a specified destination is based on the travel cost over every alternative link plus the average cost from the next junction to the destination. Recall that we use $c_l(y_l)$ to represent the travel cost along link e_l with traffic flow y_l. For each $d \in \mathcal{D}$ and each $i \in \mathcal{N}$, by using the splitting rates, the average cost from junction v_i to destination v_d can be defined by

$$\bar{c}_d^d(y) = 0, \tag{12}$$

$$\bar{c}_i^d(y) = \sum_{l\in\mathcal{L}:t(l)=i} \beta_l^d \left(c_l(y_l) + \bar{c}_{h(l)}^d(y)\right) \text{ for } i \in \mathcal{N}\backslash\{d\}. \tag{13}$$

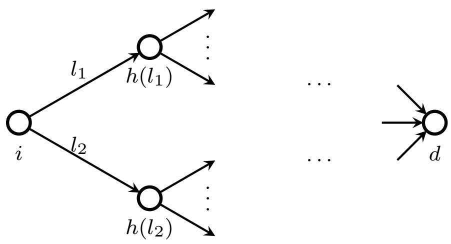

Figure 3 An example showing the definition of average costs.

The information sent to each driver at junction v_i to destination v_d is

$$\mathcal{I}_i^d = \{c_l(y_l) + \bar{c}_{h(l)}^d(y) : t(l) = i\}. \tag{14}$$

Based on this information, the drivers make routing decisions junction-by-junction.

Example 1. For a better understanding of the definition of average costs in (12) and (13), we consider a junction with two leaving links as shown in Figure 3.

Suppose that the average costs from $v_{h(l_1)}$ and $v_{h(l_2)}$ to v_d are $\bar{c}_{h(l_1)}^d(y)$ and $\bar{c}_{h(l_2)}^d(y)$, respectively. Also represent the travel costs over links e_{l_1} and e_{l_2} are $c_{l_1}(y_{l_1})$ and $c_{l_2}(y_{l_2})$, respectively. Then, the average cost from v_i to v_d can be calculated by

$$\begin{aligned}
&\frac{y_{l_1}^d(c_{l_1}(y_{l_1}) + \bar{c}_{h(l_1)}^d(y)) + y_{l_2}^d(c_{l_2}(y_{l_2}) + \bar{c}_{h(l_2)}^d(y))}{y_{l_1}^d + y_{l_2}^d} \\
&= \frac{y_{l_1}^d(c_{l_1}(y_{l_1}) + \bar{c}_{h(l_1)}^d(y)) + y_{l_2}^d(c_{l_2}(y_{l_2}) + \bar{c}_{h(l_2)}^d(y))}{x_i^d} \\
&= \frac{y_{l_1}^d}{x_i^d}(c_{l_1}(y_{l_1}) + \bar{c}_{h(l_1)}^d(y)) + \frac{y_{l_2}^d}{x_i^d}(c_{l_2}(y_{l_2}) + \bar{c}_{h(l_2)}^d(y)) \\
&= \beta_{l_1}^d(c_{l_1}(y_{l_1}) + \bar{c}_{h(l_1)}^d(y)) + \beta_{l_2}^d(c_{l_2}(y_{l_2}) + \bar{c}_{h(l_2)}^d(y)),
\end{aligned} \tag{15}$$

which is in accordance with the definition in (13).

By using the splitting rates at the junctions to represent the drivers' behavior, the drivers' response to the guidance information is modeled to satisfy:

1) For each $i \in \mathcal{N}$ and each $d \in \mathcal{D}$, the splitting rates β_l^d with $l \in \mathcal{L}$ with $t(l) = i$ are negatively correlated based on the comparison of the related costs, that is,

$$\sum_{l \in \mathcal{L}: t(l) = i} \dot{\beta}_l^d(c_l(y_l) + \bar{c}_{h(l)}^d(y)) \leqslant 0, \tag{16}$$

where equality holds only when $\dot{\beta}_l^d = 0$ for all $l \in \mathcal{L}$ satisfying $t(l) = i$.

2) For each $i \in \mathcal{N}$ and each $d \in \mathcal{D}$, $\dot{\beta}_l^d = 0$ holds for all $l \in \mathcal{L}$ satisfying $t(l) = i$ if and only if for each $l \in \mathcal{L}$ satisfying $t(l) = i$: either

$$c_l(y_l) + \bar{c}_{h(l)}^d(y) = \bar{c}_i^d(y), \tag{17}$$

or

$$c_l(y_l) + \bar{c}_{h(l)}^d(y) > \bar{c}_i^d(y) \text{ and } \beta_l^d = 0. \tag{18}$$

Here, condition 1) means that in the presence of drivers' response uncertainties, the splitting rates are only negatively correlated based on the comparison of the cost information. The changing rates of the splitting rates are not specified; see also Example 2. Condition 2) means that the splitting rates at a junction do not change if the costs of the routes from the junction that are used by the traffic through the junction are equal to each other and the other routes have higher costs. It should be noted that

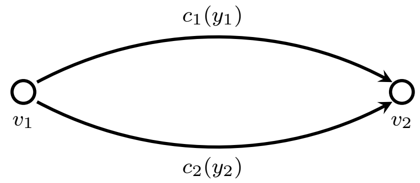

Figure 4 A network composed of two junctions connected by two links.

with conditions 1) and 2) satisfied, the splitting rates and thus the link traffic flows y_l^k can always be guaranteed to be non-negative.

By considering y as the state, we can employ a differential inclusion to represent the dynamics of the closed-loop dynamic traffic assignment system:

$$\dot{y} \in F(y), \tag{19}$$

where $F : \mathbb{R}^{Mm} \rightsquigarrow \mathbb{R}^{Mm}$ is a convex, compact and upper semicontinuous set-valued map, which guarantees the existence of the solution to the differential inclusion [33]. In Section 4, we will prove the validity of the route guidance strategy by showing that y ultimately accurately converges to UE.

Remark 1. In many of the previous results of deterministic dynamic traffic assignment, the drivers' response is usually modeled by certain differential equations with route flows, link flows or splitting rates as state variables. The driver response model proposed in this paper is directly formulated by using the splitting rates, and significantly different from many of the previous results, the derivatives of the splitting rates are only required to follow negative cost gradients, formulated by inequality (16). This means flexibility to deal with the uncertainties caused by the reluctance of the drivers' response. At the same time, we are able to avoid the problems of path-based models raised by [10].

Remark 2. We will mainly use the implicit representation of system dynamics by (16)–(18) to analyze the convergence property of the closed-loop traffic assignment system (19). From control point of view, this leads to a new problem of system analysis significantly different from the traditionally considered problems with explicit representations. Based on the new tool developed in this paper, more general systems with implicit representations could be studied in the future.

Remark 3. In our model, average costs are employed for route guidance. Such information is available as long as the travel costs and the traffic flows over all the alternative routes are measurable with the advanced traffic information and management system.

Example 2. To compare the model in this paper with the models studied in some of the previous results, we consider a simple network composed of two junctions v_1, v_2 connected by two links e_1, e_2 as shown in Figure 4. We also assume that v_1 is the only origin and v_2 is the only destination, that is, there is only one OD pair $od_1 = (v_1, v_2)$. Denote x_*^1 as the travel demand of od_1.

Clearly, according to the definitions in Section 2, $t(1) = t(2) = 1$ and $h(1) = h(2) = 2$. The traffic flows on the two links are y_1 and y_2, and the travel costs are $c_1(y_1)$ and $c_2(y_2)$, respectively. At v_1, the splitting rates of the traffic flows to destination v_2 are denoted by β_1^2 and β_2^2, respectively.

For this simple network, with our model, the information sent to the drivers at v_1 going to v_2 is

$$\mathcal{I}_1^2 = \{c_l(y_l) : t(l) = 1\} = \{c_1(y_1), c_2(y_2)\}. \tag{20}$$

Then, according to our model, the splitting rates β_1^2 and β_2^2 satisfy $\beta_1^2 + \beta_2^2 = 1$ and

$$\dot{\beta}_1^2 c_1(y_1) + \dot{\beta}_2^2 c_2(y_2) \leqslant 0, \tag{21}$$

where inequality holds only when $\dot{\beta}_1^2 = \dot{\beta}_2^2 = 0$. Moreover, $\dot{\beta}_1^2 = \dot{\beta}_2^2 = 0$ if and only if either

$$c_1(y_1) = c_2(y_2), \tag{22}$$

or there is an $l \in \{1, 2\}$ such that

$$c_l(y_l) > c_{(3-l)}(y_{(3-l)}) \text{ and } \beta_l^2 = 0. \tag{23}$$

For this simple network, with the models proposed in [13–15], the traffic flows should satisfy

$$\dot{y}_1 = -ky_1 \max\{(c_1(y_1) - c_2(y_2)), 0\} + ky_2 \max\{(c_2(y_2) - c_1(y_1)), 0\}, \tag{24}$$

$$\dot{y}_2 = -ky_2 \max\{(c_2(y_2) - c_1(y_1)), 0\} + ky_1 \max\{(c_1(y_1) - c_2(y_2)), 0\}, \tag{25}$$

where k is a positive constant. Equivalently, we have for $l = 1, 2$,

$$\dot{y}_l = \begin{cases} -ky_1(c_l(y_l) - c_{(3-l)}(y_{(3-l)})), & \text{if } c_l(y_l) \geqslant c_{(3-l)}(y_{(3-l)}); \\ -ky_{(3-l)}(c_l(y_l) - c_{(3-l)}(y_{(3-l)})), & \text{if } c_l(y_l) < c_{(3-l)}(y_{(3-l)}). \end{cases} \tag{26}$$

For the simple network, by using the definition of splitting rates $\beta_l^2 = y_l/x_*^2$ for $l = 1, 2$, it can be directly verified that the evolution defined by (26) satisfies (21)–(23). Moreover, based on (21)–(23), we are able to consider more general dynamics. For example, Eq. (26) could be modified as

$$\dot{y}_l = \begin{cases} -k\varphi_1(y_1)\varphi_2(c_l(y_l) - c_{(3-l)}(y_{(3-l)})), & \text{if } c_l(y_l) \geqslant c_{(3-l)}(y_{(3-l)}); \\ -k\varphi_1(y_{(3-l)})\varphi_2(c_l(y_l) - c_{(3-l)}(y_{(3-l)})), & \text{if } c_l(y_l) < c_{(3-l)}(y_{(3-l)}). \end{cases} \tag{27}$$

Here, $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ can be any monotone function satisfying $\varphi_1(0) = 0$ and $\varphi_2 : \mathbb{R} \to \mathbb{R}$ can be any function satisfying $\varphi_2(r)r > 0$ for all $r \neq 0$. More generally, in the presence of uncertainties, we may further replace φ_2 with a set-valued map $\breve{\varphi}_2 : \mathbb{R} \rightsquigarrow \mathbb{R}$ satisfying $rz > 0$ for all $z \in \breve{\varphi}_2(r)$ and all $r \neq 0$. In this case, the "=" in (27) should be replaced by "∈", which leads to a differential inclusion.

For the simple network shown in Figure 4, the model considered in this paper captures the essential idea that the drivers switch to an alternative route if travel cost over the current route is higher, and can be considered as a generalization of the models proposed in the previous results [13, 14], by taking into the drivers' response uncertainties. Moreover, our model does not assume single-origin single-destination, and when the network involves more junctions and links, our carefully designed route guidance strategy can still guarantee the achievement of UE.

Example 3. Consider a junction v_i that has three outgoing links $e_{l_1}, e_{l_2}, e_{l_3}$ such that $\{l_1, l_2, l_3\} = \{l \in \mathcal{L} : t(l) = i\}$. Then, the splitting rates $[\beta_{l_1}^d, \beta_{l_2}^d, \beta_{l_3}^d]^{\mathrm{T}}$ satisfying (9) and (10) remain on the simplex S_i^d defined by $\beta_l^d \geqslant 0$ for $l = l_1, l_2, l_3$ and $\beta_{l_1}^d + \beta_{l_2}^d + \beta_{l_3}^d = 1$ during the process of dynamic traffic assignment. Moreover, the splitting rates are negatively correlated with respect to the cost information such that $\dot{\gamma}_i^{d\mathrm{T}} C_i^d \leqslant 0$ with $\gamma_i^d := [\beta_{l_1}^d, \beta_{l_2}^d, \beta_{l_3}^d]^{\mathrm{T}}$ and $C_i^d := [c_{l_1}(y_{l_1}) + \bar{c}_{h(l_1)}^d(y), c_{l_2}(y_{l_2}) + \bar{c}_{h(l_2)}^d(y), c_{l_3}(y_{l_3}) + \bar{c}_{h(l_3)}^d(y)]^{\mathrm{T}}$, as shown in Figure 5. Note that the inequality condition is flexible and does not require specific $\dot{\gamma}_i^d$.

4 Main results

In this section, we verify the validity of the proposed route guidance strategy by showing the convergence of the traffic flows to UE in the presence of the uncertainties. First, we present two lemmas on the properties of the dynamic traffic assignment system. Then, we prove the convergence to UE by using LaSalle's invariance principle. An extension of the main result to the SO case is also developed.

4.1 Technical lemmas

Lemma 1 presents a property on the boundedness and non-negativeness of the solutions of the system.

Lemma 1. Given specific travel demands x_*^k with $k \in \mathcal{K}$, there exists a compact set

$$\Omega = \{y : 0 \leqslant y_l^k \leqslant x_*^k \text{ for } l \in \mathcal{L},\ k \in \mathcal{K}\}, \tag{28}$$

such that $y(t) \in \Omega$ for all $t \geqslant 0$.

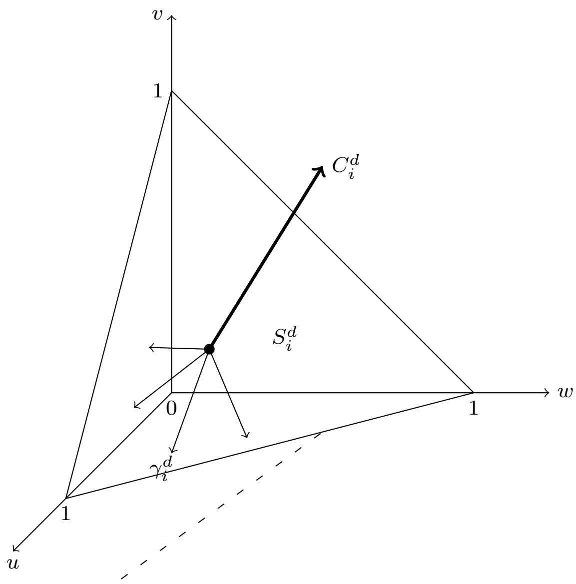

Figure 5 The simplex S_i^d and the negative correlation of $\dot{\gamma}_i^d$ with respect to C_i^d.

Since the routes in the traffic network are non-repeating sequences of vertices, Lemma 1 can be proved by using directly the non-negativeness of the link flows (due to the nonnegativeness of the splitting rates) and the flow conservation property (2). Based on Lemma 1, we can consider Ω as a positively invariant set of the dynamic traffic assignment system. Such property will be used for convergence analysis based on LaSalle's invariance principle. For the concepts and results related to the notions of invariant set, please refer to [22].

Lemma 2 presents another technical result that, together with Lemma 1, will be used for convergence analysis.

Lemma 2. Consider the traffic network defined in Section 2 and the average costs to destinations defined in (12)–(13). Then, for each $d \in \mathcal{D}$, it holds that

$$\sum_{l\in\mathcal{L}} \sum_{k\in\mathcal{K}:d(k)=d} \dot{y}_l^k c_l(y_l) = \sum_{l\in\mathcal{L}} \sum_{k\in\mathcal{K}:d(k)=d} x_{t(l)}^k \dot{\beta}_l^d \left(c_l(y_l) + \bar{c}_{h(l)}^d(y)\right), \tag{29}$$

where $\bar{c}^k(y) := \bar{c}_{o(k)}^{d(k)}(y)$ for $k \in \mathcal{K}$.

Proof. Throughout the proof, l always takes values from $\mathcal{L}$ and k always takes values from $\mathcal{K}$. For convenience of notations, we omit $l \in \mathcal{L}$ and $k \in \mathcal{K}$. For each $d \in \mathcal{D}$, based on the traffic network model, we have

$$\begin{aligned}
\sum_{k:d(k)=d} \sum_{i\in\mathcal{N}} \dot{x}_i^k \bar{c}_i^d &= \sum_{k:d(k)=d} \sum_{i\in\mathcal{N}\backslash\{d\}} \left(\dot{x}_i^k \sum_{l:t(l)=i} \beta_l^d \left(c_l + \bar{c}_{h(l)}^d\right)\right) \\
&= \sum_{i\in\mathcal{N}\backslash\{d\}} \left(\left(\sum_{k:d(k)=d} \dot{x}_i^k\right) \sum_{l:t(l)=i} \beta_l^d \left(c_l + \bar{c}_{h(l)}^d\right)\right) \\
&= \sum_{i\in\mathcal{N}\backslash\{d\}} \sum_{l:t(l)=i} \left(\beta_l^d \sum_{k:d(k)=d} \dot{x}_{t(l)}^k \left(c_l + \bar{c}_{h(l)}^d\right)\right).
\end{aligned} \tag{30}$$

Recall the definition of splitting rate in (8). By taking the derivatives of both sides of (8), we have

$$\sum_{k:d(k)=d} \dot{y}_l^k = \dot{\beta}_l^d \sum_{k:d(k)=d} x_{t(l)}^k + \beta_l^d \sum_{k:d(k)=d} \dot{x}_{t(l)}^k. \tag{31}$$

By comparing the same term in (30) and (31), we have

$$\begin{aligned}
\sum_{k:d(k)=d}\sum_{i\in\mathcal{N}}\dot{x}_i^k\bar{c}_i^d &= \sum_{i\in\mathcal{N}\setminus\{d\}}\sum_{l:t(l)=i}\sum_{k:d(k)=d}\dot{y}_l^k\left(c_l+\bar{c}_{h(l)}^d\right)\\
&\quad-\sum_{i\in\mathcal{N}\setminus\{d\}}\sum_{l:t(l)=i}\left(\dot{\beta}_l^d\sum_{k:d(k)=d}x_{t(l)}^k\left(c_l+\bar{c}_{h(l)}^d\right)\right)\\
&=\sum_{k:d(k)=d}\sum_{l\in\mathcal{L}}\dot{y}_l^k c_l+\sum_{i\in\mathcal{N}\setminus\{d\}}\sum_{l:t(l)=i}\sum_{k:d(k)=d}\dot{y}_l^k\bar{c}_{h(l)}^d\\
&\quad-\sum_{k:d(k)=d}\sum_{l\in\mathcal{L}}x_{t(l)}^k\dot{\beta}_l^d\left(c_l+\bar{c}_{h(l)}^d\right)\\
&=\sum_{k:d(k)=d}\sum_{l\in\mathcal{L}}\dot{y}_l^k c_l+\sum_{k:d(k)=d}\sum_{j\in\mathcal{N}\setminus\{o(k)\}}\left(\bar{c}_j^d\sum_{l:h(l)=j}\dot{y}_l^k\right)\\
&\quad-\sum_{k:d(k)=d}\sum_{l\in\mathcal{L}}x_{t(l)}^k\dot{\beta}_l^d\left(c_l+\bar{c}_{h(l)}^d\right)\\
&=\sum_{k:d(k)=d}\sum_{l\in\mathcal{L}}\dot{y}_l^k c_l+\sum_{k:d(k)=d}\sum_{j\in\mathcal{N}\setminus\{o(k)\}}\dot{x}_j^k\bar{c}_j^d\\
&\quad-\sum_{k:d(k)=d}\sum_{l\in\mathcal{L}}x_{t(l)}^k\dot{\beta}_l^d\left(c_l+\bar{c}_{h(l)}^d\right). \qquad (32)
\end{aligned}$$

Note that

$$\sum_{k:d(k)=d}\sum_{i\in\mathcal{N}}\dot{x}_i^k\bar{c}_i^d-\sum_{k:d(k)=d}\sum_{j\in\mathcal{N}\setminus\{o(k)\}}\dot{x}_j^k\bar{c}_j^d=\sum_{k:d(k)=d}\dot{x}_{o(k)}^k\bar{c}_{o(k)}^d=\sum_{k:d(k)=d}\dot{x}_*^k\bar{c}^k=0. \qquad (33)$$

The proof is concluded by substituting (33) into (32).

Remark 4. In the proof of Lemma 2, we only used the properties of our traffic network model given in Section 2, and thus the property presented in Lemma 2 does not depend on the driver response model.

4.2 Convergence to UE

LaSalle's invariance principle is used for the convergence analysis of the dynamic traffic assignment system to UE. One may find details of LaSalle's invariance principle from [22] and the original papers [34, 35]. Although the invariance principle was originally developed for differential equations, it has been extended to systems represented by differential inclusions [36,37], as given by Theorem 1 in this paper. See also [38] for a nice literature review of the invariance principle for nonlinear and more general hybrid nonlinear systems represented by inclusions. The concepts related to differential inclusions used in Theorem 1 can also be found in [33].

Theorem 1. Let $\Omega\subseteq D$ be a compact set that is positively invariant with respect to (19), that is, every solution of system (19) starting from Ω remains in Ω for all $t\geqslant 0$. Let $V:\Omega\to\mathbb{R}$ be a continuously differentiable function such that $\max_{v^d\in\nabla VF(y)}v^d\leqslant 0$ in Ω. Then, every trajectory in Ω converges to the largest invariant set in the closure of $E:=\{y\in\Omega:0\in\nabla VF(y)\}$.

We define the following function for the convergence analysis:

$$V(y)=\sum_{l\in\mathcal{L}}\int_0^{y_l}c_l(\xi)\mathrm{d}\xi. \qquad (34)$$

Note that $V(y)$ defined in (34) is in accordance with the Beckmann objective function [39].

Taking the derivative of $V(y)$ along the trajectories of system (19) and using Lemma 2, we have

$$\begin{aligned}\dot{V}(y) &= \sum_{l\in\mathcal{L}} \dot{y}_l c_l(y_l)\\ &= \sum_{d\in\mathcal{D}}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}:d(k)=d} \dot{y}_l^k c_l(y_l)\\ &= \sum_{d\in\mathcal{D}}\sum_{l\in\mathcal{L}}\sum_{k\in\mathcal{K}:d(k)=d} x_{t(l)}^k \dot{\beta}_l^d \left(c_l(y_l)+\bar{c}_{h(l)}^d(y)\right)\\ &\leqslant 0. \end{aligned} \tag{35}$$

Given the invariance of compact set Ω defined in Lemma 1, using LaSalle's invariance principle, the traffic flows converge to the largest invariant set inside the closure of $E=\{y\in\Omega: 0\in\nabla VF(y)\}$. Given the upper semicontinuity of F, the closure of E is E itself.

Note that the third equality in (35) can be equivalently written as

$$\dot{V}(y)=\sum_{d\in\mathcal{D}}\sum_{i\in\mathcal{N}:i\neq d}\left(\left(\sum_{l\in\mathcal{L}:t(l)=i}\dot{\beta}_l^d\left(c_l(y_l)+\bar{c}_{h(l)}^d(y)\right)\right)\sum_{k\in\mathcal{K}:d(k)=d}x_i^k\right). \tag{36}$$

Based on (36), using conditions 1) and 2) defined by (16)–(18), $0\in\nabla VF(y)$ implies that for each $d\in\mathcal{D}$, $i\in\mathcal{N}\backslash\{d\}$, either $\sum_{k\in\mathcal{K}:d(k)=d}x_i^k=0$ or $\dot{\beta}_l^d=0$ for all l satisfying $t(l)=i$. This means that all link traffic flows are constant in set E, and thus the largest invariant set in E is E itself. By using Theorem 1, the link traffic flows converge to set E. In set E, for the routes of each OD pair, the total travel costs of all the routes carrying the traffic of the OD pair are equal to each other and are not larger than the total travel costs of the other routes. This means convergence to UE.

Our main result on the convergence to UE is summarized in Theorem 2.

Theorem 2. Consider the traffic network defined in Section 2 and the driver response model satisfying conditions 1) and 2) with the average costs to destinations defined in (12)–(13). Then, the traffic flows converge to UE.

4.3 An extension to SO

Under some mild conditions, by sending appropriate route guidance information to the drivers, our dynamic traffic assignment system can also achieve SO. For the traffic network studied in this paper, the SO objective is to minimize

$$W(y)=\sum_{l\in\mathcal{L}}\breve{c}_l(y_l), \tag{37}$$

where $\breve{c}_l(y_l):=y_lc_l(y_l)$.

For $k\in\mathcal{K}$, define

$$g_i^k(y)=\sum_{l\in\mathcal{L}:t(l)=i}y_l^k-x_*^k,\quad \text{for } i=o(k), \tag{38}$$

$$g_i^k(y)=\sum_{l\in\mathcal{L}:h(l)=i}y_l^k-\sum_{l\in\mathcal{L}:t(l)=i}y_l^k,\quad \text{for } i\in\mathcal{N}\backslash\{o(k),d(k)\}. \tag{39}$$

According to the traffic network model, the minimization problem is subject to the following constraints: for each $k\in\mathcal{K}$,

$$g_i^k(y)=0,\quad \text{for } i\in\mathcal{N}\backslash\{d(k)\}. \tag{40}$$

According to Wardrop's theory [2], the SO objective can be achieved by comparing the marginal travel costs of alternative routes. Our UE result is extended to SO by using this idea. Without loss of generality, we assume that each c_l is continuously differentiable for $l \in \mathcal{L}$. The marginal travel cost of link e_l is defined as

$$\hat{c}_l(y_l) = \frac{d\check{c}_l(y_l)}{dy_l}, \tag{41}$$

for $l \in \mathcal{L}$, and use $\hat{c}_l(y_l)$ instead of $c_l(y_l)$ to compute the route guidance information sent to each driver at junction v_i heading destination v_d:

$$\mathcal{I}_i^d = \{\hat{c}_l(y_l) + \tilde{c}_{h(l)}^d(y) : t(l) = i\}, \tag{42}$$

where $\tilde{c}_{h(l)}^d(y)$ is computed as for $\bar{c}_{h(l)}^d(y)$ in (12)–(13) with $c_l(y)$ replaced by $\hat{c}_l(y)$. In this case, the drivers' response is still modeled to satisfy conditions 1) and 2) in Section 3 with c_l and $\bar{c}_i$ replaced by $\hat{c}_l$ and $\tilde{c}_i$, respectively.

Based on a similar analysis as for UE, in this case, the link traffic flows converge to set $E' = \{y \in \Omega : 0 \in \nabla WF(y)\}$. Moreover, $0 \in \nabla WF(y)$ implies that all link traffic flows are constant, and for the routes of each OD pair, the total marginal travel costs of the routes that carry the traffic of the OD pair are equal to each other, and are not larger than the total marginal travel costs of the other routes.

To validate set E' as solution of the minimization problem, we employ the Lagrangian function:

$$\Lambda(y,\nu) = W(y) + \sum_{k\in\mathcal{K}} \sum_{i\in\mathcal{N}\setminus\{d(k)\}} \nu_i^k g_i^k(y), \tag{43}$$

where ν_i^k represents the Lagrangian multipliers for $k \in \mathcal{K}$, $i \in \mathcal{N}\setminus\{d(k)\}$, and ν is the vector of ν_i^k's.

By using the definition of $\tilde{c}_i^{d(k)}(y)$, if $y \in E'$, then for each $k \in \mathcal{K}$, $i \in \mathcal{N}$,

$$\tilde{c}_i^{d(k)}(y) = \hat{c}_l(y_l) + \tilde{c}_{h(l)}^{d(k)}(y), \tag{44}$$

for all $l \in \mathcal{L}$ satisfying $t(l) = i$ and $y_l \neq 0$. Thus, for all $y \in E'$, by choosing

$$\nu_i^k = -\tilde{c}_i^{d(k)}(y), \tag{45}$$

for $k \in \mathcal{K}$, $i \in \mathcal{N}\setminus\{d(k)\}$, we have

$$\frac{\partial\Lambda(y)}{\partial y_l^k} = \frac{d\check{c}_l(y_l)}{dy_l} + \nu_{t(l)}^k - \nu_{h(l)}^k = 0, \tag{46}$$

for $k \in \mathcal{K}$, $l \in \mathcal{L}$.

As discussed in Section 2, the linear constraints on the traffic flows can be represented by the equation $Ay = D$ where the rows of A are linearly independent. This means the feasible traffic flows y are always regular. Note that the objective function W is continuously differentiable given the continuous differentiability of c_l. Assume that W is convex. By using [40, Proposition 3.4.1], y is a global minimum of W if and only if there exist ν such that y minimizes $\Lambda(y,\nu)$. For our problem, the convexity of W implies the convexity of L for any specified ν. Thus, y is a global minimum of W if and only if there exist ν such that (46) holds. It can be directly checked that each $y \in E'$ satisfies (46).

Now we are ready to state our main result on the convergence to SO.

Theorem 3. Consider the traffic network defined in Section 2 and the driver response model satisfying conditions 1) and 2) with c_l and $\bar{c}_i$ replaced by $\hat{c}_l$ and $\tilde{c}_i$, respectively. Then, the traffic flows converge to SO if the objective function W defined in (37) is convex.

Remark 5. If the objective function W is strictly convex, then the solution of the optimization problem studied in this subsection is unique.

5 Simulation results

5.1 Driver response model

The model given in Section 3 only requires the negative correlation of the traffic flows with respect to the cost information. In this section, we employ a more specific model for simulation.

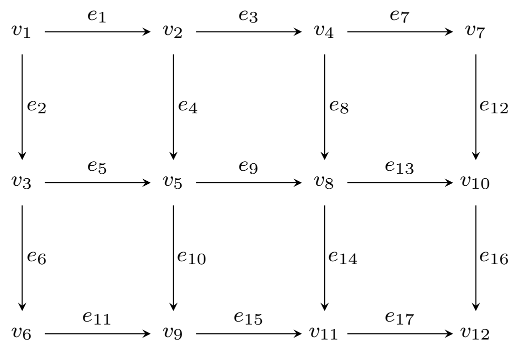

Figure 6 The traffic network for simulation.

For each junction v_i, denote $\gamma_i^d = [\beta_{l_1^i}, \ldots, \beta_{l_{D_i}^i}^d]^{\mathrm{T}}$ as the vector of splitting rates at the junction to destination d, where $\{l_1^i, \ldots, l_{D_i}^i\} = \{l \in \mathcal{L} : t(l) = i\}$ and D_i is the dimension of γ_i^d. Denote I_{D_i} as the D_i-dimensional identity matrix and $\mathbf{1}_{D_i}$ as the D_i-dimensional vector with elements being 1.

Define

$$m_i^d(t, y) = -k_i^d(t)\left(I_{D_i} - \frac{1}{D_i}\mathbf{1}_{D_i}\mathbf{1}_{D_i}^{\mathrm{T}}\right)\left(C_i(y) + \bar{C}_i(y)\right), \tag{47}$$

where $C_i(y) := [c_{l_1^i}(y_{l_1^i}), \ldots, c_{l_{D_i}^i}(y_{l_{D_i}^i})]^{\mathrm{T}}$, $\bar{C}_i(y) := [\bar{c}_{h(l_1^i)}(y), \ldots, \bar{c}_{h(l_{D_i}^i)}(y)]^{\mathrm{T}}$, and $\underline{k}_i^d \leqslant k_i^d(t) \leqslant \overline{k}_i^d$ for all $t \geqslant 0$ with $0 < \underline{k}_i^d \leqslant \overline{k}_i^d$ being constants.

For $t \geqslant 0$, we define $\acute{\gamma}_i^d(t)$ as a vector such that: For each $l = l_1^i, \ldots, l_{D_i}^i$, if $\beta_l^d(t) = 0$ and the corresponding element of $m_i^d(t, y(t))$ is negative, then $\beta_l^d(t)$ is not an element of $\acute{\gamma}_i^d(t)$; otherwise, $\beta_l^d(t)$ is an element of $\acute{\gamma}_i^d(t)$.

Correspondingly, we define $\acute{C}_i(y(t))$ and $\acute{\bar{C}}_i(y(t))$ based on $C_i(y(t))$ and $\bar{C}_i(y(t))$, respectively. The dynamics of the splitting rates contained by $\acute{\gamma}_i^d(t)$ are chosen such that

$$\dot{\acute{\gamma}}_i^d(t) = -k_i^d(t)\left(I_{\acute{D}_i} - \frac{1}{\acute{D}_i}\mathbf{1}_{\acute{D}_i}\mathbf{1}_{\acute{D}_i}^{\mathrm{T}}\right)\left(\acute{C}_i(y(t)) + \acute{\bar{C}}_i(y(t))\right), \tag{48}$$

where $\acute{D}_i$ is dimension of $\acute{\gamma}_i^d$, while the derivatives of the splitting rates not contained by $\acute{\gamma}_i^d(t)$ are zero. It can be directly verified that the specific driver response model given above satisfies conditions 1) in Section 3. From the model, it can also be observed that only when the total travel costs of the routes carrying traffic equal each other and are not larger than the total travel costs of the routes not carrying traffic (if any), the splitting rates do not change. This is in accordance with condition 2) in Section 3. It should be noted that the dynamics of γ_i^d is discontinuous with the discontinuity depending on the traffic flow pattern. It is necessary to employ differential inclusions to represent the dynamics of such systems.

5.2 Numerical simulation

In this subsection, we employ a numerical simulation to verify the main result of this paper. Consider a traffic network composed of 7 junctions and 10 links, as shown in Figure 6.

There are two OD pairs: $od_1 = (v_1, v_{11})$ and $od_2 = (v_2, v_{12})$. We define the related routes as

$$\begin{aligned}
&r_1 = (v_1, v_3, v_8, v_{14}), r_2 = (v_1, v_4, v_9, v_{14}),\\
&r_3 = (v_1, v_4, v_{10}, v_{15}), r_4 = (v_2, v_5, v_9, v_{14}),\\
&r_5 = (v_2, v_5, v_{10}, v_{15}), r_6 = (v_2, v_6, v_{11}, v_{15}),\\
&r_7 = (v_3, v_7, v_{12}, v_{16}), r_8 = (v_3, v_8, v_{13}, v_{16}),\\
&r_9 = (v_3, v_8, v_{14}, v_{17}), r_{10} = (v_4, v_9, v_{13}, v_{16}),\\
&r_{11} = (v_4, v_9, v_{14}, v_{17}), r_{12} = (v_4, v_{10}, v_{15}, v_{17}).
\end{aligned}$$

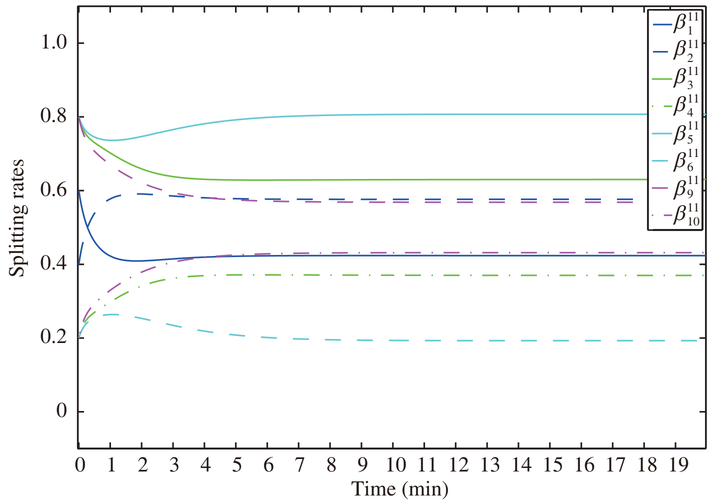

Figure 7 (Color online) The evolution of the splitting rates for od_1.

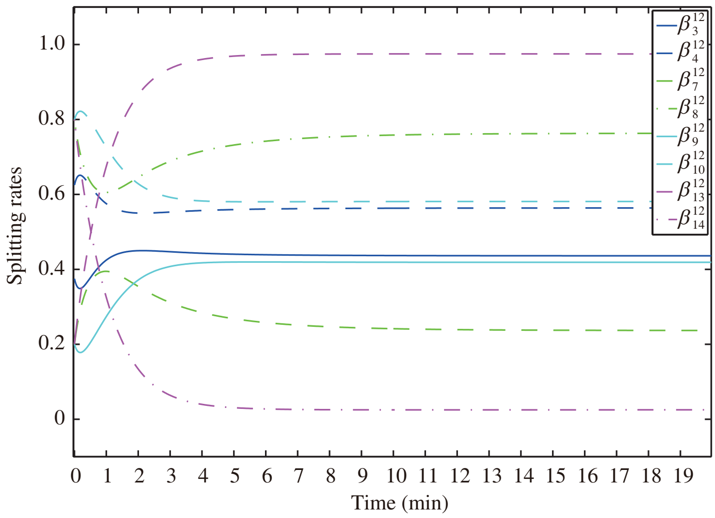

Figure 8 (Color online) The evolution of the splitting rates for od_2.

In particular, routes r_1, r_2, r_3, r_4, r_5 and r_6 correspond to od_1, and routes r_7, r_8, r_9, r_{10}, r_11 and r_{12} correspond to od_2.

The costs on the links are modeled with the BPR impedance function; see, e.g., [41] for a detailed discussion on the cost functions. The cost functions for links are chosen as follows: $c_l(y_l) = 2(0.18 + (y_l/80)^2)$ for $l = 1, 2, 3, 4, 5, 8, 9, 10, 13, 14, 15, 16, 17$ and $c_l(y_l) = 3(0.18 + (y_l/80)^2)$ for $l = 6, 7, 11, 12$. We consider the case in which the travel demands are $x^1_* = 50$ and $x^2_* = 40$ in vehicles per minute.

The initial splitting rates at the junctions are set as: $\beta^{11}_1 = 0.6$, $\beta^{11}_2 = 0.4$, $\beta^{11}_3 = 0.8$, $\beta^{11}_4 = 0.2$, $\beta^{11}_5 = 0.8$, $\beta^{11}_6 = 0.2$, $\beta^{11}_9 = 0.8$, $\beta^{11}_{10} = 0.2$, $\beta^{12}_3 = 0.375$, $\beta^{12}_4 = 0.625$, $\beta^{12}_7 = 0.2$, $\beta^{12}_8 = 0.8$, $\beta^{12}_9 = 0.2$, $\beta^{12}_{10} = 0.8$, $\beta^{12}_{13} = 0.2$ and $\beta^{12}_{14} = 0.8$.

Figures 7 and 8 show the evolutions of the splitting rates at junctions v_1, v_2, v_3 and v_4. Figures 9 and 10 show the evolution of the traffic flows and the travel costs along routes $r_1, r_2, r_3, r_4, r_5, r_6$. It can be observed that the travel costs of the routes of each OD pair converge to each other, which means the achievement of UE.

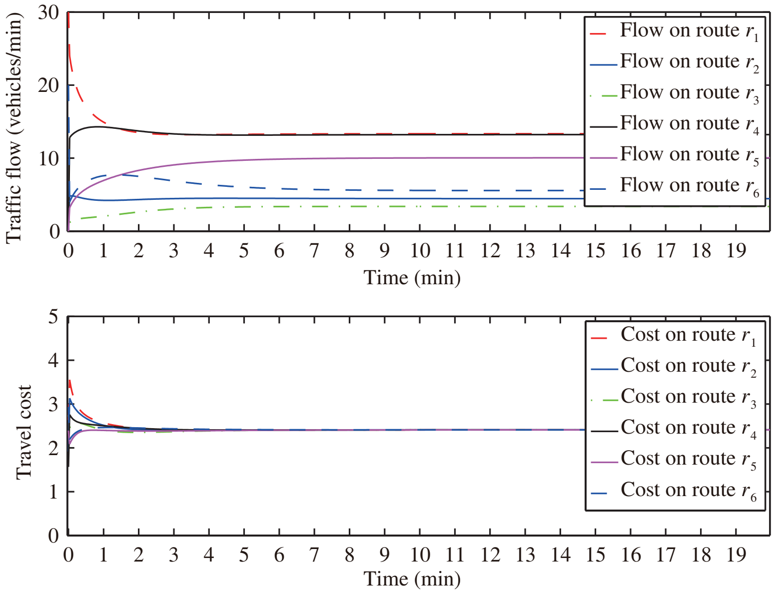

Figure 9 (Color online) The traffic flows on routes r_1, r_2, r_3, r_4, r_5 and r_6 and their corresponding travel costs.

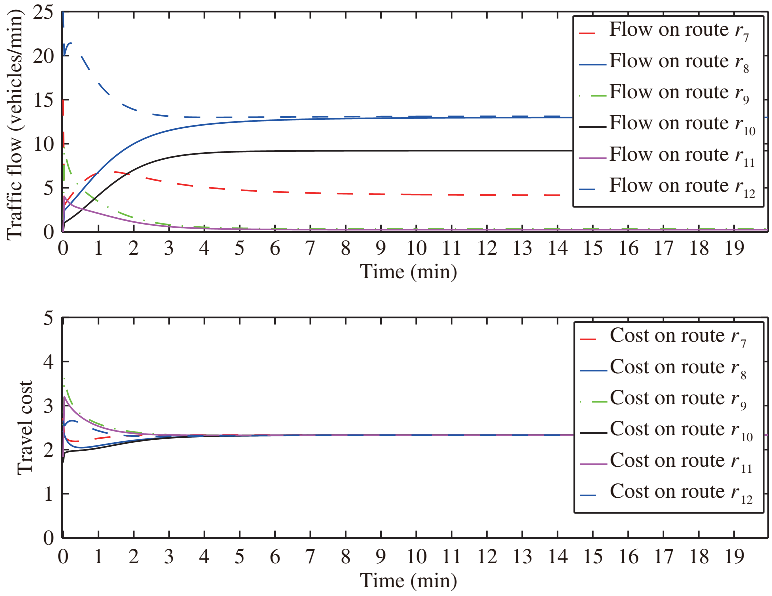

Figure 10 (Color online) The traffic flows on routes r_7, r_8, r_9, r_{10}, r_{11} and r_{12} and their corresponding travel costs.

6 Conclusion

This paper has developed a new junction-by-junction dynamic traffic assignment strategy based on a relaxed driver response model. Specifically, the drivers' response to the route guidance information is directly represented by the splitting rates at the junctions, which are simply negatively correlated with the comparison of related cost information. With the proposed dynamic traffic assignment strategy, the convergence of the traffic flows to a user equilibrium (UE) is guaranteed by using LaSalle's invariance principle. When marginal travel cost information is used instead of travel cost information, a system optimum can be achieved under a mild condition on the objective function.

In practical systems, time-delays caused by information transmission and the drivers' response cannot be neglected. To what extent the influence of time-delays can be attenuated will be studied in our future work. Other issues such as the inaccuracy of the traffic information are also interesting topics for future research. In our recent paper [42], we studied the case in which the travel demand was time-varying in a limited range and a robust convergence result was concluded. It will be interesting to investigate the robustness of the strategy developed in this paper to time-varying travel demands. Another line of future research is to investigate the possibility of applying more recent nonlinear control design methods, see, e.g., [43–48], to the emerging field of dynamic traffic assignment with distributed sensors.

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Conflict of interest The authors declare that they have no conflict of interest.

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